22 Problems

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Welcome! Today is the 26^{th} of December, and it is my birthday :D.

Today we are going to be playing a game called *22 Problems*. This game consists of 22 (mostly) **mathematical** problems and whoever has the highest score by the deadline will be the winner!

Rules

1. You must try to avoid using the internet. All books are fair game.
2. If your work is unpleasant to read, and / or difficult to mark, I shall discard it.
3. The boxed numbers in the right margin are marks.
4. Deadline: *11:59PM*, 31st of December 2023.
5. Submission: LaTeX appraised, hand-written accepted. FILENAME MUST BE YOUR FULL NAME!

Submit

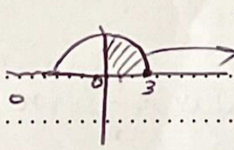
Problems

1. $\boxed{2}$

$$\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$$

$y = \sqrt{9-x^2}$

$\Rightarrow y^2 + x^2 = 9,\ y \geq 0$



$A = \frac{1}{4}\pi r^2$

$= \frac{1}{4}\pi(3)^2 = \boxed{\frac{9\pi}{4}}$

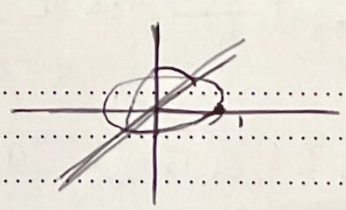
2. $\boxed{2}$

$$2\iiint_V \mathrm{d}V, V : \{(r,\theta,\phi)\,|\,0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$$

$= 2 \times \text{Volume}(V)$

$= \frac{4}{3}\pi r^3$

$= \frac{4\pi}{3}$



3. $\boxed{3}$

$$I = \int \frac{\cos x}{3 + 2\cos x}\,dx$$

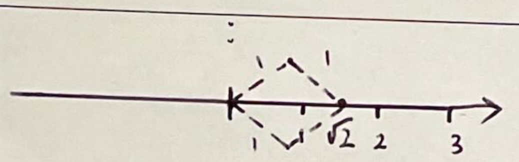
$t = \tan(x/2)$

$\Rightarrow \cos x = \frac{1-t^2}{1+t^2}$ $\qquad I = \int \frac{1-t^2}{3+3t^2+2-2t^2}\,dt$

$\Rightarrow I = \int \frac{\frac{1-t^2}{1+t^2}}{3+\frac{2(1-t^2)}{(1+t^2)}}\,dt$ $\qquad = \int \frac{1-t^2}{5+t^2}\,dt = -\int \frac{t^2+5-6}{5+t^2} = -\int 1\,dt + \int \frac{6}{5+t^2}\,dt$

$= -\tan\left(\frac{x}{2}\right) + \frac{6}{\sqrt{5}}\tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{5}}\right)$

4. Precisely mark out $\sqrt{2}$ on a number line. $\boxed{2}$



5. What is the exact value of $(\frac{3}{2})!$ $\boxed{2}$

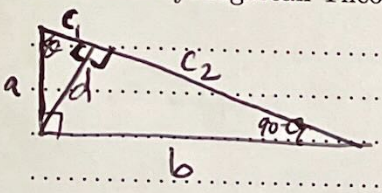
$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ (Reflection formula) $\qquad \Gamma(z+1) = z\Gamma(z)$

$\Rightarrow \Gamma(\frac{1}{2})^2 = \frac{\pi}{\sin(\frac{\pi}{2})}$ $(z = \frac{1}{2})$ $\qquad \Rightarrow \Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$

$\Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\qquad = \frac{\sqrt{\pi}}{2}$

$\Rightarrow (\frac{3}{2})! = \Gamma(\frac{5}{2}) = \boxed{\frac{3\sqrt{\pi}}{4}}$

6. Prove the Pythagorean Theorem. $\boxed{3}$



$\Delta a c_1 d \;|||\; \Delta b d c_2$

$\Rightarrow \frac{c_1}{d} = \frac{d}{c_2} = \frac{a}{b}$

$\Rightarrow (c_1+c_2)^2 = c_1^2 + c_2^2 + 2c_1c_2$

$= \left(\frac{ad}{b}\right)^2 + \left(\frac{bd}{a}\right)^2 + 2d^2$

7. Find the derivative of $\sin x$ using first principles. State any and all lemmas. $\boxed{4}$

$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$

$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos x - \sin x}{h}$ $\qquad$ lemma 1 $\left(\sin(A+B) = \sin A\cos B + \sin B\cos A\right)$

$= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\sin(h)\cos x}{h}$

$= \lim_{h\to 0} \left(\frac{\cos(h)-1}{h}\right)\sin x + \cos x$ $\qquad$ lemma 2 $\left(\lim_{h\to 0} \frac{\sin(h)}{h} = 1\right)$

$= \cos x$ $\qquad$ lemma 3 $\left(\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0\right)$

8. (a) List the first 10 terms of the Fibonacci sequence. $\boxed{1}$

1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b) Explain how this sequence is present in the **Mandelbrot Set**. 2

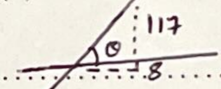
9. 3

$$\int_{\infty}^{\infty} e^{-x^2}\, dx$$

10. What does the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$ converge to? 2

11. Calculus is for everyone whilst analysis is for mathematicians. 1

12. What is the angle between the two curves $f(x) = x^4 - 5x^3$ and $g(x) = 8x - 40$ at either of their points of intersection? 2



$f(x) = g(x)$

$\Rightarrow x^4 - 5x^3 - 8x + 40 = 0$

$x^3(x-5) - 8(x-5) = 0$

$\Rightarrow (x^3 - 8)(x-5) = 0$

$\Rightarrow x = 5$ or 2

$f'(5) = 4(5)^3 - 15(5)^2$

$= 125$

$g'(5) = 8$

$\theta = \tan^{-1}(125) - \tan^{-1}(8)$

$\boxed{= 6.67°}$

13. What is the shortest path you can take from node s to node t in figure 1? 2

$s \rightarrow V_2 \rightarrow V_4 \rightarrow t$

14. What are the **complex** solutions to $\sin(z) = 2$? 2

$e^{iz} - e^{-iz} = 4i$

$\Rightarrow e^{i(a+ib)} - e^{-i(a+ib)} = 4e^{i\pi/2}$

$e^{-b}e^{ia} - e^{b}e^{-ia} = 4e^{i\pi/2}$

$e^{-b}(cis(a)) - e^{b}(cis(-a)) = 4i$

$\cos(a)(e^{-b} - e^{b}) + i\sin(a)(e^{-b} + e^{b}) = 4i$

$\Rightarrow \cos(a)(e^{-b} - e^{b}) = 0, \quad \sin(a)(e^{-b} + e^{b}) = 4$

$b = 0$ or $a = (2k+1)\frac{\pi}{2}$

impossible because $\sin(a) \neq 2$

$\Rightarrow e^{-b} + e^{b} = 4(-1)^k$

$e^{2b} + 4(-1)^k e^{b} + 1 = 0$

$e^{b} = \frac{4(-1)^k \pm \sqrt{16-4}}{2}$

$= 2(-1)^k \pm \sqrt{3}$

$\Rightarrow b = \ln(2 \pm \sqrt{3})$

$\boxed{z = (4k+1)\frac{\pi}{2} + i\ln(2 \pm \sqrt{3})}$

for $k \in \mathbb{Z}$

1 4 9 16 25 36
1 3 6 10 15, 21

15. (a) Find a closed form for the recurrence $T(n) = T(n-1) + T(n-2)$, with initial conditions $T(0) = 0$ and $T(1) = 1$. [4]

0, 1, 1, 2, 3, 5, 8, 13, 21, 34

(b) Hence find $T(27)$. [1]

16. Solve the following differential equation $y'' + 2y' + y = e^{-x}\cos(x)$ with initial value conditions of $y = 0$ and $y' = 1$. [2]

$r^2+2r+1=0 \Rightarrow (r+1)^2=0$

$y = e^{-x}(A\cos x + B\sin x)$

$2y' = -2e^{-x}(A\cos x + B\sin x) + 2e^{-x}(B\cos x - A\sin x)$

$y'' = \cancel{2}e^{-x}(A\cos x + B\sin x) - \cancel{2}e^{-x}(B\cos x - A\sin x) - \cancel{2}e^{-x}(B\cos x - A\sin x) + \cancel{2}e^{-x}(-A\cos x - B\sin x)$

$= -\cancel{2}e^{-x}(B\cos x - A\sin x)$

$y + 2y' + y'' = -e^{-x}(A\cos x + B\sin x) \Rightarrow B=0, A=-1$

$\boxed{y = Ae^{-x} - e^{-x}\cos x}$

17. What is the dot product of the functions $\sin(x)$ and $\cos(x)$Linear question. [2]

$\sin(2x) = 2\sin x\cos x$

$\langle \sin x | \cos x \rangle = \int \sin x \cos x \, dx$

$= \frac{1}{2}\int \sin(2x)\,dx = -\frac{1}{4}\cos(2x)$

18. How many permutations of the Rubiks cube exist? Give your answer as an expression. [3]

Centre squares are fixed $\Rightarrow$ 48 remaining squares.

8 corners can swap + rotate $\Rightarrow 8! \times 3^8$

12 edges can swap + rotate $\Rightarrow 12! \times 2^{12}$

$\boxed{8! \times 3^8 \times 12! \times 2^{12}}$

19. Decode using the Caesar cipher: *Urqh zdv qrw exlow lq d gdb.* [2]

rone was not built in a day

20. Calculate the length of the curve from 0 to 4 for $f(x) = x^2$. [2]

$L = \int_0^4 \sqrt{1+[f'(x)]^2}\,dx = \int_0^4 \sqrt{1+(2x)^2}\,dx = \int_0^4 \sqrt{1+4x^2}\,dx$

21. Negate the following statement and reexpress it as an equivalent positive one. EVERYONE WHO IS MAJORING IN MATH HAS A FRIEND WHO NEEDS HELP WITH HIS OR HER HOMEWORK. $\boxed{2}$

Negation: There exists a person majoring in math who does not have a friend who needs help with his or her homework.

OR There exists a person majoring in math whose friends all do not need help with homework.

22. Let the Dirichlet function be defined as: $\boxed{2}$

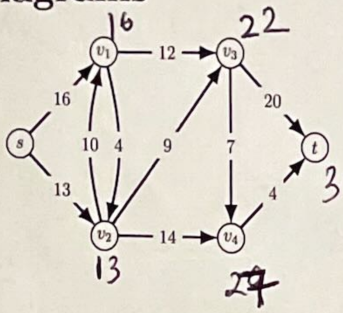
$$D(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

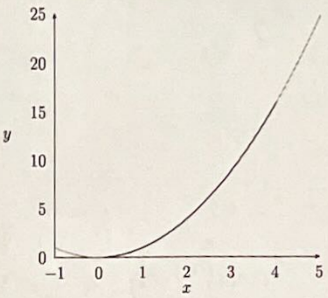
Thus evaluate $\int_0^1 D(x), dx$.

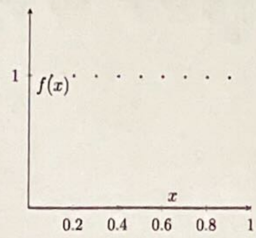
$\int_0^1 D(x)dx = \mu$ (Set of Rationals $\in [0,1]$)

$= 0$ (Since rationals are countable).

Diagrams







Marking

Question:	1	2	3	4	5	6	7	8	9	10	11	12
Points:	2	2	3	2	2	3	4	3	3	2	1	2
Score:												

Question:	13	14	15	16	17	18	19	20	21	22		Total
Points:	2	2	5	2	2	3	2	2	2	2		53
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