22 Problems

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Welcome! Today is the 26^{th} of December, and it is my birthday :D.

Today we are going to be playing a game called *22 Problems*. This game consists of 22 (mostly) **mathematical** problems and whoever has the highest score by the deadline will be the winner!

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Rules

1. You must try to avoid using the internet. All books are fair game.
2. If your work is unpleasant to read, and / or difficult to mark, I shall discard it.
3. The boxed numbers in the right margin are marks.
4. Deadline: *11:59PM*, 31st of December 2023.
5. Submission: LaTeX appraised, hand-written accepted. FILENAME MUST BE YOUR FULL NAME!

Submit

Problems

1. $$\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$$ [2]

2. $$2\iiint_V \mathrm{d}V, V : \{(r,\theta,\phi)\,|\,0 \le r \le 1,\, 0 \le \theta \le 2\pi,\, 0 \le \phi \le \pi\}$$ [2]

3. $$\int \frac{\cos x}{3+2\cos x}\,\mathrm{d}x$$ [3]

4. Precisely mark out $\sqrt{2}$ on a number line. [2]

5. What is the exact value of $(\frac{3}{2})!$ [2]

6. Prove the Pythagorean Theorem. [3]

7. Find the derivative of $\sin x$ using first principles. State any and all lemmas. [4]

8. (a) List the first 10 terms of the Fibonacci sequence. [1]

   (b) Explain how this sequence is present in the **Mandelbrot Set**. [2]

9. [3]

$$\int_{\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$$

10. What does the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ...$ converge to? [2]

11. Calculus is for _____ whilst analysis is for _____. [1]

12. What is the angle between the two curves $f(x) = x^4 - 5x^3$ and $g(x) = 8x - 40$ at either of their points of intersection? [2]

13. What is the shortest path you can take from node s to node t in figure 1? [2]

14. What are the **complex** solutions to $\sin(z) = 2$? [2]

15. (a) Find a closed form for the recurrence $T(n) = T(n-1) + T(n-2)$, with initial conditions $T(0) = 0$ and $T(1) = 1$. [4]

    (b) Hence find $T(27)$. [1]

16. Solve the following differential equation $y'' + 2y' + y = e^{-x}\cos(x)$ with initial value conditions of $y = 0$ and $y' = 1$. [2]

17. What is the dot product of the functions $\sin(x)$ and $\cos(x)$Linear question. [2]

18. How many permutations of the Rubiks cube exist? Give your answer as an expression. [3]

19. Decode using the Caesar cipher: *Urqh zdv qrw exlow lq d gdb.* [2]

20. Calculate the length of the curve from 0 to 4 for $f(x) = x^2$. [2]

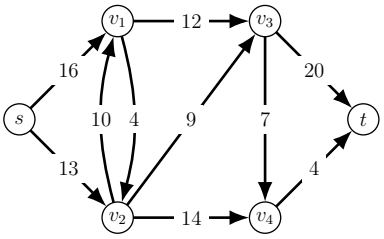
21. Negate the following statement and reexpress it as an equivalent positive one. Everyone who is majoring in math has a friend who needs help with his or her homework. [2]

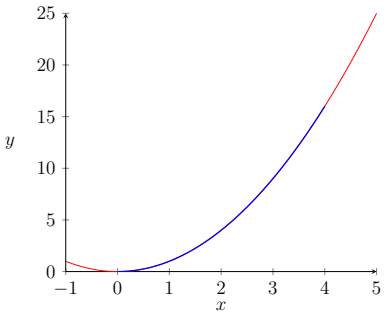
22. Let the Dirichlet function be defined as: [2]

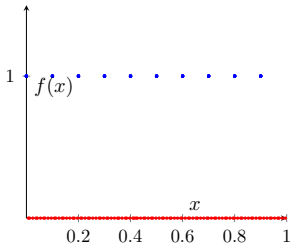
$$\mathrm{D}(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Thus evaluate $\int_0^1 D(x), \mathrm{d}x$.

Diagrams







Marking

Question:	1	2	3	4	5	6	7	8	9	10	11	12
Points:	2	2	3	2	2	3	4	3	3	2	1	2
Score:												

Question:	13	14	15	16	17	18	19	20	21	22		Total
Points:	2	2	5	2	2	3	2	2	2	2		53
Score:												