24 Problems

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Contents

Rules

As is tradition, the prize pool has increased (to $300 this year).

I have collapsed first and second place into a winner-takes-all arrangement

(c'est la vie).

Furthermore, there are additional changes to the structure of this Game:

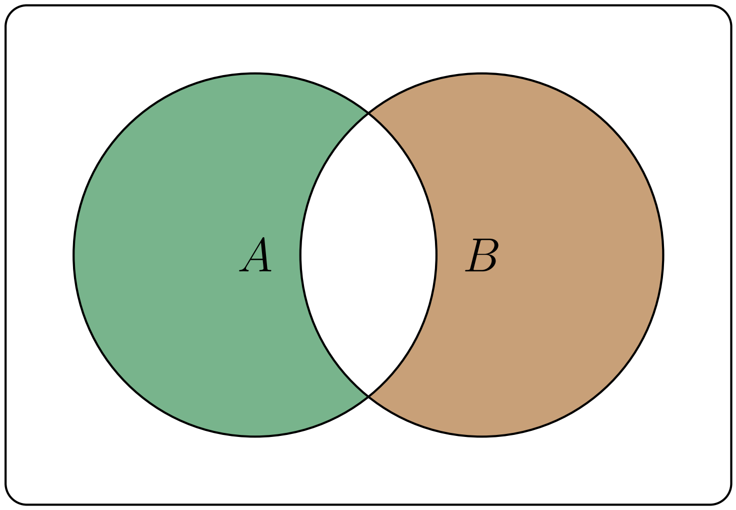
1. you must now *pass* the problem set to be awarded the prize money;
2. you may submit your solutions to the problem set at any point in the future;
3. if you plagiarise work, I reserve the right to ban you from all subsequent competitions — grim trigger
4. the problem set is also available on my website
   - there you will find an alternative presentation of these problems upheld with MathJaX, TikZ and JavaScript.
   - the url for the problem set is https://abaj.ai/projects/bday-problems/24th/problems
   - my solutions will be available from the start of 2026; by viewing them you *forfeit the prize money* (obviously)
5. Good luck!

Submit

§ Problems

1. Given that A and B are sets like so: 4

   A and B are non-empty sets



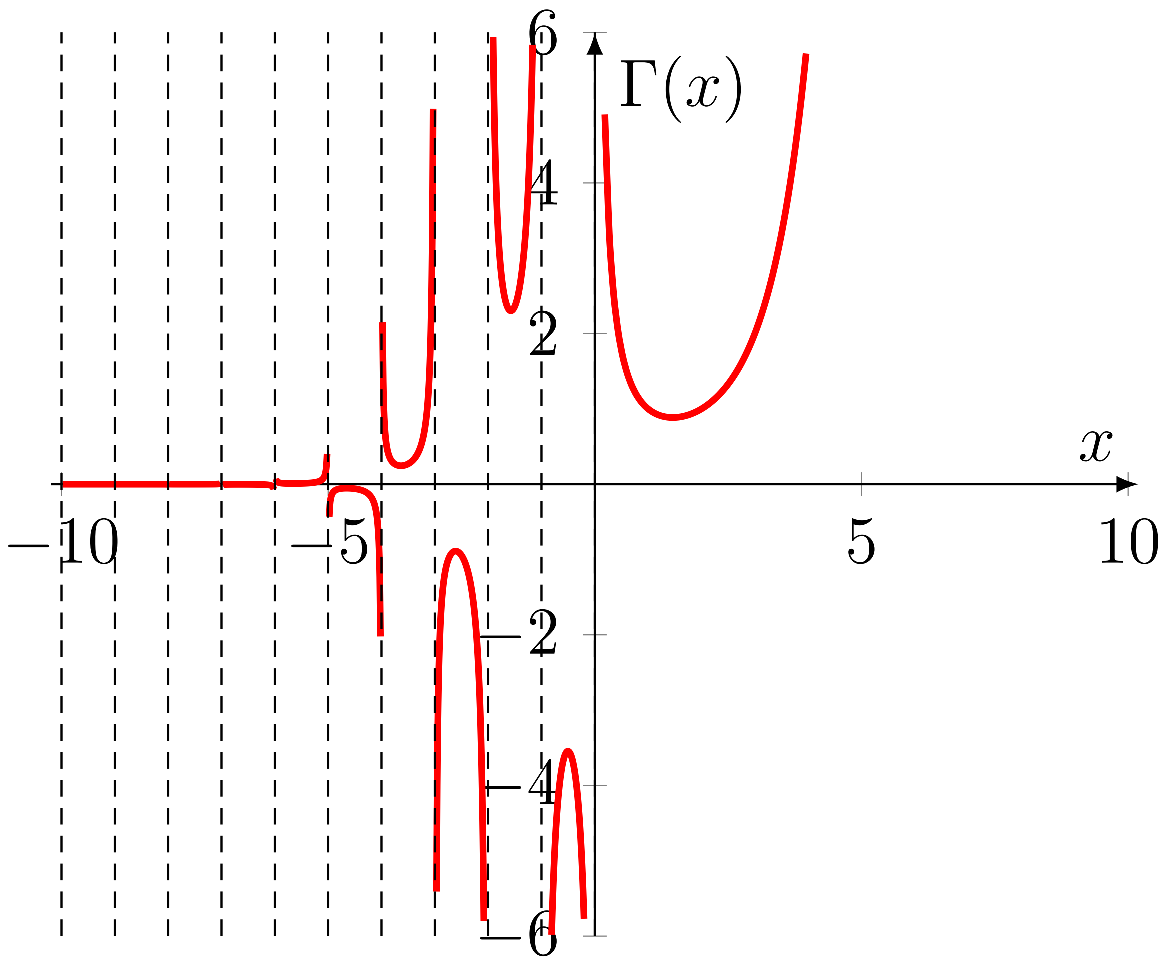
   (a) Give an expression for the conditional probability $P(A|B)$ 1

   (b) Hence or otherwise derive Bayes' (first) rule for conditional probability: 1

$$P(A|B) = P(B|A) \times \frac{P(A)}{P(B)}$$

   (c) Prove that the symmetric difference $A \Delta B = (A - B) \cup (B - A)$ is the same as $(A \cup B) - (A \cap B)$. 2

2. The Gamma Function 9



(a) Recall the integral of $\boxed{1}$

$$\int e^{-x}\mathrm{d}x$$

(b) Apply integration by parts on $\boxed{2}$

$$\int xe^{-x}\mathrm{d}x.$$

Factorise your result.

(c) These above are special cases of $\alpha = 1$ and $\alpha = 2$. More generally we can compute the integration of $\boxed{3}$

$$\int x^{\alpha-1}e^{-x}\mathrm{d}x.$$

(d) Fixing the limits of integration leads to the Gamma function, defined by $\boxed{3}$

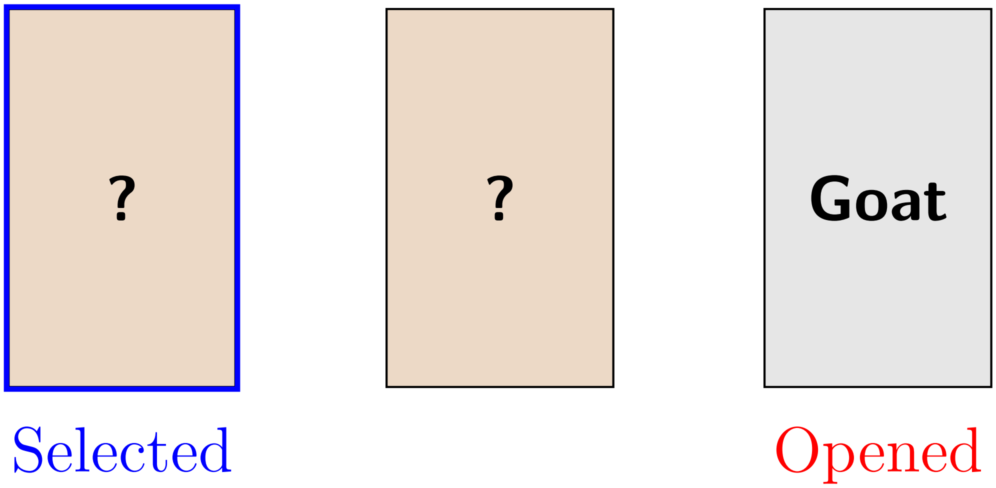
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\mathrm{d}x, \quad \text{for } \alpha > 0$$

Show that with (??), it follows that

$$\Gamma(\alpha) = (\alpha - 1) \times \Gamma(\alpha - 1).$$

Therefore, we have for any integral k:
$\Gamma(k) = (k-1) \times (k-2) \times \ldots \times 2 \times \Gamma(1)$, and $\Gamma(1) = 1$ due to (??).

3. Monty Hall Problem: [5]



   (a) Suppose there are three curtains. Behind one curtain there is a nice prize, while behind the other two there are worthless prizes (perhaps 1 car and 2 goats). A contestant selects one curtain at random, and then Monte Hall opens one of the other two curtains to reveal a worthless prize (goat). Hall then expresses the willingness to trade the curtain that the contestant has chosen for the other curtain that has not been opened. Should the contentestant switch curtains or stick or stick with the one that she has? [2]

   (b) Now consider the variant where the host does not know which door hides the car. What is the **posterior** probability of winning the prize / car if our contestant switches? [3]

4. (a) Prove that there are equally as many **natural numbers** as there are **integers**, i.e. that $|\mathbb{Z}| = |\mathbb{N}|$. [2]

   (b) Prove that there are equally as many **integers** as there are **rational numbers**, i.e. that $|\mathbb{Z}| = |\mathbb{Q}|$. [2]

   (c) Prove that the *real numbers* are *uncountable*; i.e. that $|\mathbb{R}| \neq |\mathbb{N}|$. [3]

5. (a) What does the sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$ converge to? (If at all) [2]

   (b) What does the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converge to? (If at all) [2]

6. Which is larger asymptotically as $n \to \infty$? [4]

$$2^n \ll n!$$

OR

$$2^n \gg n!$$

Give a proof by induction.

7. (a) Find the eigenspaces of the following matrices: [12][4]

   1. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
   2. $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$
   3. $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$
   4. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   (b) Determine if the following matrices are diagonalisable. If so, determine their diagonal form and a basis with respect to which the transformation matrices are diagonal. [8]

   If they are not diagonal, give reasons why not.

   1. $\begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$
   2. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
   3. $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$
   4. $\begin{bmatrix} 5 & 4 & 2 & 1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$

8. Consider the following bivariate distribution $p(x, y)$ of two discrete random variables X and Y. [4]

	y_1	0.01	0.02	0.03	0.1	0.1
Y	y_2	0.05	0.1	0.05	0.07	0.2
	y_3	0.1	0.05	0.03	0.05	0.04
		x_1	x_2	x_3	x_4	x_5
				X		

(a) Compute the marginal distributions $p(x)$ and $p(y)$. [2]

(b) Compute the conditional distributions $p(x|Y = y_1)$ and $p(y|X = x_3)$. [2]

9. Consider two random variables, x, y with joint distribution $p(x, y)$. Show that [3]

$$\mathbb{E}_X[x] = \mathbb{E}_Y[\mathbb{E}_X[x|y]]$$

Here, $\mathbb{E}_X[x|y]$ denotes the expected value of x under the conditional distribution $p(x|y)$.

10. Last year we qualitatively described a number of Probability Distributions, this year we shall discover more results. [7]

To begin, find the Probability Mass/Density Functions for the following distributions:

1. Binomial
2. Bernoulli
3. Geometric
4. Poisson
5. Exponential
6. Gamma
7. Normal

11. (a) What is the formula for the Moment Generating Function? What about the kth (central) MGF? [23]
Hint.

$$\varphi_X(s) = \mathbb{E}\,(?) = \int ?$$

(b) Hence, or otherwise derive the **Expectation**, **Variance** and **Moment Generating Function** for all 7 distributions in Q10. Show minimal working. [21]

12. Shortest distance between an arbitrary point and hyperplane given by $wx + b = 0$ [4]

13. Consider minimising the strictly convex quadratic function [4]

$$q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{G}\mathbf{x} + \mathbf{d}^T\mathbf{x} + c$$

on $\mathbb{R}^n$, where $n > 1$, $\mathbf{d}$ is a constant $n \times 1$ vector, $\mathbf{G}$ is a constant $n \times n$ symmetric positive definite matrix and c is a scalar. Let $\mathbf{x}^* \in \mathbb{R}^n$ be the global minimiser for $q(\mathbf{x})$, $\mathbf{x}^{(1)} \in \mathbb{R}^n$ and $\mathbf{x}^{(1)} \neq \mathbf{x}^*$.

Consider applying **Newton's method** to $q(\mathbf{x})$ starting at $\mathbf{x}^{(1)}$

(a) Write down the Newton direction at $\mathbf{x}^{(1)}$ and show that it is a descent direction. $\boxed{2}$

(b) How many iteration(s) will Newton's method take to reach the minimiser $\mathbf{x}^*$ of $q(\mathbf{x})$. Give reasons for your answer. $\boxed{2}$

14. Consider the following inequality constrained optimisation problem $\boxed{14}$

$$\begin{aligned} \text{(IP)} \quad \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimise}} \quad & \mathbf{a}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{x}^T Q\mathbf{x} - 1 \le 0, \end{aligned} \tag{8}$$

where $n \ge 1$, Q is a symmetric and **positive definite** $n \times n$ matrix, $\mathbf{a} \in \mathbb{R}^n, \mathbf{a} \neq \mathbf{0}$ and the constraint function $\mathbf{c}(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x} - 1$.

(a) Write down the gradient $\nabla c(\mathbf{x})$ and the Hessian $\nabla^2 c(\mathbf{x})$. $\boxed{2}$

(b) Show that $c(\mathbf{x})$ is a convex function. $\boxed{2}$

(c) Show that the problem (IP) is a convex optimisation problem. $\boxed{2}$

(d) Show that $\mathbf{x}^* = \frac{-Q^{-1}\mathbf{a}}{\sqrt{\mathbf{a}^T Q^{-1}\mathbf{a}}}$ is a constrained stationary point for the problem (IP). $\boxed{2}$

(e) Determine whether $\mathbf{x}^* = \frac{-Q^{-1}\mathbf{a}}{\sqrt{\mathbf{a}^T Q^{-1}\mathbf{a}}}$ is a local minimiser, global minimiser or neither for the problem (IP). $\boxed{2}$

(f) Write down the Wolfe dual problem for (IP). $\boxed{2}$

(g) What is the optimal objective value of the Wolfe dual problem in part f)? Give reasons for your answer. You do not need to solve the Wolfe dual problem. $\boxed{2}$

15. Consider the following equality constrained optimisation problem $\boxed{15}$

$$\begin{aligned} \text{(EP)} \quad \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \mathbf{a}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^T\mathbf{x} - 1 = 0, \end{aligned}$$

where $n \ge 1$, $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{0}$ and $\|\mathbf{a}\|_2 = \sqrt{\mathbf{a}^T\mathbf{a}}$. Let $\mathbf{z}^* = \frac{-\mathbf{a}}{\|\mathbf{a}\|_2}$.

where $n \ge 1$, Q is a symmetric and **positive definite** $n \times n$ matrix, $\mathbf{a} \in \mathbb{R}^n, \mathbf{a} \neq \mathbf{0}$ and the constraint function $\mathbf{c}(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x} - 1$.

(a) Show that global minima of (EP) must exist. $\boxed{2}$

(b) Show that $\mathbf{z}^*$ is a regular feasible point for (EP). $\boxed{2}$

(c) Show rigorously that the feasible region $\Omega = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{x} - 1 = 0\}$ of (EP) is not a convex set. $\boxed{2}$

(d) Verify that $\mathbf{z}^*$ is a constrained stationary point for (EP). $\boxed{3}$

(e) Using the second-order sufficient optimality conditions, show that $\mathbf{z}^*$ is a strict local minimiser for (EP). [3]

(f) Show that $\mathbf{z}^*$ is a global minimiser for (EP). [3]

16. (a) Prove that [0/2]

Proposition.

$$|a| \leq b \iff -b \leq a \leq b$$

□

(b) Prove that [2]

Proposition.

$$\forall a, b \in \mathbb{R}, \quad |a| \cdot |b| = |a \cdot b|$$

□

(c) Complete the following [2]

Proposition.

Two real numbers, a and b are equal if and only if, for every $\epsilon > 0$, $|a - b| < \epsilon$.

□

17. Complete the following [3]

Proposition.

Let $\mathcal{A} \subseteq \mathbb{R}$ and $m \in \mathcal{A}$ be the minimum of $\mathcal{A}$, then $\inf \mathcal{A} = m$.

□

18. (a) State and prove the Cauchy-Schwarz inequality. [7/4]

Proposition.

□

(b) Hence or otherwise, state and prove the Triangle Inequality. [3]

Proposition.

□

19. (a) Give the definitions of these elementary analysis facts: 14 8

Definition: Metric Space

Definition: Epsilon Ball

Definition: Interior Point

Definition: A subset Y in (X, d) is **open** if

(b) Complete the following 6

Proposition.

Every ϵ-ball in a metric space is open.

□

20. Infinite-dimensional vector spaces 12

(a) What are the sets c_{00}, c_0, ℓ^p and ℓ^∞? Give examples of sequences in each. 8

Definition: c_{00}

Example: c_{00}

Definition: c_0

Example: c_0

Definition: ℓ^p

Example: ℓ^p

Definition: ℓ^∞

Example: ℓ^∞

Remark.

Relationships:

$$c_{00} \subsetneq c_0 \subsetneq \ell^p \subsetneq \ell^\infty \quad \text{(for any } 1 \leq p < \infty\text{)}$$

(b) Give the definition and at least 2 examples for each each: 4

Definition: Hilbert Space

Example: Hilbert Space

Definition: Banach Space

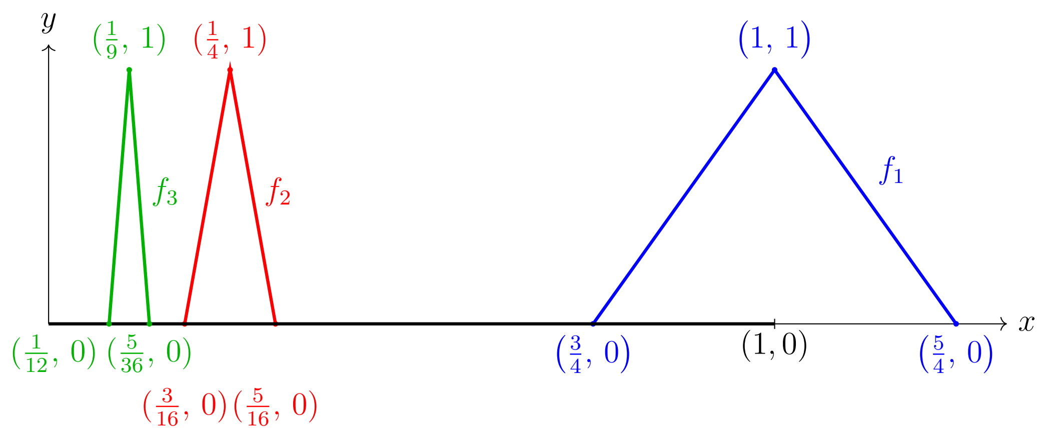
Example: Banach Space

21. (a) State **pointwise convergence** for a sequence of functions. 8

(b) State **uniform convergence** for a sequence of functions 2

(c) Does the following function converge pointwise? What about uniformly? 2
For each integer $k \geq 1$ define

$$f_k : [0,1] \longrightarrow \mathbb{R}, \qquad f_k(x) = \max\{0, 1 - 4k^2|\,x - \tfrac{1}{k^2}|\}.$$



(d) What does uniform convergence imply about a series? 2

22. (a) Give the definition of a topological space (X, τ). 7

(b) Give the definitions of these Topological spaces: 3
Definition: Co-countable Topology

Definition: Co-finite Topology

Definition: Discrete Topology

(c) Are limits unique in a topological space? What about in a metric space? 2

23. Let $(X_1, \ldots, X_n) \overset{\text{i.i.d}}{\sim} \mathcal{P}(\lambda)$. 14

(a) Let 3

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

be an estimator of λ. Find the bias, standard error and Mean Squared Error of $\bar{X}_n$.

(b) Let 2

$$\widehat{T}_n = \frac{X_1 + 2X_2 + \ldots + nX_n}{1 + 2 + \cdots + n}.$$

Compute the bias and MSE of $\widehat{T}_n$ as an estimator of λ.

(c) Compare $\widehat{T}_n$ to $\bar{X}_n$ in terms of MSE. Which estimator is preferable? 2
Intuitively, why is one better than the other?

(d) Find the moment estimator for λ. $\boxed{2}$

(e) Find the maximum likelihood estimator for λ. $\boxed{3}$

(f) Find the Fisher information $\mathcal{I}(\lambda)$ $\boxed{2}$

24. What is the maximum straight-line distance in an N-dimensional unit hyper-cube? $\boxed{3}$