CHAPTER

4

Number and Algebra

Lines and linear equations

Coordinate geometry takes the surprising approach of using algebra to solve geometric problems.

This chapter continues the development of coordinate geometry begun in *ICE-EM Mathematics Year* 9.

Each point in the plane is represented by an ordered pair (x, y) and each line is the set of points that satisfies a linear equation $ax + by + c = 0$.

The gradient of a line allows us to answer most questions about parallelism and perpendicularity. In principle, every geometric problem can be solved using coordinate geometry.

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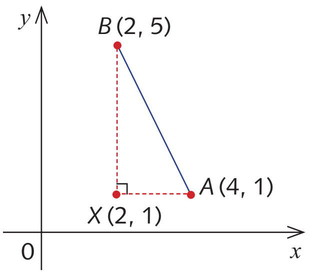
4A Distance between two points and midpoint of an interval

Distance formula

Consider the points $A(4, 1)$ and $B(2, 5)$ in the number plane. The length of the interval AB can be found using Pythagoras' theorem. This is called the **distance** between the points A and B.



Form the right-angled triangle ABX, as shown, where X is the point $(2, 1)$. Then:

$$\begin{aligned} BX &= 5 - 1 \quad \text{and} \quad & AX &= 4 - 2 \\ &= 4 & &= 2 \end{aligned}$$

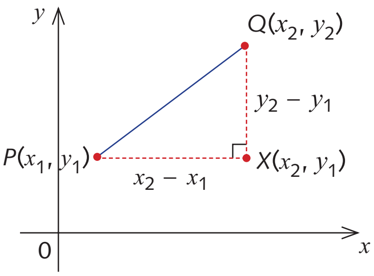
By Pythagoras' theorem:

$$\begin{aligned} AB^2 &= AX^2 + BX^2 \\ &= 2^2 + 4^2 \\ &= 20 \\ AB &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

The length of interval AB is $2\sqrt{5}$.

The general case

We can use the above idea to obtain a formula for the distance between any two points. Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points, as shown opposite.



Form the right-angled triangle PQX, where X is the point (x_2, y_1). Then:

$$PX = x_2 - x_1 \quad \text{and} \quad QX = y_2 - y_1$$

By Pythagoras' theorem:

$$\begin{aligned} PQ &= PX^2 + QX^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$

You will notice that our diagram assumes that $x_2 - x_1$ and $y_2 - y_1$ are positive. If either or both are negative, it is not necessary to change the formula, as we are squaring. In other words:

$$PQ^2 = (\text{square of the difference of } x\text{-values}) + (\text{square of the difference of } y\text{-values})$$

Therefore:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In practice, we sometimes work out the square of PQ and then take the square root.

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Example 1

Use the distance formula to find the distance between each pair of points.

a $A(2,\ 3)$ and $B(5,\ 7)$

b $A(1, 2)$ and $B(-1, -3)$

c $A(2,\ 4)$ and $B(5,\ 4)$

d $A(-2, 3)$ and $B(-4, -3)$

Solution

a
$$\begin{aligned} AB^2 &= (5-2)^2 + (7-3)^2 \\ &= 3^2 + 4^2 \\ &= 25 \\ AB &= 5 \end{aligned}$$

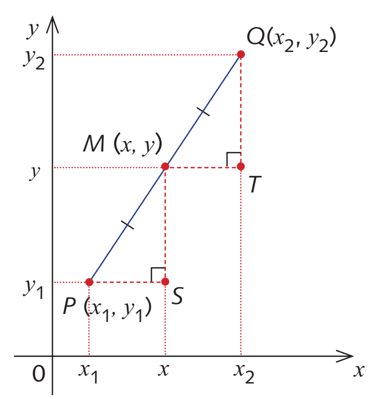
b
$$\begin{aligned} AB^2 &= (-1-1)^2 + (-3-2)^2 \\ &= (-2)^2 + (-5)^2 \\ &= 4 + 25 \\ &= 29 \\ AB &= \sqrt{29} \end{aligned}$$

c
$$\begin{aligned} AB^2 &= (5-2)^2 + (4-4)^2 \\ &= 3^2 + 0^2 \\ &= 9 \\ AB &= 3 \end{aligned}$$

d
$$\begin{aligned} AB^2 &= [-4-(-2)]^2 + (-3-3)^2 \\ &= (-2)^2 + (-6)^2 \\ &= 4 + 36 \\ &= 40 \\ AB &= \sqrt{40} \\ &= 2\sqrt{10} \end{aligned}$$

Midpoint formula

We can find a formula for the midpoint of any interval. Let $P(x_1,\ y_1)$ and $Q(x_2,\ y_2)$ be two points and let $M(x,\ y)$ be the midpoint of the interval PQ.



The triangles PMS and MQT are congruent triangles (AAS), so $PS = MT$ and $MS = QT$.

Hence, the x-coordinate of M is the average of x_1 and x_2,

Therefore, $x = \dfrac{x_1 + x_2}{2}$.

The y-coordinate of M is the average of y_1 and y_2.

Therefore, $y = \dfrac{y_1 + y_2}{2}$.

The coordinates of M are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

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Example 2

Find the coordinates of the midpoint, M, of interval AB, where A and B have coordinates $(-2, 6)$ and $(3, -7)$, respectively.

Solution

x-coordinate of $M = \frac{-2 + 3}{2} = \frac{1}{2}$ y-coordinate of $M = \frac{6 + (-7)}{2} = -\frac{1}{2}$

The coordinates of M are $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

Distance between two points and midpoint of an interval

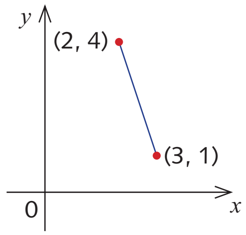
Consider two points, $P(x_1, y_1)$ and $Q(x_2, y_2)$.

- The distance between the points P and Q is given by the expression $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.
  That is, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The midpoint M of the interval PQ has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Exercise 4A

1 Find the distance between the labelled points in each diagram.

a



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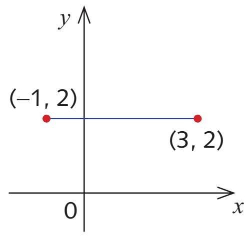

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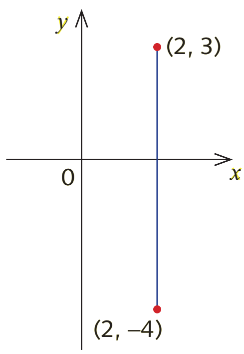

Example 1

2 Find the distance between points A and B.

a $A(1, 2),\ B(0, 0)$

b $A(-1, 6),\ B(4, 8)$

c $A(-2, 8),\ B(6, 4)$

d $A(-2, -6),\ B(3, 2)$

e $A(-3, 4),\ B(4, 3)$

f $A(-3, -4),\ B(3, 4)$

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Example 2

3 Find the midpoint of the interval AB.

a $A(-1, 2),\ B(3, 6)$

b $A(2, 8),\ B(-1, 2)$

c $A(2, 4),\ B(6, 8)$

d $A(-2, -6),\ B(-4, -8)$

e $A(-1, 5),\ B(2, 7)$

f $A(-12, 16),\ B(2, 8)$

4 **a** Find the midpoint M of the interval AB where the coordinates of A are $(6, 2)$ and the coordinates of B are $(6, 8)$.

b Let C be the point $(10, 5)$. Find the distance between:

i A and C

ii A and B

c Describe triangle ABC.

5 **a** The distance between two points $A(2, u)$ and $B(-3, 4)$ is 5. Find the value of u.

b The distance between two points $P(4, -2)$ and $Q(v, -5)$ is $\sqrt{34}$. Find the possible values for v. Draw a diagram to illustrate the result.

c The distance between two points $A(3, -2)$ and $B(w, 4)$ is 10. Find the possible values for w. Draw a diagram to illustrate the result.

6 The triangle ABC has vertices $A(0, 0)$, $B(3, 0)$ and $C(3, 4)$.

a Find the distance between A and C.

b Find the midpoint M of AC.

c Find the length of:

i AM

ii BM

iii CM

7 **a** $M(4, 2)$ is the midpoint of the interval AC, where C has coordinates $(12, 3)$. Find the coordinates of A.

b $M(10, -2)$ is the midpoint of the interval AC, where A has coordinates $(-2, 6)$. Find the coordinates of C.

8 Show that ΔPQR is a right-angled triangle where the coordinates of P, Q and R are $(3, 3)$, $(3, -1)$ and $(6, 3)$, respectively.

9 Show that the triangle with vertices $X(-3, 1)$, $Y(0, 2)$ and $Z(-2, 4)$ is isosceles.

10 Show that the points $A(-1, -3)$, $B(4, 0)$, $C(5, 7)$ and $D(0, 2)$ are the vertices of a rhombus.

11 A, B, C and D are the points $(0, -5)$, $(-4, -1)$, $(4, 3)$ and $(-8, -9)$, respectively. Show that AB and CD bisect each other.

12 The points $A(-5, 0)$, $B(-3, -4)$, $C(2, 1)$ and $D(0, 5)$ are the vertices of a quadrilateral. Show that $ABCD$ is a parallelogram.

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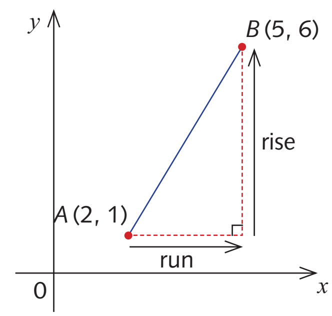
4B Gradient

Gradient of an interval

The **gradient** of an interval AB is defined as $\frac{\text{rise}}{\text{run}}$, where the **rise** is the change in the y-values as you move from A to B and the **run** is the change in the x-values as you move from A to B.



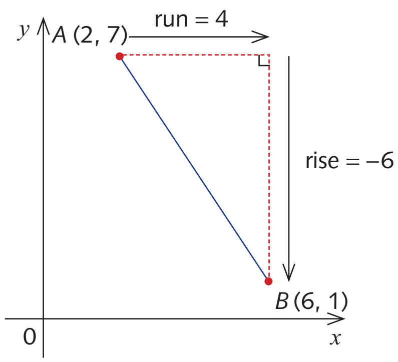
For the points $A(2, 1)$ and $B(5, 6)$:

$$\begin{aligned}\text{gradient of interval } AB &= \frac{\text{rise}}{\text{run}}\\ &= \frac{6-1}{5-2}\\ &= \frac{5}{3}\end{aligned}$$

Notice that as you move from A to B along the interval, the y-value increases as the value increases. This means the gradient is **positive**.

In the diagram to the right:

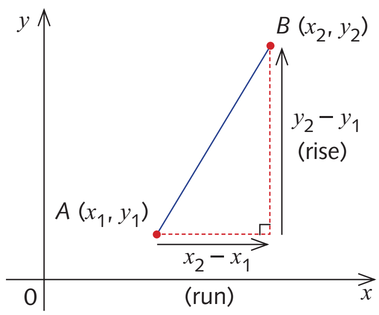
$$\begin{aligned}\text{gradient of interval } AB &= \frac{\text{rise}}{\text{run}}\\ &= \frac{1-7}{6-2}\\ &= \frac{-6}{4}\\ &= -\frac{3}{2}\end{aligned}$$



The rise from A to B is negative and the run from A to B is positive, so the gradient is **negative**.

In general, provided $x_2 \neq x_1$:

$$\begin{aligned}\text{gradient of interval } AB &= \frac{\text{rise}}{\text{run}}\\ &= \frac{y_2 - y_1}{x_2 - x_1}\end{aligned}$$



Since $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$, it does not matter which point we take as the first and which point we take as the second.

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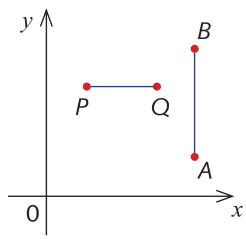
If the rise is zero, the interval is horizontal, as shown by the interval PQ at the right. The gradient of the interval is zero.

If the run is zero, the interval is vertical, as shown by the interval AB at the right. The interval does not have a gradient.



Gradient of PQ is zero.
Gradient of AB is not defined.

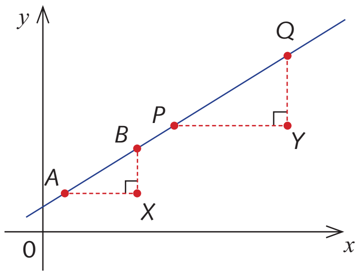
Gradient of a line

The **gradient of a line** is defined to be the gradient of any interval within the line.

Any two intervals on a line have the same gradient. We can prove this as follows.

Triangle ABX is similar to triangle PQY, as the corresponding angles are equal by parallel lines. Therefore:

$$\frac{QY}{PY} = \frac{BX}{AX} \quad \text{(Ratio of sides in similar triangles.)}$$



That is, the intervals have the same gradient. Therefore, the definition of the gradient of a line makes sense.

Example 3

Find the gradient AB.

a $A(3, -2)$, $B(2, -6)$

b $A(-1, -3)$, $B(-2, 6)$.

Solution

a
$$\begin{aligned}\text{Gradient} &= \frac{-6-(-2)}{2-3}\\ &= \frac{-6+2}{-1}\\ &= 4\end{aligned}$$

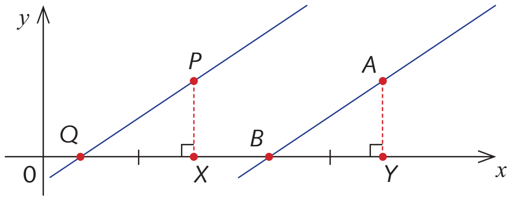
b
$$\begin{aligned}\text{Gradient} &= \frac{-3-6}{-1-(-2)}\\ &= \frac{-9}{-1+2}\\ &= -9\end{aligned}$$

Parallel lines

If two non-vertical lines are **parallel**, then they have the same gradient. Conversely, if two lines have the same gradient, then they are parallel.

We can prove this as follows.

In the diagram on the right, two lines are drawn and the right-angled triangles PQX and ABY are drawn, with $QX = BY$.



If the lines are parallel, then $\angle PQX = \angle ABY$ (corresponding angles).

The two triangles are congruent by the AAS test. Hence, $PX = AY$, so $\dfrac{AY}{BY} = \dfrac{PX}{QX}$.

The gradients are equal.

Conversely, if the gradients are equal, then $PX = AY$.

The triangles are congruent by the SAS test.

Hence, the corresponding angles PQX and ABY are equal and the lines are parallel.

Note: This proof does not work for lines that are parallel to one of the axes.

Example 4

Show that the line passing through the points $A(6, 4)$ and $B(7, 11)$ is parallel to the line passing through $P(0, 0)$ and $Q(1, 7)$.

Solution

$$\text{Gradient of } AB = \frac{11-4}{7-6} = \frac{7}{1} = 7$$

$$\text{Gradient of } PQ = \frac{7-0}{1-0} = \frac{7}{1} = 7$$

The two lines have the same gradient, so they are parallel.

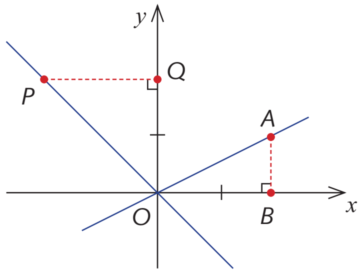
Perpendicular lines

Two lines are **perpendicular** if the product of their gradients is −1 (or if one is vertical and the other horizontal). Conversely, if two lines are perpendicular (but are not parallel with the axes), then the product of their gradients is −1.

Here is a proof of this remarkable result.

Draw two lines passing through the origin, with one of the lines having positive gradient and the other negative gradient.

Form right-angled triangles OPQ and OAB, with $OQ = OB$.



$$\text{Gradient of the line } OA = \frac{AB}{BO}$$

$$\text{Gradient of the line } OP = -\frac{OQ}{PQ}$$

$$\begin{aligned}\text{Product of gradients} &= -\frac{OQ}{PQ} \times \frac{AB}{BO} \\ &= -\frac{OQ}{PQ} \times \frac{AB}{OQ} \quad (\text{since } OB = OQ) \\ &= -\frac{AB}{PQ}\end{aligned}$$

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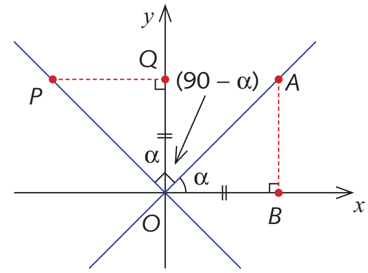
If the lines are perpendicular, then $\angle POQ = \angle AOB$.

Therefore, triangles OPQ and OAB are congruent (AAS), so $PQ = AB$ and the product of the gradients is -1.

Conversely, if the product of the gradients is -1, then $AB = PQ$ since, by the above, the product of the gradients $= -\frac{AB}{PQ}$.



This implies that the triangles OBA and OQP are congruent (SAS). Therefore, $\angle POQ = \angle AOB$ and so $\angle AOP = 90° - \alpha + \alpha = 90°$.

We have now proved the result for lines through the origin. However, this will suffice for any pair of lines in the plane (not parallel with the axes).

Example 5

Show that the line through the points $A(6, 0)$ and $B(0, 12)$ is perpendicular to the line through $P(8, 10)$ and $Q(4, 8)$.

Solution

$$\text{Gradient of } AB = \frac{12-0}{0-6} = -2$$

$$\text{Gradient of } PQ = \frac{10-8}{8-4} = \frac{2}{4} = \frac{1}{2}$$

$$(\text{Gradient of } AB) \times (\text{Gradient of } PQ) = -2 \times \frac{1}{2} = -1$$

Hence, the lines are perpendicular.

Gradient of non-vertical lines

- The **gradient of an interval**, AB, connecting the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\frac{y_2 - y_1}{x_2 - x_1}$.
- The **gradient of a line** is defined as the gradient of any interval within the line.
- Two lines are **parallel** if they have the same gradient. Conversely, if two lines are parallel, then they have the same gradient.
- Two lines are **perpendicular** if the product of their gradients is –1 (or if one is vertical and the other horizontal). Conversely, if two lines are perpendicular, then the product of their gradients is –1.

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Exercise 4B

Example 3

1 Find the gradient of each interval.

a $A(6, 3),\ B(2, 0)$ **b** $A(-2, 6),\ B(0, 10)$ **c** $A(-1, 10),\ B(6, -4)$

d $A(2, 3),\ B(-4, 5)$ **e** $A(6, 7),\ B(-2, -3)$ **f** $A(10, 0),\ B(0, 10)$

g $A(10, 0),\ B(0, -10)$ **h** $A(4, 3),\ B(6, 3)$ **i** $A(4, -3),\ B(-5, 10)$

Example 4

2 Show that the line passing through $A(1,\ 6)$ and $B(2,\ 7)$ is parallel to the line passing through $X(-1, 6)$ and $Y(2, 9)$.

3 The line passing through the points $(1, 4)$ and $(3, a)$ has gradient 2. Find the value of a.

4 The line passing through the points $(-4, 6)$ and $(b, 2)$ has gradient $\frac{1}{2}$. Find the value of b.

5 Complete:

	Coordinates of *A*	Coordinates of *B*	Gradient of *AB*
a	(2, 1)	(5, 13)	…
b	(–1, 3)	(0, –1)	…
c	(–1, 2)	(2, …)	2
d	(–4, 10)	(2, …)	$-\frac{1}{2}$
e	(…, 5)	(7, 9)	$\frac{2}{3}$
f	(…, –4)	(1, –13)	–3

Example 5

6 Show that the line passing through the points $A(5, 60)$ and $B(-1, 12)$ is perpendicular to the line passing through $P(7, 10)$ and $Q(23, 8)$.

7 Find the gradient of a line perpendicular to a line with gradient:

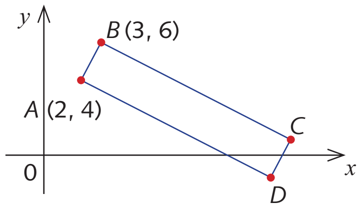
a 6 **b** $-\frac{1}{2}$ **c** $\frac{3}{2}$ **d** $-\frac{4}{5}$ **e** -1

8 *ABCD* is a rectangle.

a Find the gradient of interval *AB*.

b Find the gradient of interval *CD*.

c Find the gradient of interval *AD*.



9 **a** Plot the four points $A(0, 0),\ B(3, 0),\ C(5, 2)$ and $D(2, 2)$.

b Use gradients to show that *ABCD* is a parallelogram.

c Find the midpoint of *DB* and explain why it is the same as the midpoint of *AC*.

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10 The vertices of a quadrilateral $ABCD$ are the points $(-4, -2)$, $(3, 9)$, $(8, 1)$ and $(2, -3)$, respectively. E, F, G and H are the midpoints of AB, BC, CD and DA, respectively. Show that $EFGH$ is a parallelogram.

11 $A(3, 6)$ and $B(4, 7)$ are two adjacent vertices of a square $ABCD$.

a Find the length of each side of the square.

b Find the gradient of AB.

c Find the gradient of CD.

12 In each part, find the gradients of intervals AB and BC, and state whether A, B and C lie on the same line (are collinear) or not.

a $A(3, 6)$, $B(-1, 4)$, $C(4, 11)$

b $A(3, 8)$, $B(2, 5)$, $C(1, 2)$

c $A(4, 11)$, $B(-1, -4)$, $C(2, 5)$

d $A(4, 5)$, $B(-1, -6)$, $C(3, 7)$

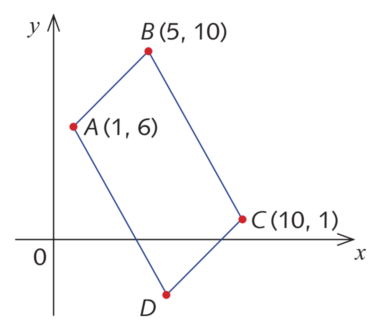
13 $ABCD$ is a parallelogram.

a Find the gradient of interval AB.

b Find the gradient of interval CD.

c Find the coordinates of D.

d Find the coordinates of the midpoints of AC and BD.



4C Gradient–intercept form and the general form of the equation of a line

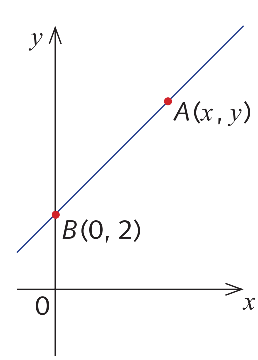
Gradient–intercept form

In earlier work, we have seen that the equation $y = mx + b$ represents a line with gradient m and y-intercept b. This is called the **gradient–intercept form** of the equation of a line. Conversely, every non-vertical line has an equation of the form $y = mx + b$.

To illustrate this, consider the line with gradient 3 and y-intercept 2. That is, $m = 3$ and $b = 2$.

Let $A(x, y)$ be any point on this line.

$$\begin{aligned}\text{Gradient of interval } AB &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y-2}{x-0} \\ &= \frac{y-2}{x}\end{aligned}$$



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We know the gradient of the line is 3. Therefore:

$$\frac{y-2}{x} = 3$$
$$y - 2 = 3x$$
$$y = 3x + 2$$

Hence, the equation of the line is $y = 3x + 2$.

The equation relates the x- and y-coordinates of any point on the line.

In general lines with gradient m and y-intercept b have equation $y = mx + b$. Conversely the points whose coordinates satisfy th equation $y = mx + b$ always lie on a line with gradient m and y-intercept b.

Example 6

a The gradient of a line is -6 and the y-intercept is 2. Find the equation of the line.

b The equation of a line is $y = -7x + 3$. State the gradient and y-intercept.

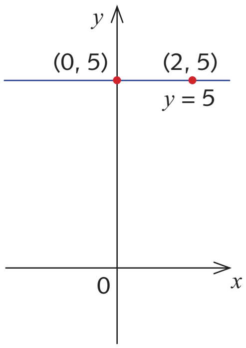
Solution

a The equation of the line is $y = -6x + 2$.

b The gradient is -7 and the y-intercept is 3.

Horizontal lines

All points on a horizontal line have the same y-coordinate, but the x-coordinate can take any value. Thus, the equation of the horizontal line through the point $(0, 5)$ is $y = 5$. The equation of the horizontal line through the point $(2, 5)$ is also $y = 5$.

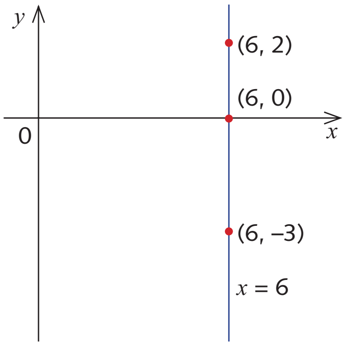


In general, the equation of the horizontal line through $P(a, b)$ is $y = b$.

A horizontal line has gradient 0 because all y-values are the same.

Vertical lines

All points on a vertical line have the same x-coordinate, but the y-coordinate can take any value. Thus, the equation of the vertical line through the point $(6, 0)$ is $x = 6$.



In general, the equation of the vertical line through $P(a, b)$ is $x = a$ or $x - a = 0$. Note that because this line does not have a gradient, it cannot be written in the form $y = mx + b$.

The form of the equation for a vertical or a horizontal line sometimes seems strange. It becomes clearer if we realise that the equation $x = a$ is shorthand for the statement $\{(x, y) : x = a\}$. This is read as the 'set of points (x, y) such that $x = a$'. Similarly, the equation $y = b$ is shorthand for the statement $\{(x, y) : y = b\}$. This is read as the 'set of points (x, y) such that $y = b$'.

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The general form of an equation of a line

The equation $y = 2x - 3$ can be written as $-2x + y + 3 = 0$.

The equation $2x - 3y = 6$ can be written as $2x - 3y - 6$ and the equation $x = 6$ can be written as $x - 6 = 0$.

The **general form** for the equation of a line is $ax + by + c = 0$, where a, b and c are constants, and either $a \neq 0$ or $b \neq 0$. The equation of *every line* can be written in general form.

Example 7

Write each equation in general form.

a $y = -\frac{2}{3}x + 4$

b $y = -\frac{4}{5}x + \frac{2}{3}$

Solution

a

$$y = -\frac{2}{3}x + 4$$
$$3y = -2x + 12$$
$$2x + 3y - 12 = 0$$

b

$$y = -\frac{4}{5}x + \frac{2}{3}$$
$$15y = -12x + 10$$
$$12x + 15y - 10 = 0$$

The general form is not unique. For example, the line $2x + 3y - 12 = 0$ is the same as the line $20x + 30y - 120 = 0$.

Example 8

Write the equation of each line in gradient–intercept form, and state its gradient and y-intercept.

a $2x + y + 6 = 0$

b $3x - 2y + 7 = 0$

Solution

a $2x + y + 6 = 0$

$$y = -2x - 6$$

gradient $= -2$

y-intercept is -6

b $3x - 2y + 7 = 0$

$$3x + 7 = 2y$$
$$y = \frac{3}{2}x + \frac{7}{2}$$

gradient $= \frac{3}{2}$

y-intercept is $\frac{7}{2}$

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Sketching a line given its equation

A line can be sketched if the coordinates of two points are known.

For lines that are not parallel to one of the axes and do not pass through the origin, a useful procedure to sketch the line is to find the intercepts with the axes. Find the x-intercept by substituting $y = 0$, and find the y-intercept by substituting $x = 0$.

A non-vertical line passing through the origin has an equation of the form $y = mx$. A second point on the line can be determined from the equation by substituting a non-zero value of x into the equation. This is recommended because it identifies the steepness of the line.

Example 9

Sketch the graph of:

a $y = 2x + 4$

b $y = -3x + 8$

c $2x + 3y + 12 = 0$

d $3x + 2y = 10$

e $x = 4$

f $y = -3x$

Solution

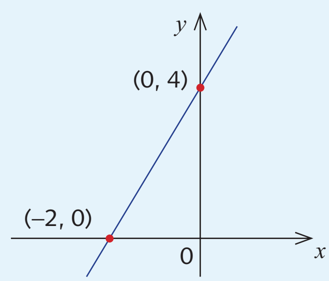
a $y = 2x + 4$

When $x = 0$, $y = 4$

When $y = 0$, $2x + 4 = 0$

$$x = \frac{-4}{2}$$

$$= -2$$



b $y = -3x + 8$

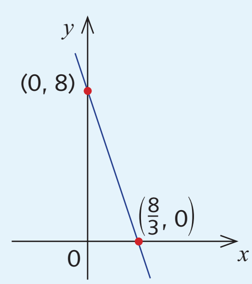
When $x = 0$, $y = 8$

When $y = 0$, $-3x + 8 = 0$

$$-3x = -8$$

$$x = \frac{8}{3}$$

$$x = 2\tfrac{2}{3}$$



c $2x + 3y + 12 = 0$

When $x = 0$, $3y + 12 = 0$

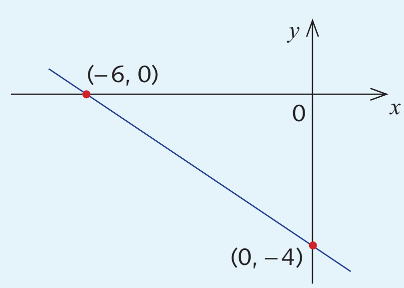
$$3y = -12$$

$$y = -4$$

When $y = 0$, $2x + 12 = 0$

$$2x = -12$$

$$x = -6$$



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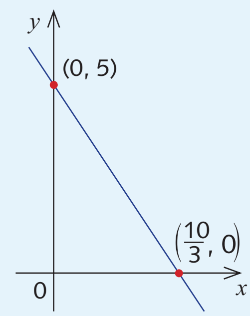
d $3x + 2y = 10$

When $x = 0, 2y = 10$

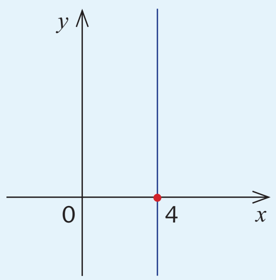
$y = 5$

When $y = 0, 3x = 10$

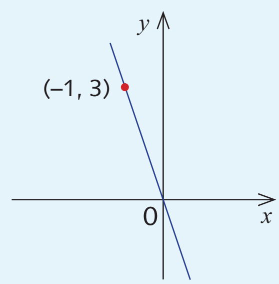
$x = \frac{10}{3}$



e $x = 4$



f $y = -3x$



Given one coordinate of a point on a line, the equation of a line can be used to find the other coordinate.

Example 10

The following points lie on the line with equation $5x - 4y = 20$. Find the value of each pronumeral.

a $(0, a)$ **b** $(b, 0)$ **c** $(d, -6)$ **d** $(f, 10)$

Solution

a $5 \times 0 - 4a = 20$

$a = -5$

b $5b - 4 \times 0 = 20$

$5b = 20$

$b = 4$

c $5d - 4 \times (-6) = 20$

$5d + 24 = 20$

$5d = -4$

$d = -\frac{4}{5}$

d $5f - 4 \times 10 = 20$

$5f = 60$

$f = 12$

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Gradient–intercept form and the general form of the equation of a line

- The **gradient–intercept form** of the equation of a line is $y = mx + b$, where m is the gradient and b is the y-intercept.
- **The general form** of the equation of a line is $ax + by + c = 0$, where a, b and c are constants, and $a \neq 0$ or $b \neq 0$.

Exercise 4C

Example 6

1 Write the gradient and y-intercept of each line.

a $y = 4x + 2$ **b** $y = -\frac{2}{3}x + 5$ **c** $y = -7x + 10$ **d** $y = -\frac{4}{11}x + \frac{2}{3}$

2 Write the equation of the line with the given gradient and y-intercept.

a gradient $= 8$, y-intercept is 3 **b** gradient $= 11$, y-intercept is 5

c gradient $= -6$, y-intercept is -7 **d** gradient $= -\frac{3}{4}$, y-intercept is $\frac{2}{5}$

3 Sketch the graph of each line.

a $y = 1$ **b** $y = 2$ **c** $y = -2$ **d** $y = -3$

e $x = 2$ **f** $x = 4$ **g** $x = -1$ **h** $x = -3$

i $y + 3 = 0$ **j** $y - 1 = 0$ **k** $x + 3 = 0$ **l** $x - 1 = 0$

m Which of these lines have a gradient and what is it?

Example 7

4 Express each equation in general form.

a $y = -2x + 6$ **b** $y = -\frac{2}{3}x + 11$ **c** $y = -\frac{3}{5}x - \frac{2}{3}$ **d** $y = \frac{4}{7}x + \frac{1}{6}$

e $y = -\frac{2}{5}x - \frac{4}{5}$ **f** $y = \frac{2}{5}x - \frac{3}{10}$ **g** $x = \frac{1}{3}y + 4$ **h** $\frac{3}{4}x = \frac{4}{3}y - 3$

Example 8

5 Express each equation in gradient–intercept form.

a $3x - 2y = 6$ **b** $5x + 2y + 10 = 0$ **c** $3y - 2x + 12 = 0$

d $6y - x + 18 = 0$ **e** $15y - 2x + 18 = 0$ **f** $2x - 3y + 12 = 0$

g $5x + 4y + 20 = 10$ **h** $6x - 4y - 24 = 0$ **i** $3x - 5y + 15 = 0$

6 Find the gradient and y-intercept in each case.

a $2y - 3x = 12$ **b** $4x + y + 24 = 0$ **c** $3x + 8y + 48 = 0$

d $4x - 7y + 56 = 0$ **e** $11x + 4y = 44$ **f** $10x - 5y = -20$

g $3x - 7y + 42 = 0$ **h** $2x - 7y = 14$ **i** $-10x - 2y = -40$

Example 9a, b, c, d

7 Find the x- and y-intercepts in each case.

a $y = 2x - 10$ **b** $y = 3x + 11$ **c** $3x + 8y = 48$

d $5x - 4y + 80 = 0$ **e** $3x - 7y - 42 = 0$ **f** $5x - 2y + 11 = 0$

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8 Sketch the graph of each line by first finding the x- and y-intercepts.

a $y = -2x + 12$ **b** $3y = -2x + 24$ **c** $6x - 3y = 18$

d $y = x + 18$ **e** $y = 2x - 11$ **f** $3x - 7y = 20$

g $4x - 7y = 28$ **h** $7x - 2y = 11$ **i** $8x - 4y + 20 = 0$

Example 9e, f

9 **a** Give the equation of the line parallel to the y-axis and passing through the point $(1, 5)$.

b Give the equation of the line parallel to the x-axis and passing through the point $(-2, 5)$.

c Give the equation of the line parallel to the y-axis and passing through the point $(-4, -7)$.

Example 9e, f

10 Sketch the graph of:

a $x = 3$ **b** $y = 2x$ **c** $y = -2x$ **d** $y = 4$ **e** $y = -4x$ **f** $y = \frac{1}{4}x$

Example 10

11 The following points lie on the line with equation $3x - 12y = 30$. Find the value of each pronumeral.

a $(0, a)$ **b** $(b, 0)$ **c** $(1, c)$ **d** $(d, -6)$ **e** $(4, e)$ **f** $(f, 10)$

12 The following points lie on the line with equation $y = -\frac{1}{2}x - 4$. Find the value of each pronumeral.

a $(0, a)$ **b** $(b, 0)$ **c** $(1, c)$ **d** $(d, -6)$ **e** $(4, e)$ **f** $(f, 10)$

13 A line has equation $y = -4x + c$. The point $(6, 10)$ is on the line. Find the value of c.

14 A line has equation $2x - by + 7 = 0$. The point $(6, -5)$ is on the line. Find the value of b.

15 A line has equation $ax - 3y + 15 = 0$ and gradient 4. Find the value of a.

16 A line has equation $3x - by + 10 = 0$ and gradient $-\frac{1}{2}$. Find the value of b.

4D Point–gradient form of an equation of a line

Equation of a line given the gradient and a point on the line

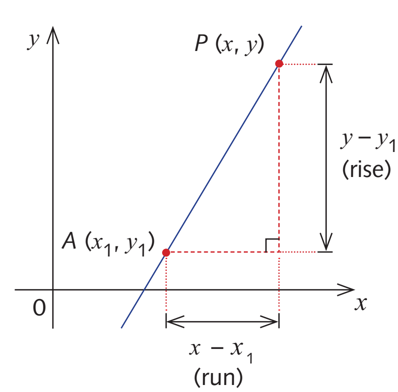
Suppose that we know the gradient m of a line and a point $A(x_1, y_1)$ on the line. Let $P(x, y)$ be any point on the line. Then:

$$m = \frac{y - y_1}{x - x_1}$$

and so:

$$y - y_1 = m(x - x_1)$$

This equation is called the **point–gradient** form of the line.



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Example 11

Find the equation of the line that is parallel to the line with equation $y = -2x + 6$ and:

a passes through the point $A(1, 10)$

b passes through the point $B(-1, 0)$

Solution

The gradient of the line $y = -2x + 6$ is -2.

a Therefore the line through the point $A(1, 10)$ parallel to $y = -2x + 6$ has equation:

$$y - 10 = -2(x - 1)$$
$$y - 10 = -2x + 2$$
$$y = -2x + 12 \text{ or } 2x + y - 12 = 0$$

b The line through the point $B(-1, 0)$ parallel to $y = -2x + 6$ has equation:

$$y - 0 = -2(x + 1)$$
$$y = -2x - 2 \text{ or } 2x + y + 2 = 0$$

Give the equation of the line in gradient–intercept form or general form – whichever you prefer.

Example 12

Find the equation of the line ℓ that is perpendicular to the line with equation $y = \frac{2}{3}x - 3$ and passes through the point $P(1, 6)$.

Solution

The gradient of $y = \frac{2}{3}x - 3$ is $\frac{2}{3}$.

$\frac{2}{3} \times \left(-\frac{3}{2}\right) = -1$, so the gradient of ℓ is $-\frac{3}{2}$

Since it passes through the point $(1, 6)$, the equation of the line is:

$$y - 6 = -\frac{3}{2}(x - 1)$$
$$2y - 12 = -3(x - 1)$$
$$2y - 12 = -3x + 3$$
$$3x + 2y - 15 = 0 \text{ or } y = -\frac{3}{2}x + \frac{15}{2}$$

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Equation of a line given two points

In Section 4B, we saw that the gradient m of a line passing through two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

We can now find the equation of a line, given the coordinates of two points on the line, as follows:

- Find the gradient of the line.
- Use the point–gradient form with either one of the points.

Example 13

Find the equation of the line passing through (2, 6) and (–3, 7).

Solution

$$\text{Gradient of line} = \frac{7-6}{-3-2} = -\frac{1}{5}$$

The point–gradient form with $m = -\dfrac{1}{5}$ and $(x_1, y_1) = (2, 6)$ gives:

$$y - 6 = -\frac{1}{5}(x - 2)$$

$5y - 30 = -(x - 2)$ (Multiply both sides by 5.)

$5y - 30 = -x + 2$

Hence, $x + 5y - 32 = 0$ is the general form of the line.

Check that both points lie on the line.

Note: the same equation can be established using $(x_1, y_1) = (-3, 7)$

The point–gradient form

- The equation of a line, **given the gradient m and one point $A(x_1, y_1)$** on the line, is:
  $y - y_1 = m(x - x_1)$
- To find the equation of a line, **given two points $A(x_1, y_1)$ and $B(x_2, y_2)$** use the point–gradient formula:
  $y - y_1 = m(x - x_1)$, where the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

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Exercise 4D

Example 11

1 a Find the equation of the line with gradient 6 that passes through the point $(5, 6)$.

b Find the equation of the line with gradient -4 that passes through the point $(2, 6)$.

c Find the equation of the line with gradient $\frac{1}{2}$ that passes through the point $(-1, 8)$.

d Find the equation of the line parallel to the line $y = -3x + 8$ and passing through the point $(1, 8)$.

e Find the equation of the line with gradient 0 that passes through the point $(-3, 6)$.

f Find the equation of the line parallel to the line $x = -4$ and passing through the point $(-7, 11)$.

Example 12

2 a Find the equation of the line perpendicular to the line $y = -3x + 6$ and passing through the point $(-2, 8)$.

b Find the equation of the line perpendicular to the line $x + 2y = 6$ and passing through the point $(1, 2)$.

c Find the equation of the line perpendicular to the line $2x - y = 6$ and passing through the point $(6, -3)$.

3 a Find the midpoint of the interval AB, where the coordinates of A and B are $(2, -1)$ and $(3, 6)$, respectively.

b Find the gradient of the line that passes through points A and B.

c Find the equation of the line AB.

d Find the equation of the perpendicular bisector of the interval AB.

4 a Find the equation of the line with gradient -4 that passes through the point $(0, -6)$.

b Find the equation of the line with gradient -4 that passes through the point $(3, 8)$.

c Find the equation of the line that passes through the points $(-4, 8)$ and $(-6, -2)$.

d Find the equation of the line that is parallel to the line $y = -2x + 3$ and passes through the point with coordinates $(-1, -10)$.

Example 13

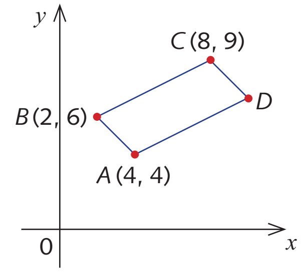
5 Find the equation of the line that passes through the two given points in each case.

a $(0, -4)$ and $(4, 0)$ b $(-3, 0)$ and $(0, -9)$ c $(2, 4)$ and $(-6, 12)$

d $(6, 3)$ and $(7, 3)$ e $(1, 4)$ and $(1, 8)$ f $(0, -3)$ and $(4, 6)$

6 Show that the points $A(1, 1)$, $B(3, 11)$ and $C(-2, -14)$ all lie on the same line (are collinear) and find the equation of this line. Do this by finding the equation of AB and checking that point C lies on the line.

7 $ABCD$ is a parallelogram with vertices $A(4, 4)$, $B(2, 6)$ and $C(8, 9)$. Find:



a the equation of the line BC b the equation of the line AB

c the equation of the line AD d the gradient of the line CD

e the distance AB f the distance CD

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8 $A(1, 1)$, $B(1, 6)$, $C(6, 6)$ and $D(6, 1)$ are the vertices of a square $ABCD$.

a Find the midpoint of:

i AC **ii** BD

b Find the gradient of AC and BD, and hence show that AC is perpendicular to BD.

4E Review of simultaneous linear equations

In this section, we revise the standard methods for solving simultaneous linear equations. The solutions are the coordinates of the point of intersection of the two lines given by the linear equation.

Solving a pair of simultaneous equations means finding the values of x and y that satisfy both equations.

Example 14

Find the coordinates of the point of intersection of the lines $y = x - 1$ and $y = 2x - 3$ and sketch the lines on the one set of axes.

Solution

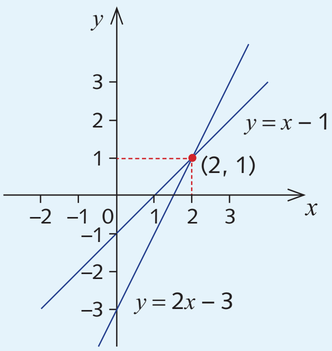
At the point (x, y) of intersection of the graphs, the y-coordinates of both graphs are the same.

Therefore,
$$\begin{aligned} x - 1 &= 2x - 3 \\ x + 2 &= 2x \\ x &= 2 \end{aligned}$$

Substituting into either equation gives $y = 1$.

The coordinates of the point of intersection are $(2, 1)$.

The solution of the simultaneous equations $y = x - 1$ and $y = 2x - 3$ is $x = 2$ and $y = 1$.



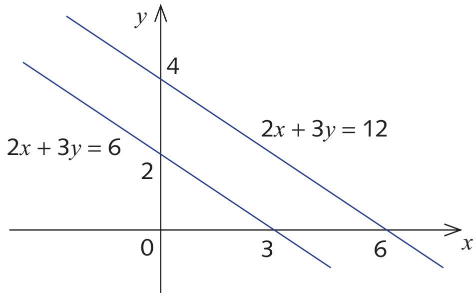
Note that the process outlined above can be done without the graph.

Lines that are parallel and lines that coincide

Simultaneous linear equations do not always have a unique solution. There are two geometric situations in which lines do not intersect at a single point.

Parallel lines

The equations $2x + 3y = 12$ and $2x + 3y = 6$ represent parallel lines. There are no solutions to this pair of simultaneous equations since the lines do not meet.



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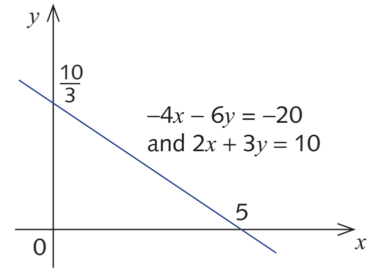
Lines that coincide

Sometimes we have two equations that represent the same line. The equations $-4x - 6y = -20$ and $2x + 3y = 10$ represent the same line. This can be checked by showing that the lines have the same intercepts. We say that there are infinitely many solutions to this pair of equations since every point on the line satisfies both equations.



Solution by substitution

We recall that to solve a pair of simultaneous equations that involve pronumerals x and y, we can make either x or y the subject of one of the equations and substitute into the other equation. This method of solving a pair of simultaneous equations is called the **substitution method**. This method was used in Example 14.

Example 15

Solve this pair of equations for x and y.

$$x = 2y - 3 \quad (1)$$
$$2x - 3y = 7 \quad (2)$$

Solution

Substitute for x into equation (2), using equation (1):

$$\begin{aligned} 2(2y - 3) - 3y &= 7 \\ 4y - 6 - 3y &= 7 \\ y - 6 &= 7 \\ y &= 13 \end{aligned}$$

Using equation (1) gives:

$$\begin{aligned} x &= 2 \times 13 - 3 \\ &= 23 \end{aligned}$$

Thus the solution is $x = 23$, $y = 13$.

That is, the corresponding lines meet at $(23, 13)$.

Note: You should always check your answers by substituting into both of the original equations.

Solution by elimination

The other standard method for solving simultaneous equations is called the **elimination method**. This method relies on combining the two equations so that one of the pronumerals is eliminated; that is, we add or subtract multiples of the two equations to eliminate one pronumeral.

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Example 16

Solve this pair of equations for x and y.

$$3x + y = 13 \qquad (1)$$
$$x - y = 3 \qquad (2)$$

Solution

Adding equations (1) and (2) gives:

$$4x = 16 \qquad (3)$$
$$x = 4$$

Substituting into equation (1) gives:

$$12 + y = 13$$
$$y = 1$$

Therefore, the solution is $x = 4,\ y = 1$.

That is, the corresponding lines meet at (4, 1).

Example 17

Solve this pair of equations for x and y.

$$2x + 3y = 14 \qquad (1)$$
$$2x - y = 6 \qquad (2)$$

Solution

Subtracting the two equations will eliminate x and produce a single equation involving only y.

$$(1) - (2): \quad 4y = 8$$
$$y = 2$$

Substituting $y = 2$ into equation (1) gives:

$$2x + 6 = 14$$
$$2x = 8$$
$$x = 4$$

Hence, the solution is $x = 4,\ y = 2$.

That is, the corresponding lines meet at (4, 2).

Sometimes it is necessary to multiply both sides of an equation by a factor to enable a pronumeral to be eliminated. Remember that the new equation formed is equivalent to the first. This is shown in the following example.

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Example 18

Solve this pair of equations for x and y.

$x - 3y = 2 \quad (1)$

$4x + y = 21 \quad (2)$

Solution

We make the coefficients of y the same by multiplying equation (2) by 3. Then we have:

$x - 3y = 2 \quad (1)$

$(2) \times 3: \quad 12x + 3y = 63 \quad (3)$

Now y can be eliminated by adding the two equations.

$(1) + (3): \quad 13x = 65$

$x = 5$

Substituting into equation (1) gives:

$5 - 3y = 2$

$-3y = -3$

$y = 1$

Hence, the solution is $x = 5$, $y = 1$ and the corresponding lines meet at $(5, 1)$.

In the next example, it is necessary to find two new equivalent equations in order to eliminate a pronumeral.

Example 19

Solve this pair of equations for x and y.

$3x + 5y = 1 \quad (1)$

$5x + 3y = 7 \quad (2)$

Solution

We choose to eliminate x, so we proceed as follows.

$(1) \times 5: \quad 15x + 25y = 5 \quad (3)$

$(2) \times 3: \quad 15x + 9y = 21 \quad (4)$

$(3) - (4): \quad 16y = -16$

$y = -1$

Substituting into equation (1) gives:

$3x - 5 = 1$

$3x = 6$

$x = 2$

The solution is $x = 2$, $y = -1$ and the corresponding lines meet at $(2, -1)$.

Check that $(2, -1)$ satisfies both equations (1) and (2).

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Review of simultaneous linear equations

- A pair of simultaneous equations has either one, zero or infinitely many solutions. These cases occur, respectively, when the two lines meet at a point, are parallel or coincide.
- A pair of simultaneous equations can be solved using either the **substitution method** or the **elimination method**.
  - In the **substitution method**, make x or y the subject of one equation and substitute into the other equation.
  - In the **elimination method**, add or subtract suitable multiples of the two equations to eliminate one pronumeral.

Exercise 4E

Example 14

1 For each pair of equations, sketch the graphs and find the coordinates of the point of intersection.

a $y = 3x + 1$
$y = 2x + 2$

b $y = 3 - 2x$
$y = x - 3$

c $y = 2x + 1$
$y = 5x + 3$

Example 15

2 For each pair of equations, solve using the substitution method.

a $y = 3x$
$2x - 3y = 9$

b $x = 2y$
$3x + 2y = 6$

c $y = 2x + 1$
$x - 3y = 4$

d $x = 1 - 3y$
$4x - 3y = 12$

e $y = 1 - 2x$
$y = 5x + 2$

f $x = \frac{y}{3} + 2$
$7x - 5y = 10$

Example 16, 17, 18, 19

3 For each pair of equations, solve using the elimination method.

a $x - y = 3$
$2x + y = 9$

b $3x + y = 5$
$5x - y = 3$

c $x + y = 1$
$2x + y = 4$

d $2x + 3y = 4$
$5x + 3y = 1$

e $2x - 3y = 4$
$2x + y = 12$

f $3x + 2y = 5$
$3x + 5y = 26$

g $2x + y = 4$
$3x + 2y = 7$

h $4x - y = 5$
$3x + 4y = -1$

i $x + 2y = 2$
$3x + 5y = 3$

j $2x - 3y = 7$
$3x + 2y = 4$

k $5x + 4y = 20$
$2x + 5y = 10$

l $7x - 5y = 15$
$3x - 4y = 13$

m $2x - 3y = -9$
$3x - 2y = -1$

n $2x + 3y = 2$
$3x + 7y = -7$

o $2x + 5y = -35$
$3x - 2y = 8$

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4 Solve each pair of equations.

a $y = 5x - 1$
$2x - 7y = 35$

b $7x - 9y = 63$
$5x + 8y = 40$

c $4x + 7y - 24 = 0$
$6x + 9y - 17 = 0$

d $x = 3 - 4y$
$7y - 3x = 21$

e $7x - 11y = 48$
$5x - 6y = 27$

f $\frac{1}{2}x - \frac{2}{3}y = 4$
$\frac{2}{3}x + \frac{3}{4}y = 7$

g $y = \frac{2}{5}x - 8$
$y = -\frac{2}{7}x + 3$

h $y = 3x - 2$
$2x + 3y = 4$

i $x + 7y = 0$
$3x - 4y = 24$

j $3x - 7y - 42 = 0$
$2x - 3y - 18 = 0$

k $x = 3y - 5$
$2x - 3y = 21$

l $y = \frac{1}{4}x + 7$
$y = -\frac{3}{5}x - 4$

5 The line $y = 2x$ intersects the line $y = x + 6$ at the point A. Find the equation of the line that passes through A and has gradient 3.

6 The line $y = 2x - 4$ intersects the line $y = -3x + 6$ at the point B. Find the equation of the line that passes through B and is:

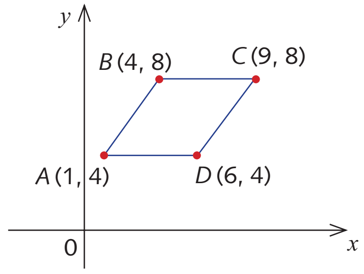
a parallel to the x-axis

b parallel to the y-axis

7 The line $y = x$ intersects the line $y = 2x + 1$ at the point A, and intersects the line $y = -3x + 12$ at the point B. The line $y = 2x + 1$ meets the line $y = -3x + 12$ at the point C. Find the coordinates of the vertices of triangle ABC.

8 The line that passes through the points $A(0, 2)$ and $B(1, 4)$ meets the line that passes through the points $C(1, 8)$ and $D(-1, 10)$ at the point E. Find the equations of the lines AB and CD and hence find the coordinates of E.

9 $ABCD$ is a rhombus.



a Find the equation of:

i AC

ii BD

b Use the results of part **a** to find the coordinates of the point of intersection of AC and BD.

c Show that the point of intersection is the midpoint of both AC and BD and that AC is perpendicular to BD.

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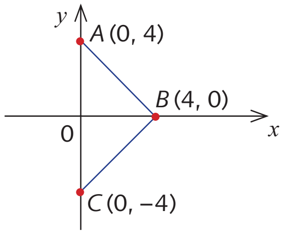
10 $A(0,4)$, $B(4,0)$ and $C(0,-4)$ are the vertices of triangle ABC.



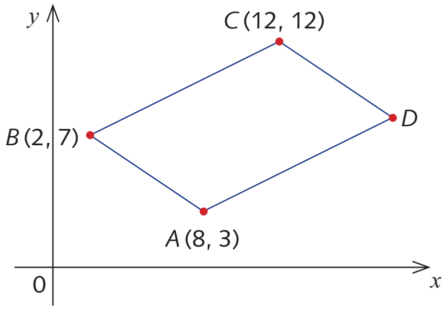
a Find the equation of the perpendicular bisector of interval:

i AB **ii** BC

b Find the coordinates of the intersection of the two perpendicular bisectors found in part **a**.

11 The line with equation $y = mx + 3$ intersects the line with equation $3x + 4y + 12 = 0$ at the point $\left(1, -\frac{15}{4}\right)$ Find the value of m.

12 The diagram opposite shows a parallelogram $ABCD$ in which A is the point with coordinates $(8,3)$, B is the point with coordinates $(2,7)$ and C is the point with coordinates $(12,12)$. X is a point on BC such that AX is perpendicular to BC. Find:



a the equation of the line AD

b the equation of the line AX

c the coordinates of X

d the distance AX

e the distance BC

f the area of the parallelogram

13 Find a and b if $ax - 10y = 8$ and $6x + by = 12$ represent the same line.

14 Show that the straight lines $2x - 3y = 7$, $3x - 4y = 13$ and $8x - 11y = 33$ meet at a point.

15 Find the equation of the straight line that passes through the origin and the point of intersection of the lines:

a $x - y - 4 = 0$ and $7x + y + 20 = 0$ **b** $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$

16 The line $ax - by + 3 = 0$ is parallel to the line $3x + 2y - 4 = 0$ and passes through the point $(1, -2)$. Find a and b.

17 Given three points, $A(0,5)$, $B(8,7)$ and $C(4,1)$, calculate the coordinates of the point of intersection of the perpendicular bisectors of the lines AB and BC.

18 In the quadrilateral $ABCD$, the points A, B and D are at $(3,3)$, $(0,1)$ and $(6,2)$, respectively. The line BD bisects the line AC at right-angles at the point M.

a Find the equation of BD and of AC.

b Calculate the coordinates of M.

c Calculate the length AM.

d Find the area of quadrilateral $ABCD$.

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4F Solving word problems using simultaneous equations

In this section, we look at how simultaneous equations can be used to solve problems expressed in words.

When solving a problem expressed in words:

- introduce pronumerals
- translate all the relevant facts into equations
- solve the equations and check your solutions
- write a conclusion in words.

Example 20

The attendance at an evening performance of a local theatre production was 420 people and the box office receipts were $3840. Admission costs were $13 for each adult and $4 for each child. How many of each type of ticket were sold?

Solution

Let c be the number of child tickets sold, and let a be the number of adult tickets sold.

$$\begin{aligned} c + a &= 420 \quad &(1) \\ 4c + 13a &= 3840 \quad &(2) \\ (1) \times 4: \quad 4c + 4a &= 1680 \quad &(3) \\ (2) - (3): \quad 9a &= 2160 & \\ a &= 240 & \end{aligned}$$

Substituting in equation (1) gives:

$$\begin{aligned} c + 240 &= 420 \\ c &= 180 \end{aligned}$$

Hence, 240 adult tickets and 180 child tickets were sold.

The above question could also be solved by using one variable. For example, if we let x be the number of children, then the number of adults is $420 - x$.

Exercise 4F

Solve each of these problems by introducing two pronumerals and forming a pair of simultaneous equations.

1 The sum of two numbers is 112 and their difference is 22. Find the two numbers.

2 In a game of netball, the winning team won by 9 goals. In total, 83 goals were scored in the game. How many goals did each team score?

3 A father is 28 years older than his daughter. In six years' time, he will be three times her age. Find their present ages.

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Example 20

4 A stallholder at a local market sells articles at either \$2 or \$5 each. On a particular market day, he sold 101 articles and took \$331 in revenue. How many articles were sold at each price?

5 Four times Brian's age exceeds Andrew's age by 20 years and one-third of Andrew's age is less than Brian's age by two years. Find their ages.

6 A ball of string of length 150 m is cut into 8 pieces of one length and 5 pieces of another length. The total length of three of the first 8 pieces exceeds that of two of the second 5 pieces by 2 m. Find the length of the pieces.

7 A manufacturer of lawn fertiliser produces bags of fertiliser in two sizes, standard and jumbo. To transport bags to retail outlets, he uses a van with a carrying capacity of one tonne. He discovers that he can transport either 110 standard bags and 60 jumbo bags or 50 standard bags and 100 jumbo bags at any one time. Find the weight of each type of bag.

8 Ten thousand tickets were sold for a concert. Some tickets sold for \$80 each and the remainder sold for \$60 each. If the total receipts were \$640 000, how many tickets of each price were sold?

9 The cooling system of Ennio's car contains 7.5 L of coolant, which is $33\frac{1}{3}\%$ antifreeze. How much of this solution must be drained from the system and replaced with 100% antifreeze so that the solution in the cooling system will contain 50% antifreeze?

10 A motorist travelled a total distance of 432 km and had an average speed of 80 km/h on highways and an average speed of 32 km/h while passing through towns. If the journey took 6 hours, find how long the motorist spent travelling on highways.

11 A car leaves Melbourne at 8 a.m., travelling at a constant speed of 80 km/h. It is followed at 10 a.m. by another car travelling on the same road at a constant speed of 110 km/h. At what time will the second car overtake the first?

12 One alloy of iron contains 52% iron and another contains 36% iron. How many tonnes of each alloy should be used to make 200 tonnes of 40% iron alloy?

13 Two aeroplanes pass each other in flight while travelling in opposite directions. Each aeroplane continues on its flight for 45 minutes, after which time the aeroplanes are 840 km apart. The speed of the first aeroplane is $\frac{3}{4}$ of the speed of the other aeroplane. Calculate the average speed of each aeroplane.

14 Six model horses and 7 model cows can be bought for \$250. Thirteen model cows and 11 model horses can be bought for \$460. What is the cost of each model animal?

15 If 1 is added to the numerator of a fraction $\frac{a}{b}$, it simplifies to $\frac{1}{5}$. If 1 is subtracted from the denominator, it simplifies to $\frac{1}{7}$. Find the fraction $\frac{a}{b}$.

16 A hiker walks a certain distance. If he had gone 1 km/h faster, he would have walked the distance in $\frac{4}{5}$ of the time. If he had walked 1 km/h slower, he would have taken $2\frac{1}{2}$ hours longer to travel the distance. Find the distance.

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Review exercise

1 Find the distance between the points A and B.

a $A(1, 6),\ B(3, -2)$ **b** $A(1, 5),\ B(7, 5)$ **c** $A(-1, 6),\ B(-1, -6)$

d $A(-2, -8),\ B(-1, -3)$ **e** $A(2, 7),\ B(-3, 10)$ **f** $A(-1, 6),\ B(7, 10)$

2 Find the midpoint of the interval AB.

a $A(1, 6),\ B(2, -4)$ **b** $A(2, 3),\ B(-4, 6)$ **c** $A(1, -10),\ B(-2, 10)$

d $A(-1, -3),\ B(10, 13)$ **e** $A(-2, 6),\ B(-1, 7)$ **f** $A(3, -4),\ B(6, -2)$

3 Find the gradient of the line that passes through each pair of points.

a $(1, 2)$ and $(5, 18)$ **b** $(2, 3)$ and $(4, 9)$ **c** $(-2, 1)$ and $(1, 10)$

d $(1, -2)$ and $(3, 0)$ **e** $(-3, -4)$ and $(0, -2)$ **f** $(-1, -2)$ and $(1, -7)$

g $(0, -6)$ and $(-2, 0)$ **h** $(3, 5)$ and $(7, 5)$ **i** $(6, -3)$ and $(2, -3)$

4 The line passing through the points $(-1, 6)$ and $(4, b)$ has gradient -2. Find the value of b.

5 Write down the gradient and y-intercept for each equation.

a $y = 2x + 4$ **b** $y = x - 4$ **c** $y = \frac{1}{2}x + 1$

d $y = -2x + 5$ **e** $y = -x + 6$ **f** $y = 2 - x$

g $y = 4 - 3x$ **h** $3x + y = 4$ **i** $2x - 3y = 6$

j $3x + 4y = 12$ **k** $y = 2x$ **l** $y = 3x$

m $y = -4x$ **n** $-3x + 2y = 0$ **o** $2y = -3x + 6$

6 Sketch the graph of each equation by finding the x- and y-intercepts.

a $x + y = 4$ **b** $2x + y = 2$ **c** $3x + 4y = 12$

d $2x - y = 4$ **e** $3x - 2y = 6$ **f** $\frac{3}{2}x - y = 12$

7 Sketch the graph of each equation.

a $y = 2x - 3$ **b** $y = 3x - 2$ **c** $2x - y = 1$

d $2x - 5y = 10$ **e** $x = 2y + 1$ **f** $x = 3y - 2$

g $y = 4 - x$ **h** $y = 1 - 3x$ **i** $x = 2$

j $x = -1$ **k** $y - 3 = 0$ **l** $y = -2$

m $y = 2(x + 1)$ **n** $x = \frac{y + 1}{3}$ **o** $\frac{x}{2} + \frac{y}{3} = 1$

p $\frac{x}{4} - y = 1$ **q** $\frac{2x}{3} - \frac{3y}{2} = 1$ **r** $\frac{1}{2}x - 2y = 3$

8 **a** Find the equation of the line with gradient -6 that passes through the point $(1, 5)$.

b Find the equation of the line that is perpendicular to the line with equation $3x + 2y = 8$, and that passes through the point $(-1, 4)$.

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9 Find the equation of the line that passes through the points:

a $(5, 6)$ and $(-4, 10)$ **b** $(3, 4)$ and $(-2, 8)$ **c** $(-2, 6)$ and $(1, 10)$

10 Solve each pair of simultaneous equations.

a $5x + 3y = 15$
$x - y = 6$

b $\frac{x}{3} + \frac{y}{5} = 1$
$3x + 5y = 15$

11 Solve each pair of simultaneous equations.

a $y = 3x + 2$
$y = x - 4$

b $y = 2 - 3x$
$y = 10 + x$

c $y = 5 - x$
$y = 10 - 2x$

d $3x - y = 2$
$y + 3x = 4$

e $\frac{x}{3} = 6 - \frac{y}{3}$
$\frac{3x}{4} - 3 = 2y$

f $2y - \frac{5x}{3} = -4$
$5x + \frac{y}{2} = \frac{21}{2}$

12 The vertices of ΔABC are $A(3, 4)$, $B(8, 10)$, $C(5, -1)$.

a Find the equation of the perpendicular bisector of:

i AB **ii** BC

b Find the coordinates of the point of intersection of the two perpendicular bisectors.

13 The equation of the perpendicular bisector of AB is $3y = 2x - 1$. The coordinates of A are $(1, 4)$. Find the coordinates of B.

14 Show that the points $(2, 0)$, $(5, 3)$, $(3, 6)$ and $(0, 3)$ are the vertices of a parallelogram. Find the equation of each of its sides.

15 Show that the points $(1, 4)$, $(-4, -1)$ and $(2, 3)$ are the vertices of a right-angled triangle.

16 **a** Prove that the points $(3, -2)$, $(7, 6)$, $(-1, 2)$ and $(-5, -6)$ are the vertices of a rhombus.

b Find the length of each of the diagonals of the rhombus.

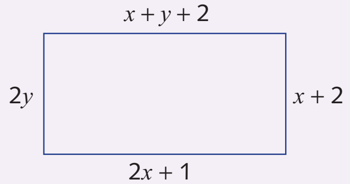
17 Find the two numbers whose sum is 138 and whose difference is 88.

18 Six stools and four chairs cost \$580 but five stools and two chairs cost \$350. Find the cost of each chair and each stool.

19 Three points have coordinates $A(1, 2)$, $B(3, 10)$ and $C(p, 8)$. Find the values of p if:

a A, B and C are collinear **b** AC is perpendicular to AB

20 Find the perimeter of the rectangle shown below.



21 Prove that the lines $2y - x = 2$, $y + x = 7$ and $y = 2x - 5$ are concurrent. (That is, they intersect at only one point.)

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Challenge exercise

1 Tom begins in Mildura and travels a distance of 300 km to Broken Hill at a constant speed of 80 km/h. Steve, beginning at the same time, travels at a constant speed of 100 km/h from Mildura to Broken Hill with a 30-minute rest after travelling 150 km.

a Let d be the distance (in km) from Mildura, and t the time (in hours) after Tom and Steve leave Mildura. On a single set of axes, draw graphs to illustrate the journeys of Tom and Steve (d against t).

b From the graphs, find:

i when and where Tom overtakes Steve

ii when and where Steve overtakes Tom

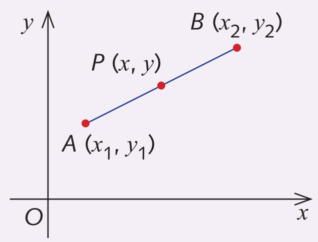
iii the distance Tom still has to travel to Broken Hill at the time Steve arrives at Broken Hill

2 For the interval AB, the coordinates of A and B are (x_1, y_1) and (x_2, y_2), respectively.

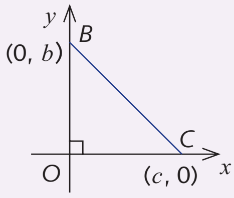
a If M is a point on AB such that $AM : MB = 3 : 1$, find the coordinates of M.

b If N is a point on AB such that $AN : NB = 3 : 2$, find the coordinates of N.

3 The point P divides the interval AB in the ratio $m : n$. That is $AP : PB = m : n$. Find the coordinates of P.



4 $O(0, 0)$, $B(0, b)$ and $C(c, 0)$ are the vertices of a right-angled triangle, with the right angle at O.

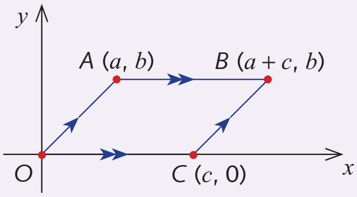


a Find the coordinates of the midpoint M of BC.

b Find the distances:

i OM **ii** MB **iii** MC

5 $OABC$ is a parallelogram.



a Find the equations of:

i OB **ii** AC

b Find the coordinates of the midpoints of:

i OB **ii** AC

Note that the diagonals of the parallelogram bisect each other.

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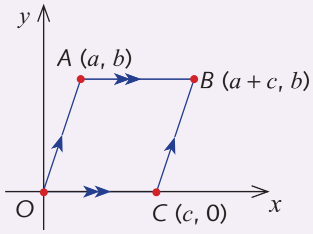
6 $OABC$ is a rhombus, with vertices $O(0, 0)$, $A(a, b)$, $B(a + c, b)$ and $C(c, 0)$.



a Find the gradients of the lines:

i OB **ii** AC

b Show that (gradient of OB) × (gradient of AC) $= \dfrac{b^2}{a^2 - c^2}$.

c Find the length of OA.

d Use the fact that $OA = OC$ to show that $c^2 = a^2 + b^2$, and hence that OB is perpendicular to AC.

7 AE, BF and CD are the medians of ΔABC. They are concurrent at the point X. We also have:

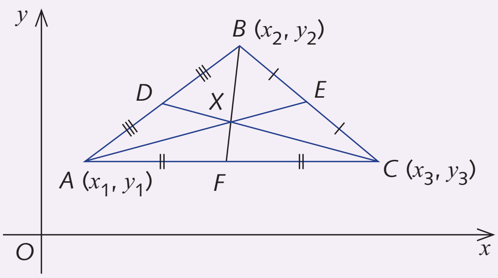
$BX = 2XF$

$CX = 2XD$

$AX = 2XE$

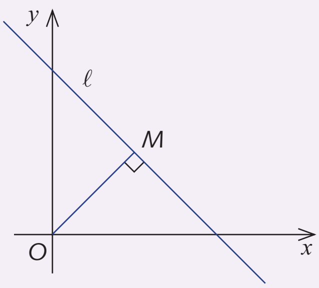
Show that X has coordinates:

$$\left(\frac{1}{3}(x_1 + x_2 + x_3), \frac{1}{3}(y_1 + y_2 + y_3)\right)$$



8 The line ℓ has equation $ax + by + c = 0$.

Show that $OM = \dfrac{c}{\sqrt{a^2 + b^2}}$.



9 $A(2, 6)$, $B(8, 11)$ and $C(4, 4)$ are the vertices of ΔABC.

Line BC intersects the x-axis at P.

Line CA intersects the x-axis at Q.

Line AB intersects the x-axis at R.

Show that $\dfrac{BP}{PC} \times \dfrac{CQ}{QA} \times \dfrac{AR}{RB} = 1$.

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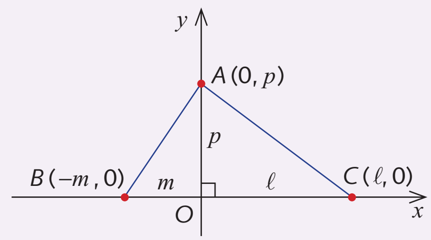
10 We shall prove that the altitudes of a triangle are concurrent.



For triangle ABC, we choose a set of axes with the origin O on BC so that BOA is a right-angle.

Let $OA = p$, $OB = m$ and $OC = \ell$, so that the coordinates of A, B and C are $(0, p)$, $(-m, 0)$ and $(\ell, 0)$.

a Find the gradient of lines AB and CA.

b Find the equation of the line that is perpendicular to AB and passes through C (the altitude from C to AB).

c Find the equation of the line that is perpendicular to AC and passes through B (the altitude from B to AC).

d Show that the three altitudes of the triangle intersect at $\left(0, \frac{m\ell}{p}\right)$. That is, the altitudes are concurrent.

11 **a** Show that the area of a triangle ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is $\pm\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$.

b Show that the area of a quadrilateral whose vertices taken in order are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ is

$\pm\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4)$,

where the sign is chosen to provide a positive answer.

12 Two lines have equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

a Show that the lines are parallel if $a_1b_2 = a_2b_1$.

b Show that the lines are perpendicular if $a_1a_2 + b_1b_2 = 0$.

13 **a** Show that the line passing through the point (x_1, y_1) and parallel to the line $ax + by + c = 0$ is $ax + by = ax_1 + by_1$.

b Show that the line passing through the point (x_1, y_1) and perpendicular to the line $ax + by + c = 0$ is $bx - ay = bx_1 - ay_1$.

14 Show that the three lines:

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$

are concurrent if $a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0$. The converse is always true.

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