CHAPTER

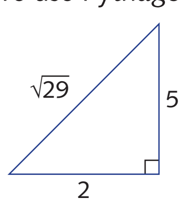
2

Number and Algebra

Review of surds

In this chapter, we revise our work on **surds**, which are a special class of irrational numbers that you studied in *ICE-EM Mathematics Year 9*.

Surds, such as $\sqrt{29}$, arise when we use Pythagoras' theorem.



$$5^2 + 2^2 = 29$$

The values of the trigonometric ratios of some common angles are surds.

For example, $\cos 30^\circ = \dfrac{\sqrt{3}}{2}$.

Surds also arise when solving quadratic equations and can often be treated as if they are pronumerals. Skills in manipulating surds strengthen algebraic skills.

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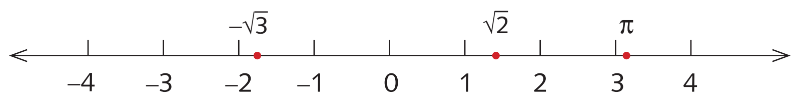
2A Irrational numbers and surds

Irrational numbers

When we apply Pythagoras' theorem, we often obtain numbers such as $\sqrt{2}$. The numbers $\sqrt{2}$ and $\sqrt{3}$ are examples of **irrational** numbers and so is π, the number that arises from circles. Note that $\sqrt{2}$ means the positive square root of 2.

Recall that a **rational** number is a number that can be written as $\frac{p}{q}$, where p is an integer and q is a non-zero integer.

A real number is a point on the number line. Every rational number is real but, as we have seen, not every real number is rational. A real number that is not rational is called **irrational**.



As we have mentioned, every real number is a point on the number line and, conversely, every point on the number line is real.

Surds can always be approximated by decimals, but working with exact values enables us to see important relationships and gives insights that would be lost if we approximated everything.

Surds

We can take the n^{th} root of any positive number a. The n^{th} root of a, written as $\sqrt[n]{a}$, is the positive number whose n^{th} power is a. Thus, $\sqrt[n]{a} = b$ is equivalent to the statement $b^n = a$.

If $\sqrt[n]{a}$ is irrational, then it is called a **surd**. If $\sqrt[n]{a}$ is rational, then a is the n^{th} power of a rational number. Hence, $\sqrt{3}$, $\sqrt[3]{5}$ and $\sqrt[5]{7}$ are surds. On the other hand, $\sqrt[3]{8} = 2$ and $\sqrt[4]{81} = 3$, so they are not surds.

Approximations to surds can be found using a calculator.

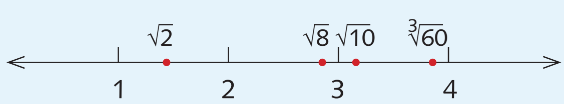
Example 1

Use your calculator to arrange the surds $\sqrt{8}$, $\sqrt{10}$, $\sqrt{2}$ and $\sqrt[3]{60}$ in order of size on the number line.

Solution

We use a calculator to find an approximation to each number, correct to two decimal places.

$\sqrt{8} \approx 2.83$ $\quad$ $\sqrt{10} \approx 3.16$ $\quad$ $\sqrt{2} \approx 1.41$ $\quad$ $\sqrt[3]{60} \approx 3.91$



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Constructing some surds geometrically

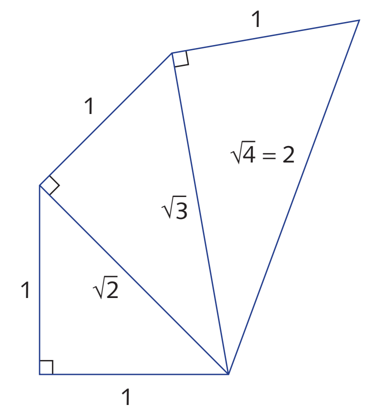
We can use Pythagoras' theorem to construct lengths of $\sqrt{2}$, $\sqrt{3}$ and so on.

Using a ruler, draw a length of 1 unit.

Using your ruler and compasses or set square, draw a right angle at the end of your interval and mark off 1 unit.

Joining the endpoints, we have a length $\sqrt{2}$ units by Pythagoras' theorem.

If we now draw an interval of length 1 unit perpendicular to the hypotenuse, as shown in the diagram, and form another right-angled triangle, then the new hypotenuse is $\sqrt{3}$ units in length. We can continue this process, as shown, to construct the numbers $\sqrt{5}$, $\sqrt{6}$ and so on.



Irrational numbers and surds

- Every **real** number is a point on the number line and, conversely, every point on the number line is a real number.
- Every **rational** number is a real number. A real number that is not rational is called an **irrational** number.
- If a is a positive rational number and $\sqrt[n]{a}$ is irrational, then $\sqrt[n]{a}$ is called a **surd**.

Arithmetic with surds

We will review the basic rules for working with square roots.

When we write $2\sqrt{3}$, we mean $2 \times \sqrt{3}$. As in algebra, we can omit the multiplication sign.

If a and b are positive numbers, then:

$$(\sqrt{a})^2 = a$$
$$\sqrt{a^2} = a$$
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$
$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

For example:

$$(\sqrt{11})^2 = 11$$
$$\sqrt{3^2} = 3$$
$$\sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7} = \sqrt{21}$$
$$\sqrt{35} \div \sqrt{5} = \sqrt{\frac{35}{5}} = \sqrt{7}$$

The first two of these rules remind us that, for positive numbers, squaring and taking a square root are **inverse processes**. For example:

$$(\sqrt{7})^2 = 7 \text{ and } \sqrt{7^2} = 7$$

Also, $\sqrt{\pi^2} = \pi$ and $(\sqrt{\pi})^2 = \pi$. Note that $\sqrt{\pi}$ is not a surd.

Take the surd $\sqrt{12}$. We can factor out the perfect square 4 from 12, and write:

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \times 3}\\ &= \sqrt{4} \times \sqrt{3} \qquad (\sqrt{ab} = \sqrt{a} \times \sqrt{b})\\ &= 2\sqrt{3}\end{aligned}$$

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Hence, $\sqrt{12}$ and $2\sqrt{3}$ are equal. We will regard $2\sqrt{3}$ as a **simpler form** than $\sqrt{12}$, since the number under the square root sign is smaller.

To **simplify** a surd (or a multiple of a surd), we write it so that the number under the square root sign has no factors that are perfect squares (apart from 1). For example:

$$\sqrt{12} = 2\sqrt{3}$$

We shall also refer to any rational multiple of a surd as a surd. For example, $4\sqrt{7}$ is a surd.

In mathematics, we are often instructed to leave our answers in **surd form**. This means that we should not approximate the answer using a calculator, but leave the answer – in simplest form – expressed using square roots, cube roots etc. This is also called **giving the exact value** of the answer.

We can use our knowledge of factorising whole numbers to simplify surds.

Example 2

Simplify:

a $\sqrt{50}$

b $\sqrt{27}$

Solution

a $\sqrt{50} = \sqrt{25} \times \sqrt{2}$
$= 5\sqrt{2}$

b $\sqrt{27} = \sqrt{9} \times \sqrt{3}$
$= 3\sqrt{3}$

We look for factors of the number under the square root sign that are perfect squares. Sometimes we may need to do this in stages.

Example 3

Simplify:

a $\sqrt{588}$

b $7\sqrt{243}$

c $6\sqrt{162}$

Solution

a
$$\begin{aligned}\sqrt{588} &= \sqrt{4 \times 147}\\ &= 2\sqrt{147}\\ &= 2\sqrt{49 \times 3}\\ &= 2 \times 7\sqrt{3}\\ &= 14\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}7\sqrt{243} &= 7\sqrt{3^5}\\ &= 7\sqrt{3^4 \times 3}\\ &= 7 \times 9\sqrt{3}\\ &= 63\sqrt{3}\end{aligned}$$

c
$$\begin{aligned}6\sqrt{162} &= 6\sqrt{81 \times 2}\\ &= 6\sqrt{9^2 \times 2}\\ &= 54\sqrt{2}\end{aligned}$$

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In some problems, we need to reverse this process.

Example 4

Express each as the square root of a whole number.

a $5\sqrt{7}$

b $7\sqrt{6}$

Solution

a
$$\begin{aligned} 5\sqrt{7} &= \sqrt{5^2} \times \sqrt{7} \\ &= \sqrt{25 \times 7} \\ &= \sqrt{175} \end{aligned}$$

b
$$\begin{aligned} 7\sqrt{6} &= \sqrt{49 \times 6} \\ &= \sqrt{294} \end{aligned}$$

Example 5

Simplify each expression.

a $\sqrt{5} \times \sqrt{7}$

b $\sqrt{3} \times \sqrt{11}$

c $\sqrt{5} \times \sqrt{30}$

d $\sqrt{3} \times \sqrt{15}$

Solution

a $\sqrt{5} \times \sqrt{7} = \sqrt{35}$

b $\sqrt{3} \times \sqrt{11} = \sqrt{33}$

c
$$\begin{aligned} \sqrt{5} \times \sqrt{30} &= \sqrt{150} \\ &= \sqrt{25 \times 6} \\ &= 5\sqrt{6} \end{aligned}$$

d
$$\begin{aligned} \sqrt{3} \times \sqrt{15} &= \sqrt{45} \\ &= \sqrt{9 \times 5} \\ &= 3\sqrt{5} \end{aligned}$$

Example 6

Simplify each expression.

a $\sqrt{15} \div \sqrt{3}$

b $\dfrac{\sqrt{70}}{\sqrt{14}}$

Solution

a
$$\begin{aligned} \sqrt{15} \div \sqrt{3} &= \sqrt{\frac{15}{3}} \\ &= \sqrt{5} \end{aligned}$$

b
$$\begin{aligned} \frac{\sqrt{70}}{\sqrt{14}} &= \sqrt{\frac{70}{14}} \\ &= \sqrt{5} \end{aligned}$$

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Algebra of surds

- If a and b are positive numbers, then:

$$\left(\sqrt{a}\right)^2 = a$$
$$\sqrt{a^2} = a$$
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$
$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

- A surd is in its **simplest form** if the number under the square root sign has no factors that are perfect squares (apart from 1).
- To simplify a surd, take out any square factors.

  For example, $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$

Exercise 2A

Example 1

1 Arrange the irrational numbers $\sqrt{3}, \sqrt{6}, \sqrt[3]{30}$ and $\sqrt[5]{60}$ in order of size on the number line.

Example 2, 3

2 Simplify:

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{32}$ **d** $\sqrt{50}$ **e** $\sqrt{54}$
f $\sqrt{108}$ **g** $\sqrt{98}$ **h** $\sqrt{200}$ **i** $\sqrt{288}$ **j** $\sqrt{147}$
k $\sqrt{112}$ **l** $\sqrt{175}$ **m** $\sqrt{245}$ **n** $\sqrt{294}$ **o** $\sqrt{225}$
p $\sqrt{900}$ **q** $\sqrt{450}$ **r** $\sqrt{800}$ **s** $\sqrt{1000}$ **t** $\sqrt{1728}$

Example 3

3 Simplify:

a $2\sqrt{75}$ **b** $4\sqrt{125}$ **c** $6\sqrt{99}$ **d** $3\sqrt{150}$ **e** $2\sqrt{720}$
f $5\sqrt{245}$ **g** $6\sqrt{32}$ **h** $7\sqrt{50}$ **i** $11\sqrt{108}$ **j** $56\sqrt{100}$
k $7\sqrt{75}$ **l** $3\sqrt{176}$ **m** $2\sqrt{208}$ **n** $5\sqrt{275}$ **o** $4\sqrt{300}$

Example 4

4 Express each as the square root of a whole number.

a $2\sqrt{2}$ **b** $3\sqrt{5}$ **c** $7\sqrt{3}$ **d** $6\sqrt{6}$ **e** $10\sqrt{3}$
f $4\sqrt{10}$ **g** $11\sqrt{5}$ **h** $7\sqrt{50}$ **i** $6\sqrt{3}$ **j** $3\sqrt{20}$

Example 5

5 Simplify:

a $\sqrt{2} \times \sqrt{3}$ **b** $\sqrt{7} \times \sqrt{11}$ **c** $\sqrt{8} \times \sqrt{5}$ **d** $\sqrt{3} \times \sqrt{13}$ **e** $\sqrt{6} \times \sqrt{8}$

Example 6

6 Simplify:

a $\frac{\sqrt{10}}{\sqrt{2}}$ **b** $\frac{\sqrt{10}}{\sqrt{5}}$ **c** $\frac{\sqrt{30}}{\sqrt{5}}$ **d** $\frac{\sqrt{50}}{\sqrt{10}}$ **e** $\frac{\sqrt{18}}{\sqrt{6}}$ **f** $\frac{\sqrt{24}}{\sqrt{3}}$

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7 Simplify:

a $\sqrt{2} \times \sqrt{7}$ **b** $\sqrt{2} \times \sqrt{8}$ **c** $\sqrt{3} \times \sqrt{6}$ **d** $\left(\sqrt{2}\right)^3 \times \left(\sqrt{3}\right)^2$

e $\sqrt{12} \div \sqrt{2}$ **f** $\left(\sqrt{5}\right)^2 - \sqrt{5^2}$ **g** $\sqrt{5^3} \times \sqrt{5}$ **h** $\sqrt{18} \div \sqrt{2}$

8 Complete:

a $\sqrt{5} \times \ldots = \sqrt{30}$ **b** $\sqrt{12} \times \ldots = \sqrt{36}$ **c** $\sqrt{15} \times \ldots = \sqrt{45}$

d $\dfrac{\sqrt{100}}{\ldots} = \sqrt{20}$ **e** $\dfrac{\ldots}{\sqrt{11}} = \sqrt{3}$ **f** $\dfrac{\sqrt{20}}{\ldots} = 2$

9 **a** Find the area of a rectangle with height $\sqrt{7}$ cm and width $\sqrt{3}$ cm.

b Find the area of a triangle with base $\sqrt{6}$ cm and height $\sqrt{5}$ cm.

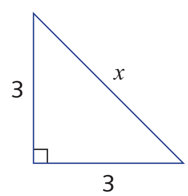
c Find the area of a square with side length $\sqrt{17}$ cm.

d A square has area 11 cm^2. What is the length of each side?

e A square has area 63 cm^2. What is the length of each side?

10 Use Pythagoras' theorem to find the value of x in exact form.

a



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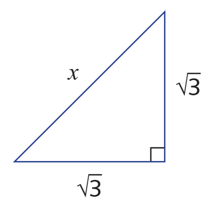

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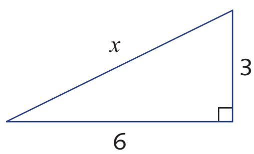

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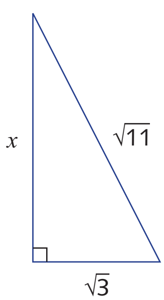

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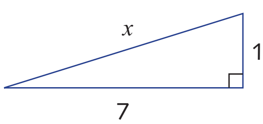
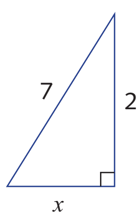


f



11 A square has side length $4\sqrt{7}$ cm. Find:

a the area of the square **b** the length of a diagonal

12 A rectangle has length 7 cm and width $\sqrt{3}$ cm. Find:

a the area of the rectangle **b** the length of a diagonal

2B Addition and subtraction of surds

Consider the calculation $4\sqrt{7} + 5\sqrt{7} = 9\sqrt{7}$. We can think of this as 4 lots of $\sqrt{7}$ plus 5 lots of $\sqrt{7}$ equals 9 lots of $\sqrt{7}$. This is very similar to algebra, where we write $4x + 5x = 9x$. We regard the numbers $4\sqrt{7}$ and $5\sqrt{7}$ as **like surds** since they are both multiples of $\sqrt{7}$.

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On the other hand, in algebra we cannot simplify $4x + 7y$, because $4x$ and $7y$ are not like terms. Similarly, it is not possible to write $4\sqrt{2} + 7\sqrt{3}$ in a simpler way. The surds $4\sqrt{2}$ and $7\sqrt{3}$ are **unlike surds**, since one is a multiple of $\sqrt{2}$ while the other is a multiple of $\sqrt{3}$.

We can only simplify the sum or difference of like surds.

Example 7

Simplify:

a $2\sqrt{2} + 7\sqrt{2} - 4\sqrt{2}$

b $4\sqrt{7} + 3\sqrt{5} - 2\sqrt{5} + 8\sqrt{7}$

Solution

a $2\sqrt{2} + 7\sqrt{2} - 4\sqrt{2} = 5\sqrt{2}$

b $4\sqrt{7} + 3\sqrt{5} - 2\sqrt{5} + 8\sqrt{7} = 12\sqrt{7} + \sqrt{5}$

When dealing with expressions involving surds, we should simplify the surds first and then look for like terms.

Example 8

Simplify:

a $\sqrt{8} + 7\sqrt{2} - \sqrt{32}$

b $\sqrt{27} + 3\sqrt{5} + \sqrt{45} - 4\sqrt{3}$

Solution

a
$$\begin{aligned}\sqrt{8} + 7\sqrt{2} - \sqrt{32} &= 2\sqrt{2} + 7\sqrt{2} - 4\sqrt{2}\\ &= 5\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}\sqrt{27} + 3\sqrt{5} + \sqrt{45} - 4\sqrt{3} &= 3\sqrt{3} + 3\sqrt{5} + 3\sqrt{5} - 4\sqrt{3}\\ &= 6\sqrt{5} - \sqrt{3}\end{aligned}$$

Addition and subtraction of surds

- Simplify each surd first, then look for like surds.
- We can add and subtract like surds.

Exercise 2B

Example 7

1 Simplify:

a $6\sqrt{2} + 7\sqrt{2}$

b $12\sqrt{3} + 13\sqrt{3}$

c $-6\sqrt{2} + 9\sqrt{2}$

d $-13\sqrt{5} - 14\sqrt{5}$

e $-19\sqrt{3} + 21\sqrt{3} - 4\sqrt{3}$

f $13\sqrt{5} - 16\sqrt{5} + 25\sqrt{5}$

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2 Simplify:

a $5\sqrt{2} + 6\sqrt{3} + 7\sqrt{2} - 4\sqrt{3}$

b $7\sqrt{7} - 4\sqrt{5} + 3\sqrt{7} - 6\sqrt{5}$

c $8\sqrt{11} - 7\sqrt{10} + 5\sqrt{11} + 4\sqrt{10}$

d $\sqrt{3} + 4\sqrt{2} - 5\sqrt{3} + 6\sqrt{2}$

e $9\sqrt{15} - 4\sqrt{7} - 3\sqrt{15}$

f $8\sqrt{5} + 5\sqrt{8} + 3\sqrt{5} - 6\sqrt{8}$

3 Complete:

a $5\sqrt{2} + \ldots = 11\sqrt{2}$

b $9\sqrt{3} + \ldots = 14\sqrt{3}$

c $6\sqrt{5} - \ldots = \sqrt{5}$

d $11\sqrt{2} - \ldots = -4\sqrt{2}$

e $7\sqrt{3} + \ldots = 2\sqrt{3}$

f $4\sqrt{5} - \ldots = 6\sqrt{5}$

g $2\sqrt{3} + 4\sqrt{5} + \ldots = 5\sqrt{3} + 8\sqrt{5}$

h $7\sqrt{11} - 6\sqrt{5} + \ldots = 8\sqrt{11} + 2\sqrt{5}$

i $9\sqrt{10} - 4\sqrt{3} + \ldots = \sqrt{10} - \sqrt{3}$

j $6\sqrt{5} + 3\sqrt{2} + \ldots = 2\sqrt{5} - 5\sqrt{2}$

Example 8

4 Simplify:

a $\sqrt{12} + \sqrt{27}$

b $\sqrt{8} + \sqrt{18}$

c $3\sqrt{8} - 4\sqrt{2}$

d $\sqrt{45} - 3\sqrt{20}$

e $3\sqrt{32} - 4\sqrt{27} + 5\sqrt{18}$

f $5\sqrt{147} + 3\sqrt{48} - \sqrt{12}$

g $3\sqrt{45} + \sqrt{20} + 7\sqrt{5}$

h $4\sqrt{63} + 5\sqrt{7} - 8\sqrt{28}$

i $\sqrt{44} + 5\sqrt{176} + 2\sqrt{99}$

j $2\sqrt{363} - 5\sqrt{243} + \sqrt{192}$

5 Simplify:

a $\sqrt{72} - \sqrt{50}$

b $\sqrt{48} + \sqrt{12}$

c $\sqrt{8} + \sqrt{2} + \sqrt{18}$

d $\sqrt{12} + 4\sqrt{3} - \sqrt{75}$

e $\sqrt{32} - \sqrt{200} + 3\sqrt{50}$

f $4\sqrt{5} - 4\sqrt{20} - \sqrt{45}$

g $\sqrt{54} + \sqrt{24}$

h $\sqrt{27} - \sqrt{48} + \sqrt{75}$

i $\sqrt{45} + \sqrt{80} - \sqrt{125}$

j $\sqrt{2} + \sqrt{32} + \sqrt{72}$

k $3\sqrt{20} - 4\sqrt{5} + \frac{1}{2}\sqrt{5}$

l $5\sqrt{18} - 3\sqrt{20} - 4\sqrt{5}$

6 Simplify:

a $\sqrt{12} + 3\sqrt{8} - 2\sqrt{27} + \sqrt{32}$

b $4\sqrt{18} - 2\sqrt{20} + 3\sqrt{5} + 6\sqrt{8}$

c $6\sqrt{12} + 9\sqrt{40} - 2\sqrt{27} - \sqrt{90}$

d $4\sqrt{27} - 3\sqrt{18} + 2\sqrt{108} - \sqrt{200}$

7 Find the value of x if:

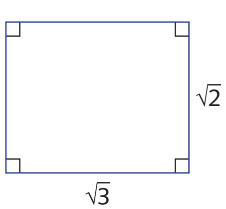
a $\sqrt{63} - \sqrt{28} = \sqrt{x}$

b $\sqrt{80} - \sqrt{45} = \sqrt{x}$

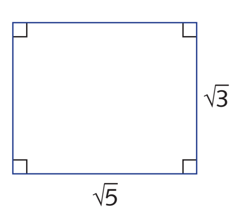
c $\sqrt{54} - 2\sqrt{24} = -\sqrt{x}$

8 For each rectangle, find (in exact form) the perimeter and area, and the length of the diagonal.

a



b



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2C Multiplication and division of surds

When multiplying two surds, we multiply the numbers outside the square root sign together and, similarly, multiply the numbers under the square root sign. A similar procedure applies for division. These procedures are captured by the following general rules:

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd},$$

where b and d are positive numbers.

$$a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}},$$

where b and d are positive numbers and $c \neq 0$.

As usual, we should always give the answer in simplest form.

Example 9

Find:

a $5\sqrt{7} \times 3\sqrt{2}$

b $15\sqrt{77} \div 3\sqrt{7}$

Solution

a $5\sqrt{7} \times 3\sqrt{2} = 15\sqrt{14}$

$\left(\text{This is } 5 \times \sqrt{7} \times 3 \times \sqrt{2}\right)$

b $15\sqrt{77} \div 3\sqrt{7} = \dfrac{15 \times \sqrt{77}}{3 \times \sqrt{7}}$

$= 5\sqrt{11}$

$\left(15 \div 3 = 5 \text{ and } \sqrt{77} \div \sqrt{7} = \sqrt{11}\right)$

Example 10

Find:

a $5\sqrt{6} \times 7\sqrt{10}$

b $\dfrac{18\sqrt{10}}{3\sqrt{5}}$

c $\left(2\sqrt{7}\right)^2$

d $\left(2\sqrt{3}\right)^3$

Solution

a $5\sqrt{6} \times 7\sqrt{10} = 35\sqrt{60}$

$= 35\sqrt{4 \times 15}$

$= 70\sqrt{15}$

b $\dfrac{18\sqrt{10}}{3\sqrt{5}} = 6\sqrt{2}$

c $\left(2\sqrt{7}\right)^2 = 2\sqrt{7} \times 2\sqrt{7}$

$= 4 \times 7$

$= 28$

d $\left(2\sqrt{3}\right)^3 = 2\sqrt{3} \times 2\sqrt{3} \times 2\sqrt{3}$

$= 4 \times 3 \times 2 \times \sqrt{3}$

$= 24\sqrt{3}$

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The distributive law

We can apply the distributive law to expressions involving surds, just as we do in algebra.

Example 11

Expand and simplify:

a $2\sqrt{5}\left(6 + 3\sqrt{5}\right)$

b $-4\sqrt{3}\left(\sqrt{6} - 2\sqrt{3}\right)$

Solution

a $2\sqrt{5}\left(6 + 3\sqrt{5}\right) = 12\sqrt{5} + 6\sqrt{25}$
$= 12\sqrt{5} + 30$

b $-4\sqrt{3}\left(\sqrt{6} - 2\sqrt{3}\right) = -4\sqrt{18} + 8 \times 3$
$= -12\sqrt{2} + 24$

In algebra, you learned how to expand brackets such as $(a + b)(c + d)$. These are known as **binomial products**. You multiply each term in the second bracket by each term in the first, then add. This means you expand out $a(c + d) + b(c + d)$ to obtain $ac + ad + bc + bd$. We use this idea again when multiplying out binomial products involving surds. (Remember to be very careful with the signs.)

Example 12

Expand and simplify:

a $\left(2\sqrt{3} - 1\right)\left(4\sqrt{3} + 2\right)$

b $\left(3\sqrt{2} - 4\sqrt{3}\right)\left(5\sqrt{3} - \sqrt{2}\right)$

c $\left(\sqrt{2} + \sqrt{3}\right)\left(\sqrt{5} - \sqrt{7}\right)$

d $\left(\sqrt{2} + \sqrt{3}\right)\left(\sqrt{2} - \sqrt{3}\right)$

Solution

a $\left(2\sqrt{3} - 1\right)\left(4\sqrt{3} + 2\right) = 2\sqrt{3}\left(4\sqrt{3} + 2\right) - 1\left(4\sqrt{3} + 2\right)$
$= 8\sqrt{9} + 4\sqrt{3} - 4\sqrt{3} - 2$
$= 24 - 2$
$= 22$

b $\left(3\sqrt{2} - 4\sqrt{3}\right)\left(5\sqrt{3} - \sqrt{2}\right) = 3\sqrt{2}\left(5\sqrt{3} - \sqrt{2}\right) - 4\sqrt{3}\left(5\sqrt{3} - \sqrt{2}\right)$
$= 15\sqrt{6} - 3\sqrt{4} - 20\sqrt{9} + 4\sqrt{6}$
$= 15\sqrt{6} - 6 - 60 + 4\sqrt{6}$
$= 19\sqrt{6} - 66$

c $\left(\sqrt{2} + \sqrt{3}\right)\left(\sqrt{5} - \sqrt{7}\right) = \sqrt{2}\left(\sqrt{5} - \sqrt{7}\right) + \sqrt{3}\left(\sqrt{5} - \sqrt{7}\right)$
$= \sqrt{10} - \sqrt{14} + \sqrt{15} - \sqrt{21}$

d $\left(\sqrt{2} + \sqrt{3}\right)\left(\sqrt{2} - \sqrt{3}\right) = \sqrt{2}\left(\sqrt{2} - \sqrt{3}\right) + \sqrt{3}\left(\sqrt{2} - \sqrt{3}\right)$
$= 2 - \sqrt{6} + \sqrt{6} - 3$
$= -1$

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Multiplication and division of surds

- For positive numbers b and d, $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$.
- For positive numbers b and d, $a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}}$, where $c \neq 0$ and $d \neq 0$.
- We can apply the distributive law to expressions involving surds.
- We can expand binomial products involving surds just as we do in algebra:
  $(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$.
- Always give the answer in simplest form.

Exercise 2C

1 Simplify:

a $\sqrt{5} \times \sqrt{3}$ **b** $\sqrt{5} \times \sqrt{11}$ **c** $\sqrt{8} \times \sqrt{14}$

d $\sqrt{5} \times \sqrt{13}$ **e** $\sqrt{6} \times \sqrt{2}$ **f** $\sqrt{18} \times \sqrt{3}$

2 Simplify:

a $\frac{\sqrt{30}}{\sqrt{6}}$ **b** $\frac{\sqrt{6}}{\sqrt{3}}$ **c** $\frac{\sqrt{42}}{\sqrt{7}}$

d $\frac{\sqrt{35}}{\sqrt{7}}$ **e** $\frac{\sqrt{33}}{\sqrt{11}}$ **f** $\frac{\sqrt{40}}{\sqrt{5}}$

Example 9

3 Simplify:

a $4\sqrt{2} \times 3\sqrt{5}$ **b** $7\sqrt{3} \times 11\sqrt{5}$ **c** $9\sqrt{5} \times 6\sqrt{7}$

d $8\sqrt{2} \times 4\sqrt{3}$ **e** $-6\sqrt{3} \times 5\sqrt{2}$ **f** $7\sqrt{2} \times \left(-4\sqrt{11}\right)$

g $-3\sqrt{7} \times \left(-4\sqrt{11}\right)$ **h** $-6\sqrt{2} \times \left(-3\sqrt{7}\right)$ **i** $11\sqrt{7} \times \left(-2\sqrt{6}\right)$

Example 9

4 Simplify:

a $\frac{12\sqrt{6}}{6\sqrt{2}}$ **b** $\frac{25\sqrt{15}}{10\sqrt{3}}$

c $\frac{8\sqrt{48}}{12\sqrt{8}}$ **d** $\frac{-20\sqrt{6}}{8\sqrt{3}}$

e $\frac{-32\sqrt{45}}{16\sqrt{15}}$ **f** $\frac{8\sqrt{3}}{24\sqrt{6}}$

5 Simplify:

a $\sqrt{2} \times \sqrt{2}$ **b** $\sqrt{11} \times \sqrt{11}$ **c** $3\sqrt{3} \times \sqrt{3}$

d $8\sqrt{5} \times \sqrt{5}$ **e** $3\sqrt{2} \times 5\sqrt{2}$ **f** $6\sqrt{3} \times 5\sqrt{3}$

g $4\sqrt{7} \times \left(-2\sqrt{7}\right)$ **h** $-2\sqrt{3} \times \left(-4\sqrt{3}\right)$ **i** $7\sqrt{6} \times \left(-3\sqrt{6}\right)$

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6 Complete:

a $2\sqrt{3} \times \ldots = 6$

b $4\sqrt{2} \times \ldots = 8$

c $5\sqrt{2} \times \ldots = 20$

d $2\sqrt{3} \times \ldots = 18$

e $2\sqrt{7} \times \ldots = 42$

f $2\sqrt{5} \times \ldots = 60$

g $8\sqrt{2} \times \ldots = 96$

h $3\sqrt{3} \times \ldots = 108$

i $2\sqrt{5} \times \ldots = 100$

j $3\sqrt{8} \times \ldots = 96$

k $\sqrt{2} \times \ldots = 64$

l $4\sqrt{5} \times \ldots = 1000$

7 Complete:

a $9\sqrt{5} \times \ldots = -27\sqrt{15}$

b $6\sqrt{2} \times \ldots = -18\sqrt{10}$

c $\dfrac{15\sqrt{6}}{\ldots} = 5\sqrt{2}$

d $\dfrac{28\sqrt{22}}{\ldots} = 4\sqrt{11}$

e $\dfrac{\ldots}{5\sqrt{7}} = 5\sqrt{5}$

f $\dfrac{\ldots}{3\sqrt{7}} = 8\sqrt{6}$

g $4\sqrt{2} \times (\ldots) + 8\sqrt{6} = 20\sqrt{6}$

h $3\sqrt{5} \times (\ldots) - 2\sqrt{10} = 16\sqrt{10}$

i $\dfrac{12\sqrt{6}}{\ldots} + 3\sqrt{2} = 7\sqrt{2}$

j $\dfrac{\ldots}{8\sqrt{3}} - 4\sqrt{5} = -2\sqrt{5}$

Example 10a, b

8 Simplify:

a $2\sqrt{3} \times 4\sqrt{2} + 8\sqrt{6}$

b $7\sqrt{3} \times 5\sqrt{5} + 8\sqrt{15}$

c $16\sqrt{6} - 2\sqrt{3} \times 5\sqrt{2}$

d $18\sqrt{10} - 3\sqrt{5} \times 4\sqrt{2}$

e $3\sqrt{6} \times 5\sqrt{5} - 8\sqrt{15} \times 4\sqrt{2}$

f $8\sqrt{20} \times 3\sqrt{2} - 5\sqrt{5} \times 5\sqrt{8}$

g $\dfrac{12\sqrt{6}}{4\sqrt{3}} + 5\sqrt{2}$

h $\dfrac{16\sqrt{15}}{4\sqrt{3}} - 8\sqrt{5}$

i $\dfrac{5\sqrt{20}}{10\sqrt{10}} + \dfrac{3\sqrt{2}}{2}$

j $\dfrac{6\sqrt{10}}{9\sqrt{5}} + \dfrac{\sqrt{2}}{3}$

Example 10c

9 Simplify:

a $\left(2\sqrt{2}\right)^2$

b $\left(2\sqrt{3}\right)^2$

c $\left(3\sqrt{5}\right)^2$

d $\left(5\sqrt{6}\right)^2$

e $\left(3\sqrt{7}\right)^2$

f $\left(2\sqrt{11}\right)^2$

g $\left(5\sqrt{10}\right)^2$

h $\left(a\sqrt{b}\right)^2$

Example 10d

10 Simplify:

a $\left(\sqrt{2}\right)^1$

b $\left(\sqrt{2}\right)^2$

c $\left(\sqrt{2}\right)^3$

d $\left(\sqrt{2}\right)^4$

e $\left(\sqrt{2}\right)^5$

f $\left(\sqrt{2}\right)^6$

g $\left(\sqrt{3}\right)^3$

h $\left(\sqrt{5}\right)^3$

i $\left(2\sqrt{2}\right)^3$

j $\left(3\sqrt{2}\right)^3$

k $\left(4\sqrt{3}\right)^3$

l $\left(2\sqrt{5}\right)^3$

m $\left(5\sqrt{5}\right)^3$

n $\left(2\sqrt{7}\right)^3$

o $\left(2\sqrt{2}\right)^6$

p $\left(3\sqrt{3}\right)^5$

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Example 11 **11** Expand and simplify:

a $\sqrt{2}(\sqrt{3}+\sqrt{5})$ **b** $\sqrt{7}(\sqrt{5}+\sqrt{6})$ **c** $\sqrt{5}(\sqrt{7}-\sqrt{2})$

d $3\sqrt{2}(2\sqrt{5}-3\sqrt{3})$ **e** $4\sqrt{3}(5\sqrt{2}+6\sqrt{5})$ **f** $4\sqrt{3}(\sqrt{2}-1)$

g $3\sqrt{5}(2\sqrt{3}+\sqrt{5})$ **h** $2\sqrt{6}(3\sqrt{3}+2\sqrt{2})$ **i** $4\sqrt{10}(3\sqrt{5}-4\sqrt{2})$

j $2\sqrt{3}(4\sqrt{3}-\sqrt{6})$ **k** $3\sqrt{2}(5\sqrt{2}+4\sqrt{10})$ **l** $3\sqrt{6}(4\sqrt{3}-\sqrt{6})$

m $4\sqrt{5}(2\sqrt{20}-3\sqrt{8})$ **n** $3\sqrt{7}(5\sqrt{35}-2\sqrt{21})$ **o** $3\sqrt{11}(2\sqrt{22}-4\sqrt{33})$

Example 12 **12** Expand and simplify:

a $(\sqrt{3}+\sqrt{2})(\sqrt{5}+\sqrt{7})$ **b** $(\sqrt{5}+\sqrt{7})(\sqrt{11}+\sqrt{6})$

c $(\sqrt{5}+\sqrt{2})(\sqrt{3}-\sqrt{7})$ **d** $(\sqrt{3}+\sqrt{6})(\sqrt{5}-\sqrt{7})$

e $(3\sqrt{2}+4\sqrt{3})(2\sqrt{2}-\sqrt{3})$ **f** $(5\sqrt{3}-\sqrt{5})(2\sqrt{3}-3\sqrt{5})$

g $(4\sqrt{5}+1)(2\sqrt{5}-3)$ **h** $(2\sqrt{3}+3\sqrt{6})(5\sqrt{2}-\sqrt{6})$

i $(3\sqrt{2}+\sqrt{7})(4\sqrt{2}-5\sqrt{7})$ **j** $(2\sqrt{7}+\sqrt{5})(\sqrt{3}-2\sqrt{5})$

k $(2\sqrt{2}-\sqrt{7})(5\sqrt{2}-2\sqrt{7})$ **l** $(3\sqrt{7}-8)(\sqrt{3}-3\sqrt{5})$

13 If $x = 2\sqrt{3}$ and $y = -3\sqrt{6}$, find:

a xy **b** $\frac{y}{x}$ **c** $x^2 + y^2$ **d** $\frac{1}{x^2}+\frac{1}{y^2}$

e x^3 **f** x^3y^2 **g** $\frac{y^2}{x^3}$ **h** $x^2 - y^2$

14 If $x = 2+\sqrt{3}$ and $y = 2-\sqrt{3}$, find:

a $x + y$ **b** $x + 2y$ **c** $3x + 2y$ **d** $x - y$

e $x - 2y$ **f** xy **g** $\sqrt{3xy}$ **h** $\sqrt{xy}$

15 Find the area and perimeter of a rectangle with:

a length $2\sqrt{3}$ and width $4\sqrt{2}$

b length $2\sqrt{3}$ and width $4\sqrt{3}$

c length $7+2\sqrt{5}$ and width $7-2\sqrt{5}$

d length $1+\sqrt{5}$ and width $2+\sqrt{5}$

16 The hypotenuse of a right-angled triangle has length $8+\sqrt{3}$. Another side has length $4\sqrt{3}+2$.

Find:

a the length of the third side

b the perimeter of the triangle

c the area of the triangle

17 A square has side length $2+5\sqrt{3}$.
Find:

a the perimeter of the square

b the area of the square

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2D Special products

In algebra, you learned the following special identities. Recognising and applying these identities is important.

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a - b)(a + b) = a^2 - b^2$$

These identities are also useful when dealing with surds, and the following examples demonstrate this use. The last identity listed above is known as the **difference of squares** identity, and we will pay particular attention to this.

Example 13

Expand and simplify:

a $\left(\sqrt{7} + \sqrt{3}\right)^2$ **b** $\left(5\sqrt{2} - \sqrt{3}\right)^2$ **c** $\left(\sqrt{3} + 5\sqrt{6}\right)^2$

Solution

a
$$\begin{aligned}\left(\sqrt{7} + \sqrt{3}\right)^2 &= \left(\sqrt{7}\right)^2 + 2\left(\sqrt{7}\right)\left(\sqrt{3}\right) + \left(\sqrt{3}\right)^2\\ &= 7 + 2\sqrt{21} + 3\\ &= 10 + 2\sqrt{21}\end{aligned}$$

b
$$\begin{aligned}\left(5\sqrt{2} - \sqrt{3}\right)^2 &= \left(5\sqrt{2}\right)^2 - 2\left(5\sqrt{2}\right)\left(\sqrt{3}\right) + \left(\sqrt{3}\right)^2\\ &= 50 - 10\sqrt{6} + 3\\ &= 53 - 10\sqrt{6}\end{aligned}$$

c
$$\begin{aligned}\left(\sqrt{3} + 5\sqrt{6}\right)^2 &= \left(\sqrt{3}\right)^2 + 2\left(\sqrt{3}\right)\left(5\sqrt{6}\right) + \left(5\sqrt{6}\right)^2\\ &= 3 + 10\sqrt{18} + 150\\ &= 153 + 30\sqrt{2}\end{aligned}$$

You should always express your answer in simplest form.

Notice what happens when we use the difference of squares identity.

Example 14

Expand and simplify:

a $\left(\sqrt{11} - \sqrt{5}\right)\left(\sqrt{11} + \sqrt{5}\right)$ **b** $\left(2\sqrt{3} + \sqrt{4}\right)\left(2\sqrt{3} - \sqrt{4}\right)$

Solution

a
$$\begin{aligned}\left(\sqrt{11} - \sqrt{5}\right)\left(\sqrt{11} + \sqrt{5}\right) &= \left(\sqrt{11}\right)^2 - \left(\sqrt{5}\right)^2\\ &= 11 - 5\\ &= 6\end{aligned}$$

b
$$\begin{aligned}\left(2\sqrt{3} + 4\right)\left(2\sqrt{3} - 4\right) &= \left(2\sqrt{3}\right)^2 - (4)^2\\ &= 12 - 16\\ &= -4\end{aligned}$$

When we apply the difference of squares identity, the answer is an integer.

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Special products

We can apply the identities

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$(a - b)(a + b) = a^2 - b^2$

to calculations involving surds.

Exercise 2D

Example 13

1 Simplify:

a $\left(\sqrt{5} + \sqrt{2}\right)^2$

b $\left(\sqrt{3} + \sqrt{7}\right)^2$

c $\left(\sqrt{2} - \sqrt{3}\right)^2$

d $\left(\sqrt{7} - \sqrt{6}\right)^2$

e $\left(2\sqrt{3} + 1\right)^2$

f $\left(3\sqrt{2} - 2\right)^2$

g $\left(2\sqrt{3} + \sqrt{2}\right)^2$

h $\left(4\sqrt{2} - 3\sqrt{3}\right)^2$

i $\left(2\sqrt{5} + 3\sqrt{7}\right)^2$

j $\left(3\sqrt{2} - 4\sqrt{5}\right)^2$

k $\left(4 - 3\sqrt{2}\right)^2$

l $\left(2 - 5\sqrt{3}\right)^2$

m $\left(\frac{1}{2} - \sqrt{3}\right)^2$

n $\left(5 - \frac{\sqrt{3}}{2}\right)^2$

Example 14

2 Simplify:

a $\left(4 - \sqrt{3}\right)\left(4 + \sqrt{3}\right)$

b $\left(\sqrt{7} + 2\right)\left(\sqrt{7} - 2\right)$

c $\left(\sqrt{5} + \sqrt{3}\right)\left(\sqrt{5} - \sqrt{3}\right)$

d $\left(\sqrt{6} - \sqrt{5}\right)\left(\sqrt{6} + \sqrt{5}\right)$

e $\left(2\sqrt{3} + 1\right)\left(2\sqrt{3} - 1\right)$

f $\left(3\sqrt{2} + 4\right)\left(3\sqrt{2} - 4\right)$

g $\left(3\sqrt{2} + 2\sqrt{3}\right)\left(3\sqrt{2} - 2\sqrt{3}\right)$

h $\left(3\sqrt{6} + 2\sqrt{5}\right)\left(3\sqrt{6} - 2\sqrt{5}\right)$

i $\left(2\sqrt{2} + \sqrt{7}\right)\left(2\sqrt{2} - \sqrt{7}\right)$

j $\left(3\sqrt{5} - 4\sqrt{3}\right)\left(3\sqrt{5} + 4\sqrt{3}\right)$

k $\left(\frac{1}{2} - \frac{1}{2}\sqrt{3}\right)\left(\frac{1}{2} + \frac{1}{2}\sqrt{3}\right)$

l $\left(3 - \frac{\sqrt{3}}{2}\right)\left(3 + \frac{\sqrt{3}}{2}\right)$

3 Find the area of a square with side length:

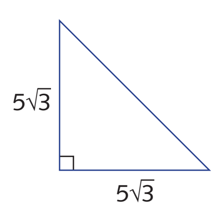
a $2 + \sqrt{3}$ **b** $2 - \sqrt{3}$ **c** $5 + 2\sqrt{3}$ **d** $5 - 2\sqrt{3}$

4 If $x = 2 + \sqrt{5}$ and $y = \sqrt{5} - 2$, find:

a xy **b** $x + y$ **c** $x^2 + y^2$ **d** $x^2 - y^2$

5 **a** Find the area of the triangle.

b Find the length of the hypotenuse.



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6 If $x = 5 + 2\sqrt{3}$ and $y = 2 + 5\sqrt{3}$, find:

a $x + y$ **b** $x - y$ **c** xy **d** $x^2 + y^2$ **e** $x^2 - y^2$

7 The two shorter sides of a right-angled triangle have length $7 + 2\sqrt{3}$ and $7 - 2\sqrt{3}$.

Find:

a the length of the hypotenuse of the triangle

b the perimeter of the triangle

c the area of the triangle

2E Rationalising denominators

In the expression $\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2}$, the first term has a square root in the denominator. This makes it difficult to tell if the surds are like surds or not.

Fractions involving surds are usually easier to deal with when the surd is in the numerator and there is a whole number in the denominator.

To express a fraction in such a way is called **rationalising the denominator**.

When we multiply the numerator and denominator of a fraction by the same number, we form an equivalent fraction. The same happens with a quotient involving surds.

Example 15

Rationalise the denominator of:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{4}{2\sqrt{5}}$ **c** $\frac{4\sqrt{3} - 1}{4\sqrt{6}}$

Solution

a
$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

b
$$\frac{4}{2\sqrt{5}} = \frac{4}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{4\sqrt{5}}{10} = \frac{2\sqrt{5}}{5}$$

c
$$\frac{4\sqrt{3} - 1}{4\sqrt{6}} = \frac{4\sqrt{3} - 1}{4\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{4\sqrt{18} - \sqrt{6}}{24} = \frac{4\sqrt{9 \times 2} - \sqrt{6}}{24} = \frac{12\sqrt{2} - \sqrt{6}}{24}$$

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Example 16

Simplify $\dfrac{2}{\sqrt{3}} + \dfrac{\sqrt{3}}{2}$.

Solution

$$\begin{aligned}\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2} &= \frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{2} \qquad \left(\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}\right)\\ &= \frac{4\sqrt{3} + 3\sqrt{3}}{6}\\ &= \frac{7\sqrt{3}}{6}\end{aligned}$$

Binomial denominators

In the expression $\dfrac{1}{\sqrt{7} - \sqrt{5}}$, it is more difficult to remove the surds from the denominator.

In the following example, we explain how this can be done by using the difference of squares identity.

In the section on special products, we saw that:

$$\begin{aligned}\left(\sqrt{7} - \sqrt{5}\right)\left(\sqrt{7} + \sqrt{5}\right) &= \left(\sqrt{7}\right)^2 - \left(\sqrt{5}\right)^2\\ &= 7 - 5\\ &= 2, \qquad \text{which is rational.}\end{aligned}$$

So:

$$\begin{aligned}\frac{1}{\sqrt{7} - \sqrt{5}} &= \frac{1}{\sqrt{7} - \sqrt{5}} \times \frac{\sqrt{7} + \sqrt{5}}{\sqrt{7} + \sqrt{5}}\\ &= \frac{\sqrt{7} + \sqrt{5}}{7 - 5}\\ &= \frac{\sqrt{7} + \sqrt{5}}{2}\end{aligned}$$

Using the difference of two squares identity in this way is an important and initially surprising technique.

Example 17

Rationalise the denominators of the following, simplifying where possible.

a $\dfrac{2\sqrt{5}}{2\sqrt{5} - 2}$

b $\dfrac{\sqrt{3} + \sqrt{2}}{3\sqrt{2} + 2\sqrt{3}}$

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Solution

a $\dfrac{2\sqrt{5}}{2\sqrt{5}-2} = \dfrac{2\sqrt{5}}{2\sqrt{5}-2} \times \dfrac{2\sqrt{5}+2}{2\sqrt{5}+2}$

$= \dfrac{20+4\sqrt{5}}{20-4}$

$= \dfrac{20+4\sqrt{5}}{16}$

$= \dfrac{4\left(5+\sqrt{5}\right)}{16}$

$= \dfrac{5+\sqrt{5}}{4}$

b $\dfrac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} = \dfrac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} \times \dfrac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$

$= \dfrac{\left(\sqrt{3}+\sqrt{2}\right)\left(3\sqrt{2}-2\sqrt{3}\right)}{18-12}$

$= \dfrac{3\sqrt{6}-6+6-2\sqrt{6}}{6}$

$= \dfrac{\sqrt{6}}{6}$

Rationalising denominators

- To rationalise the denominator of $\dfrac{2}{\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$.
- To rationalise a denominator with two terms, we use the difference of squares identity.
  - In an expression such as $\dfrac{3}{5+\sqrt{3}}$, multiply top and bottom by $5-\sqrt{3}$.
  - In an expression such as $\dfrac{\sqrt{2}}{7-3\sqrt{2}}$, multiply top and bottom by $7+3\sqrt{2}$.
- Rationalising a denominator allows us to identify like and unlike surds.

Exercise 2E

Example 15

1 Rationalise the denominator and simplify:

a $\dfrac{5}{\sqrt{3}}$ **b** $\dfrac{6}{\sqrt{2}}$ **c** $\dfrac{7}{\sqrt{7}}$

d $\dfrac{3}{\sqrt{5}}$ **e** $\dfrac{3}{\sqrt{3}}$ **f** $\dfrac{\sqrt{5}}{\sqrt{2}}$

g $\dfrac{\sqrt{6}}{\sqrt{3}}$ **h** $\dfrac{2}{3\sqrt{2}}$ **i** $\dfrac{4}{3\sqrt{6}}$

j $\dfrac{2\sqrt{3}}{3\sqrt{5}}$ **k** $\dfrac{3\sqrt{5}}{4\sqrt{3}}$ **l** $\dfrac{8+\sqrt{3}}{2\sqrt{3}}$

m $\dfrac{3-2\sqrt{2}}{5\sqrt{2}}$ **n** $\dfrac{3\sqrt{2}+4\sqrt{3}}{3\sqrt{2}}$ **o** $\dfrac{\sqrt{5}-2\sqrt{3}}{4\sqrt{2}}$

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Example 16

2 Rationalise the denominator and simplify:

a $\frac{3}{\sqrt{2}}+\frac{4}{\sqrt{3}}$ **b** $\frac{3}{\sqrt{7}}-\frac{2}{\sqrt{5}}$ **c** $\frac{3\sqrt{2}}{4\sqrt{3}}+\frac{1}{2\sqrt{5}}$ **d** $\frac{3\sqrt{2}}{\sqrt{7}}-\frac{3}{2\sqrt{2}}$

e $\frac{4}{\sqrt{7}}-\frac{2}{\sqrt{5}}$ **f** $\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{11}}$ **g** $\frac{4}{2\sqrt{3}}-\frac{5}{\sqrt{7}}$ **h** $\frac{2}{\sqrt{3}}-\frac{4}{\sqrt{11}}$

Example 17

3 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{5}-2}$ **b** $\frac{1}{7+\sqrt{6}}$ **c** $\frac{3}{\sqrt{6}+2}$ **d** $\frac{6}{2-\sqrt{3}}$ **e** $\frac{2}{\sqrt{3}-\sqrt{2}}$

f $\frac{4}{\sqrt{5}-\sqrt{2}}$ **g** $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{2}}$ **h** $\frac{\sqrt{7}}{\sqrt{3}+\sqrt{2}}$ **i** $\frac{2}{\sqrt{5}-\sqrt{3}}$ **j** $\frac{4}{\sqrt{7}-\sqrt{3}}$

k $\frac{4\sqrt{2}}{2\sqrt{2}-\sqrt{3}}$ **l** $\frac{2\sqrt{3}}{\sqrt{3}+2\sqrt{2}}$ **m** $\frac{3\sqrt{5}}{3\sqrt{2}-\sqrt{10}}$ **n** $\frac{2\sqrt{3}}{3\sqrt{2}-1}$ **o** $\frac{4\sqrt{2}}{2\sqrt{3}+3}$

p $\frac{4\sqrt{2}+\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$ **q** $\frac{2\sqrt{3}+\sqrt{5}}{2\sqrt{5}-3\sqrt{3}}$ **r** $\frac{2\sqrt{5}+1}{2\sqrt{5}-1}$ **s** $\frac{3\sqrt{2}+\sqrt{5}}{4\sqrt{2}-\sqrt{5}}$ **t** $\frac{3\sqrt{2}-\sqrt{5}}{3\sqrt{2}+\sqrt{5}}$

4 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{2\sqrt{3}-1}{2\sqrt{3}-2}$ **b** $\frac{3}{2\sqrt{5}+1}+\frac{1}{\sqrt{5}-\sqrt{2}}$

c $\frac{3\sqrt{2}}{\sqrt{2}+\sqrt{5}}+\frac{4\sqrt{2}}{3\sqrt{2}-1}$ **d** $\frac{2}{2\sqrt{5}+\sqrt{3}}-\frac{4}{2\sqrt{5}-\sqrt{3}}$

e $\frac{4}{2\sqrt{3}+\sqrt{2}}-\frac{1}{2\sqrt{3}-\sqrt{2}}$ **f** $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}+\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

5 If $x = 2\sqrt{3}$ and $y = \frac{7}{\sqrt{3}}$, find, in simplest form:

a $x + y$ **b** $x - y$ **c** xy **d** $\frac{x}{y}$ **e** $x^3 + y^3$

6 A rectangle has area 10. The length of the rectangle is $\sqrt{2}-1$. Find the width of the rectangle in simplest form.

7 If $x = \sqrt{3}+1$ and $y = \sqrt{3}+2$, find, in simplest form:

a $\frac{1}{x}+\frac{1}{y}$ **b** $\frac{1}{x^2}+\frac{1}{y^2}$ **c** x^3 **d** y^3

8 Given that $x = \frac{2}{3\sqrt{2}-1}$, find the value of each expression, giving answers in simplest form with a rational denominator.

a x^2 **b** $\frac{1}{x}$ **c** $\frac{1}{x^2}$ **d** $x+\frac{1}{x}$ **e** $x^2+\frac{1}{x}$ **f** $x^2+\frac{1}{x^2}$

9 Repeat question **8** for:

i $x = \frac{\sqrt{5}}{\sqrt{5}+1}$ **ii** $x = \frac{2\sqrt{3}}{5-2\sqrt{6}}$

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Review exercise

1 Simplify:

a $2\sqrt{2}+3\sqrt{2}-\sqrt{2}$

b $-4\sqrt{6}-3\sqrt{6}+8\sqrt{6}$

c $\sqrt{3}-2\sqrt{2}+2\sqrt{3}+\sqrt{2}$

d $\sqrt{5}-3\sqrt{2}-4\sqrt{5}+7\sqrt{2}$

2 Simplify:

a $3\sqrt{2}+2\sqrt{2}$ **b** $\sqrt{32}-\sqrt{18}$ **c** $\sqrt{28}-6\sqrt{7}$ **d** $\sqrt{75}+6\sqrt{3}$

3 Simplify:

a $2\sqrt{3}\times 5\sqrt{6}$ **b** $3\sqrt{5}\times 2\sqrt{10}$ **c** $4\sqrt{2}\times 3\sqrt{5}$ **d** $7\sqrt{6}\times 4\sqrt{7}$

4 Simplify:

a $\sqrt{72}$ **b** $\sqrt{45}$ **c** $\sqrt{24}$ **d** $\sqrt{27}$ **e** $\sqrt{80}$ **f** $\sqrt{44}$

g $3\sqrt{8}$ **h** $4\sqrt{12}$ **i** $9\sqrt{50}$ **j** $3\sqrt{108}$ **k** $10\sqrt{32}$

5 Write each as a single surd.

a $5\sqrt{3}$ **b** $4\sqrt{7}$ **c** $11\sqrt{2}$ **d** $5\sqrt{5}$

e $8\sqrt{6}$ **f** $9\sqrt{11}$ **g** $4\sqrt{13}$ **h** $4\sqrt{11}$

6 Simplify:

a $\sqrt{32}+\sqrt{50}$

b $\sqrt{20}+\sqrt{75}$

c $9\sqrt{3}-2\sqrt{27}$

d $3\sqrt{63}+5\sqrt{28}$

e $4\sqrt{20}+3\sqrt{80}-3\sqrt{45}$

f $7\sqrt{54}+5\sqrt{216}+2\sqrt{24}$

7 Simplify:

a $\sqrt{32}+4\sqrt{8}+2\sqrt{50}-3\sqrt{2}$

b $5\sqrt{32}-3\sqrt{50}+4\sqrt{8}-3\sqrt{18}$

c $7\sqrt{2}+4\sqrt{8}-3\sqrt{54}+5\sqrt{24}$

d $5\sqrt{28}-\sqrt{147}+2\sqrt{63}-5\sqrt{48}$

8 Simplify:

a $2\sqrt{3}\left(3+\sqrt{3}\right)$

b $5\sqrt{2}\left(3\sqrt{2}-2\right)$

c $4\sqrt{3}\left(2\sqrt{3}-4\sqrt{7}\right)$

d $5\sqrt{5}\left(6-2\sqrt{5}\right)$

e $3\sqrt{7}\left(4-\sqrt{7}\right)$

f $3\sqrt{3}\left(5\sqrt{3}-4\sqrt{2}\right)$

9 Expand and simplify:

a $\left(2\sqrt{2}+1\right)\left(3\sqrt{2}-2\right)$ **b** $\left(5\sqrt{3}-2\right)\left(2\sqrt{3}-1\right)$ **c** $\left(\sqrt{7}-\sqrt{5}\right)\left(\sqrt{7}+\sqrt{5}\right)$

d $\left(2\sqrt{5}-\sqrt{3}\right)\left(2\sqrt{5}+\sqrt{3}\right)$ **e** $\left(7\sqrt{2}+4\sqrt{3}\right)\left(7\sqrt{2}-4\sqrt{3}\right)$ **f** $\left(\sqrt{5}+\sqrt{3}\right)^2$

g $\left(2\sqrt{3}+\sqrt{2}\right)^2$ **h** $\left(2\sqrt{3}-\sqrt{2}\right)^2$ **i** $\left(2\sqrt{3}-\sqrt{2}\right)\left(2\sqrt{3}+\sqrt{2}\right)$

j $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$ **k** $\left(\sqrt{7}-2\right)^2$ **l** $\left(5+\sqrt{3}\right)^2$

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10 Rationalise each denominator and simplify where possible.

a $\frac{5}{\sqrt{3}}$ **b** $\frac{7}{2\sqrt{3}}$ **c** $\frac{4}{3\sqrt{2}}$ **d** $\frac{6}{7\sqrt{2}}$

e $\frac{5\sqrt{2}}{3\sqrt{6}}$ **f** $\frac{2\sqrt{3}}{4\sqrt{2}}$ **g** $\frac{6\sqrt{7}}{7\sqrt{6}}$ **h** $\frac{42\sqrt{7}}{12\sqrt{6}}$

11 Rationalise each denominator and simplify where possible.

a $\frac{1}{\sqrt{5}-\sqrt{7}}$ **b** $\frac{\sqrt{5}}{2\sqrt{5}-3}$

c $\frac{2}{3+\sqrt{5}}$ **d** $\frac{2}{3-\sqrt{5}}$

12 Rationalise each denominator.

a $\frac{1}{\sqrt{2}-1}$ **b** $\frac{1}{\sqrt{3}+2}$

c $\frac{1}{\sqrt{3}+\sqrt{2}}$ **d** $\frac{1}{\sqrt{3}-\sqrt{2}}$

13 Find integers p and q such that $\frac{\sqrt{5}}{\sqrt{5}-2} = p + q\sqrt{5}$.

14 Simplify:

a $\frac{3}{\sqrt{5}-2} + \frac{2}{\sqrt{5}+2}$

b $\frac{2}{6-3\sqrt{3}} - \frac{1}{2\sqrt{3}+3}$

15 If $x = \frac{1}{2-\sqrt{3}}$ and $y = 2+\sqrt{3}$, find, in simplest form:

a $x + y$ **b** $x - y$ **c** xy **d** $\frac{x}{y}$

16 If $x = 2+\sqrt{3}$ and $y = 4-\sqrt{3}$, find, in simplest form:

a $x + y$ **b** $x - y$ **c** $x^2 + y^2$ **d** $x^2 - y^2$

e $\frac{1}{x}$ **f** $\frac{1}{x} + \frac{1}{y}$ **g** $\frac{1}{x} - \frac{1}{y}$ **h** xy

17 A square has sides of length $2+\sqrt{3}$. Find:

a the perimeter

b the area

18 A rectangle has area 20. The width is $2+\sqrt{3}$. Find the length of the rectangle in simplest form.

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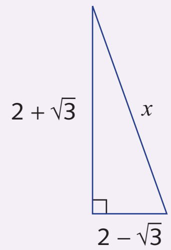
19 Find the value of x in each diagram.

a


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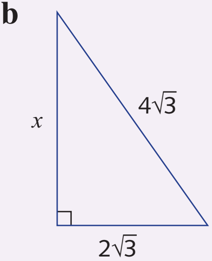

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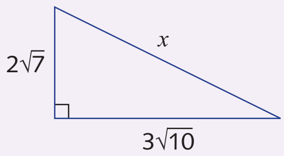

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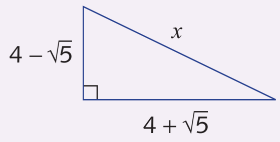

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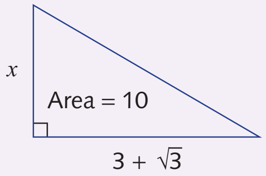

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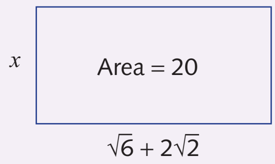

20 The number $\phi = \dfrac{\sqrt{5}+1}{2}$ is known as the **golden ratio**.

a Find:

i ϕ^2 **ii** $\dfrac{1}{\phi}$ **iii** $1+\dfrac{1}{\phi}$ **iv** ϕ^3 **v** $\phi+\dfrac{1}{\phi^2}$

b Show that:

i $\phi^2 = \phi + 1$ **ii** $\phi^3 = \phi^2 + \phi$ **iii** $\phi = 1 + \dfrac{1}{\phi}$

21 A square has area 50. Find its perimeter.

22 Simplify:

a $\dfrac{2}{\sqrt{3}-2}+\dfrac{2}{\sqrt{3}+2}$

b $\dfrac{2}{\sqrt{3}-2}-\dfrac{2}{\sqrt{3}+2}$

c $\dfrac{2}{\sqrt{3}-2}\times\dfrac{2}{\sqrt{3}+2}$

d $\dfrac{2}{\sqrt{3}-2}\div\dfrac{2}{\sqrt{3}+2}$

23 A rectangle has area 30 cm^2 and length $\sqrt{5}$ cm. Find its perimeter.

24 For $x = 3 + 2\sqrt{5}$ and $y = 3 - 2\sqrt{5}$, find:

a $x + y$

b xy

c $\dfrac{1}{x}+\dfrac{1}{y}$

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Challenge exercise

1 **a** Show that $\left(\sqrt{x}+\sqrt{y}\right)^2 = x + y + 2\sqrt{xy}$.

b Use this result to find:

i $\sqrt{16+2\sqrt{55}}$

ii $\sqrt{16-2\sqrt{55}}$

iii $\sqrt{11+2\sqrt{30}}$

2 Simplify:

a $\left(\sqrt{a+b}+a\right)\left(\sqrt{a+b}-a\right)$

b $\left(2\sqrt{1+x^2}+1\right)\left(2\sqrt{1+x^2}-1\right)$

c $\left(\sqrt{1+x}+\sqrt{1-x}\right)\left(\sqrt{1+x}-\sqrt{1-x}\right)$

d $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$

e $\left(\sqrt{a+b}+\sqrt{a-b}\right)^2$

3 Solve these equations for x. (Make sure you check your solutions.)

a $6x - \sqrt{x} = 12$

b $\sqrt{x} - \sqrt{x-5} = 1$

c $\sqrt{7x-5} - \sqrt{2x} = \sqrt{15-7x}$

d $2\sqrt{x} - \sqrt{4x-11} = 1$

e $\dfrac{6\sqrt{x}-11}{3\sqrt{x}} = \dfrac{2\sqrt{x}+1}{\sqrt{x}+6}$

f $\sqrt{9+2x} - \sqrt{2x} = \dfrac{5}{\sqrt{9+2x}}$

4 Expand $\left(\sqrt{x}-1+\dfrac{1}{\sqrt{x}}\right)^2$.

5 Express $\dfrac{1}{\sqrt[3]{5}}$ with a rational denominator.

6 Express $\dfrac{1}{1+\sqrt{3}+\sqrt{5}+\sqrt{15}}$ with a rational denominator.

Hint: Factorise the denominator.

7 Express $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ with a rational denominator.

8 Simplify $\dfrac{a^2+ab+b^2}{a+\sqrt{ab}+b}$.

9 **a** Show that $\sqrt{9\frac{9}{80}} = 9\sqrt{\dfrac{9}{80}}$ and $\sqrt{4\frac{4}{15}} = 4\sqrt{\dfrac{4}{15}}$.

b Describe all mixed numbers that have this property.

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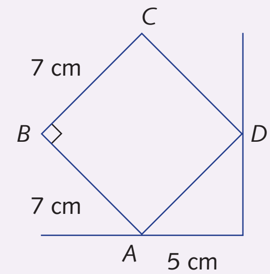
10 A square box of side 7 cm is leaning against a vertical wall, as shown below. Find the height of point *C* above the floor.



11 Simplify $\dfrac{1}{1+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{5}} + \dfrac{1}{\sqrt{5}+\sqrt{7}} + \dfrac{1}{\sqrt{7}+3}$.

12 Simplify $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} - \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+\sqrt{2-\sqrt{3}}}$.

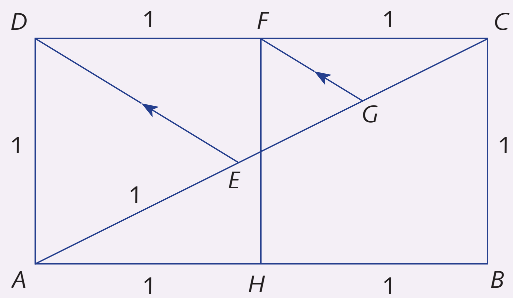
13 $[x]$ is defined as the largest integer, n, such that $n \le x$.
For example, $[1.78] = 1$ and $[2.31] = 2$.

Calculate:

a $1 + [\sqrt{2}] + [\sqrt{3}] + 2 + \ldots + [\sqrt{99}] + 10$

b $1 + [\sqrt{2}] + [\sqrt{3}] + 2 + \ldots + [\sqrt{200}]$

14 In the diagram below, squares *AHFD* and *HBCF* are drawn with common side *FH*. Diagonal *AC* is drawn and *E* is a point on *AC* such that $AE = 1$. *G* is a point on *AC* so that *FG* is parallel to *DE*.



Find:

a i AC

ii EC

iii EG

iv GC

b Show that $EG^2 + EG = 1$.

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