CHAPTER

5

Number and Algebra

Quadratic equations

Quadratic equations turn up frequently in mathematics, and being able to solve them is a fundamental skill.

The ancient Babylonians were solving quadratic equations more than 5000 years ago!

In this chapter, we will revise and extend the basic methods of solving equations based on factorising, and then explore how to solve quadratic equations when the factorising method does not work. Techniques include completing the square and determining a general formula for the solution of quadratic equations.

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5A Solution of quadratic equations

Equations that can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$, are **quadratic equations**.

In *ICE-EM Mathematics Year 9*, we learned some of the methods of solving quadratic equations.

The method you learned used the following idea. If the product of two numbers is zero, then at least one of the numbers is zero.

In symbols, if $mn = 0$, then either $m = 0$ or $n = 0$ (or both).

To solve a quadratic equation $ax^2 + bx + c = 0$, you should first attempt to factorise the quadratic expression on the left to express it as a product of two factors and then use the above idea.

Example 1

Solve each equation.

a $x^2 - 6x = 0$ **b** $x^2 - 5x + 6 = 0$

Solution

a $x^2 - 6x = 0$

$x(x - 6) = 0$

Hence, $x = 0$ or $x - 6 = 0$

so $x = 0$ or $x = 6$

b $x^2 - 5x + 6 = 0$ (Look for two numbers with a product that is 6 and that sum to -5.)

$(x - 2)(x - 3) = 0$

Hence, $x - 2 = 0$ or $x - 3 = 0$

So $x = 2$ or $x = 3$

We can check by substitution that the two numbers obtained are solutions to the original equation. For example, in Example **1b**:

If $x = 2$:

$$\begin{aligned}\text{LHS} &= x^2 - 5x + 6\\ &= 2^2 - 5 \times 2 + 6\\ &= 0\\ &= \text{RHS}\end{aligned}$$

If $x = 3$:

$$\begin{aligned}\text{LHS} &= x^2 - 5x + 6\\ &= 3^2 - 5 \times 3 + 6\\ &= 0\\ &= \text{RHS}\end{aligned}$$

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Example 2

Solve:

a $x^2 - 16 = 0$ **b** $x^2 + 16 = 0$

Solution

a There are two ways we could do this. The simpler way is to write:

$$x^2 = 16$$
$$x = 4 \quad \text{or} \quad x = -4$$

[Alternatively, we could factorise using the difference of two squares identity.

$$x^2 - 16 = 0$$
$$(x - 4)(x + 4) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = 4 \quad \text{or} \quad x = -4$$
]

b We write the equation as $x^2 = -16$. There is no solution, since the square of any real number is positive or zero. This equation has no solution.

Note: $x^2 + 16$ cannot be factorised.

Always remember to move all of the terms onto the left-hand side of the equation when you want to solve a quadratic equation by factorising.

Example 3

Solve:

a $x^2 = 17x$ **b** $x^2 = 7x - 6$

Solution

a

$$x^2 = 17x$$
$$x^2 - 17x = 0$$
$$x(x - 17) = 0$$
$$x = 0 \quad \text{or} \quad x - 17 = 0$$
$$x = 0 \quad \text{or} \quad x = 17$$

b

$$x^2 = 7x - 6$$
$$x^2 - 7x + 6 = 0$$
$$(x - 6)(x - 1) = 0$$
$$x - 6 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = 6 \quad \text{or} \quad x = 1$$

Note: A very common mistake is to 'cancel out the x' in the first line of part **a** above and obtain $x = 17$. You should never do this – you must *always* factorise. Otherwise, you will lose the solution $x = 0$.

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Quadratic equations

- If $mn = 0$, then $m = 0$ or $n = 0$ (or both).
- To solve a quadratic equation using the factorising method, move all terms to the left-hand side, factorise and use the result stated above.
- If a pronumeral is a common factor, never divide by it – instead, always factorise.

Factorising general quadratic expressions

We will review a method of factorising general quadratic expressions when the coefficients do not have a common factor. There are a number of such methods, but we will only give one method here.

To factorise, for example, $3x^2 + 11x + 6$, we go through the following steps.

First, we multiply the coefficient of x^2 by the constant term.

$$3 \times 6 = 18$$

Next, we find two numbers with product 18 and sum 11, the coefficient of x. The numbers are 9 and 2. Using these numbers:

$$\begin{aligned} 3x^2 + 11x + 6 &= 3x^2 + 9x + 2x + 6 && \text{(Split the } 11x \text{ term into } 9x + 2x.) \\ &= 3x(x + 3) + 2(x + 3) && \text{(Factorise in pairs.)} \\ &= (x + 3)(3x + 2) && \text{(Take out the common factor, } (x + 3).) \end{aligned}$$

It doesn't matter if we split $11x$ as $2x + 9x$ instead.

Thus, $3x^2 + 11x + 6 = (x + 3)(3x + 2)$.

This is the method presented in Section 3G of this book.

Example 4

Solve:

a $3x^2 + 11x + 6 = 0$ **b** $6x^2 + 7x + 2 = 0$

Solution

a

$$\begin{aligned} 3x^2 + 11x + 6 &= 0 \\ (x + 3)(3x + 2) &= 0 && \text{(Using the factorisation shown above.)} \\ x + 3 = 0 \quad &\text{or} \quad 3x + 2 = 0 \\ x = -3 \quad &\text{or} \quad x = -\frac{2}{3} \end{aligned}$$

b

$$\begin{aligned} 6x^2 + 7x + 2 &= 0 && \text{(Find two numbers that multiply to give } 6 \times 2 = 12 \\ & && \text{and add to give 7. They are 4 and 3.)} \\ 6x^2 + 4x + 3x + 2 &= 0 && \text{(Split the middle term.)} \\ 2x(3x + 2) + 1(3x + 2) &= 0 && \text{(Factorise in pairs.)} \\ (3x + 2)(2x + 1) &= 0 && \text{(Take out the common factor.)} \\ 3x + 2 = 0 \quad &\text{or} \quad 2x + 1 = 0 \\ x = -\frac{2}{3} \quad &\text{or} \quad x = -\frac{1}{2} \end{aligned}$$

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Note: It does not matter in which order we split the middle term. In the example on the previous page, we could write $4x + 3x$ or $3x + 4x$, and factorise in pairs. Try it for yourself!

Common factor

If there is a factor common to all of the coefficients in the equation, we can divide both sides by this common factor.

Example 5

Solve:

a $10x^2 - 40x - 210 = 0$ **b** $24x^2 = 46x - 10$

Solution

a
$$10x^2 - 40x - 210 = 0$$
$$x^2 - 4x - 21 = 0 \quad \text{(Divide both sides of the equation by 10.)}$$
$$(x - 7)(x + 3) = 0$$
$$x - 7 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 7 \quad \text{or} \quad x = -3$$

b
$$24x^2 = 46x - 10 \quad \text{(Divide both sides of the equation by 2}$$
$$12x^2 - 23x + 5 = 0 \quad \text{and rearrange.)}$$
$$12x^2 - 20x - 3x + 5 = 0 \quad (12 \times 5 = 60. \text{ Find two numbers with a}$$
$$4x(3x - 5) - 1(3x - 5) = 0 \quad \text{product that is 60 and sum that is } -23.$$
$$(3x - 5)(4x - 1) = 0 \quad \text{The numbers are } -20 \text{ and } -3.)$$
$$3x - 5 = 0 \quad \text{or} \quad 4x - 1 = 0$$
$$x = \frac{5}{3} \quad \text{or} \quad x = \frac{1}{4}$$

Quadratic equations of the form $ax^2 + bx + c = 0$, when $a \neq 0$

- If the coefficients have a common factor, divide through by that factor.
- To factorise a quadratic expression such as $ax^2 + bx + c$, find two numbers, α and β, whose product is ac and whose sum is b. Write the middle term as $\alpha x + \beta x$ and then factorise.
- To solve a quadratic equation using the factorising method, write the equation in the form $ax^2 + bx + c = 0$, then factorise the quadratic and write down the solutions.

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Exercise 5A

1 Solve each equation.

a $x(x+3)=0$ **b** $x(x-7)=0$ **c** $3x(x+5)=0$

d $(x-3)(x+6)=0$ **e** $(x+7)(x+9)=0$ **f** $(x-10)(x-7)=0$

g $4x(5x+4)=0$ **h** $(4x+3)(3x-2)=0$ **i** $(2x+7)(x+4)=0$

j $(2x-3)(3x+4)=0$ **k** $3(2x-5)(x+4)=0$ **l** $7(2-3x)(4-3x)=0$

Example 1a

2 Solve each quadratic equation by factorising.

a $x^2-5x=0$ **b** $x^2+7x=0$ **c** $x^2+8x=0$

d $x^2-25x=0$ **e** $x^2=-18x$ **f** $x^2=\frac{1}{2}x$

Example 1b

3 Solve each quadratic equation by factorising.

a $x^2+9x+8=0$ **b** $x^2+8x+12=0$ **c** $x^2+12x+27=0$

d $x^2+12x+36=0$ **e** $x^2-6x+8=0$ **f** $x^2+x-6=0$

g $x^2+x-30=0$ **h** $x^2+3x-40=0$ **i** $x^2+4x-60=0$

j $x^2-7x+6=0$ **k** $x^2-7x+12=0$ **l** $x^2-10x+25=0$

m $x^2-18x+32=0$ **n** $x^2-4x-21=0$ **o** $x^2-20x+100=0$

Example 2, 3

4 Solve, if possible:

a $x^2=8x$ **b** $x^2=17x-16$ **c** $3x-x^2-2=0$

d $x^2+4=0$ **e** $15=8x-x^2$ **f** $-100-x^2=0$

g $x^2=-3x$ **h** $h^2=20-h$ **i** $x^2+9=0$

j $9a-10=-a^2$ **k** $8y=y^2+7$ **l** $a^2-1=0$

Example 5a

5 Solve each equation by first dividing both sides by a common factor.

a $2x^2+6x+4=0$ **b** $3a^2-15a+18=0$ **c** $4x^2+8x-140=0$

Example 4

6 Solve:

a $2x^2+11x+12=0$ **b** $3x^2+13x+4=0$ **c** $2x^2+7x+6=0$

d $2x^2-3x-2=0$ **e** $2x^2-9x+9=0$ **f** $3x^2-10x+8=0$

g $10x^2+23x+12=0$ **h** $6x^2-17x+12=0$ **i** $8x^2=6x+5$

j $12x^2=x+6$ **k** $12x^2=5x+2$ **l** $6x^2+11x=10$

m $3x^2=19x+14$ **n** $5x^2+17x+6=0$ **o** $12x^2-31x-15=0$

p $15x^2+224x=15$ **q** $72x^2-145x+72=0$ **r** $6+5x-6x^2=0$

Example 5b

7 Solve each equation, remembering first to divide both sides by any common factor.

a $12x^2-22x+8=0$ **b** $72x^2-78x-15=0$ **c** $12x^2-21x+9=0$

d $10x^2+5x-30=0$ **e** $72x^2-228x+120=0$ **f** $90x^2=75x+60$

g $100x^2-290x+100=0$ **h** $160x^2+136x+24=0$ **i** $10x^2-25x+10=0$

j $28x^2-49x-105=0$ **k** $42m^2-2m-4=0$ **l** $8x^2+46x-70=0$

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5B Rearranging to standard form

In many mathematical problems and applications, equations arise that do not initially appear to be quadratic equations. We often need to rearrange these equations to the standard form for a quadratic equation.

Some equations involve fractions in which the pronumeral may appear in the denominator. You will need to take care when solving these. We always assume that the pronumeral cannot take a value that makes the denominator equal to zero. It is a wise idea to check that your answers are the correct solutions to the initially given equation.

Example 6

Solve:

a $1 + \frac{5}{x} = \frac{6}{x^2}$ **b** $x = \frac{5x - 4}{x}$

Solution

a

$$\begin{aligned} 1 + \frac{5}{x} &= \frac{6}{x^2} \\ x^2 + 5x &= 6 \quad \text{(Multiply both sides of the equation by } x^2\text{.)} \\ x^2 + 5x - 6 &= 0 \\ (x + 6)(x - 1) &= 0 \\ x &= -6 \quad \text{or} \quad x = 1 \end{aligned}$$

We can check that these are the correct solutions by substitution.

If $x = -6$:

$$\text{LHS} = 1 - \frac{5}{6} = \frac{1}{6} \quad \text{and} \quad \text{RHS} = \frac{6}{36} = \frac{1}{6}$$

so LHS = RHS

If $x = 1$:

$$\text{LHS} = 1 + \frac{5}{1} = 6 \quad \text{and} \quad \text{RHS} = \frac{6}{1} = 6$$

so LHS = RHS

b

$$\begin{aligned} x &= \frac{5x - 4}{x} \\ x^2 &= 5x - 4 \quad \text{(Multiply both sides by } x\text{.)} \\ x^2 - 5x + 4 &= 0 \quad \text{(Rearrange.)} \\ (x - 1)(x - 4) &= 0 \\ x - 1 &= 0 \quad \text{or} \quad x - 4 = 0 \\ x &= 1 \quad \text{or} \quad x = 4 \end{aligned}$$

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Another standard technique in algebra that is useful in the solution of equations is **cross-multiplication**. This is using the result:

$$\frac{a}{b} = \frac{c}{d} \text{ is equivalent to } ad = bc$$

Example 7

Solve $\dfrac{x-2}{3} = \dfrac{5}{x}$.

Solution

$$\begin{aligned}
\frac{x-2}{3} &= \frac{5}{x} \\
x(x-2) &= 3 \times 5 && \text{(Multiply by } 3x.) \\
x^2 - 2x &= 15 \\
x^2 - 2x - 15 &= 0 && \text{(Rearrange.)} \\
(x+3)(x-5) &= 0 \\
x + 3 = 0 \quad &\text{or} \quad x - 5 = 0 \\
x = -3 \quad &\text{or} \quad x = 5
\end{aligned}$$

Example 8

Solve $\dfrac{x+1}{x-1} - \dfrac{3}{x+2} = 1$.

Solution

$$\begin{aligned}
\frac{x+1}{x-1} - \frac{3}{x+2} &= 1 \\
(x-1)(x+2)\left[\frac{x+1}{x-1} - \frac{3}{x+2}\right] &= (x-1)(x+2) && \text{(Multiply by } (x-1)(x+2).) \\
(x+1)(x+2) - 3(x-1) &= (x-1)(x+2) \\
x^2 + 3x + 2 - 3x + 3 &= x^2 + x - 2 \\
-x + 5 &= -2 \\
-x &= -7 \\
x &= 7
\end{aligned}$$

Check solution:

When $x = 7$, LHS $= \dfrac{8}{6} - \dfrac{3}{9} = \dfrac{4}{3} - \dfrac{1}{3} = 1$ and RHS $= 1$.

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Exercise 5B

1 Solve:

a $x(x-7)=18$ **b** $x^2=4(x+8)$ **c** $x^2=\frac{1}{2}(5x+12)$

d $5(x^2+5)=6x^2$ **e** $3x^2=4(x^2+4)$ **f** $(x+1)(x-1)=2(x+1)^2$

g $x(x-3)=2x(x+1)$ **h** $(x-4)(x-2)=3$ **i** $(9+x)(9-x)=17$

j $(2x-1)(3x+1)=11$ **k** $5x(2x-3)+7(2x-3)=0$ **l** $3x-8=\frac{x^2}{4}$

Example 6, 7, 8

2 Solve:

a $x+5=\frac{14}{x}$ **b** $\frac{15}{x}=x-2$ **c** $\frac{6}{x}-x=1$

d $x-1=\frac{2}{x}$ **e** $x+\frac{6}{x}=7$ **f** $x+\frac{32}{x}=18$

g $\frac{x+1}{3}=\frac{10}{x}$ **h** $\frac{x+1}{4}=\frac{5}{x}$ **i** $x+\frac{2}{x}=-\frac{9}{2}$

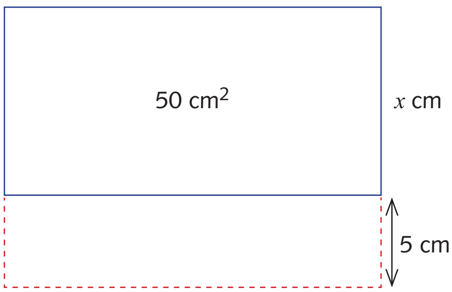
j $\frac{2}{2x-3}=\frac{x}{4x-6}$ **k** $\frac{1}{x-1}-\frac{1}{x+3}=\frac{1}{35}$ **l** $\frac{4}{x-1}-\frac{5}{x+2}=\frac{3}{x}$

m $6(4x+5)+\frac{7}{x}(4x+5)=0$ **n** $\frac{2}{x+3}+\frac{x+3}{2}=\frac{10}{3}$

3 The rectangle on the right has area 50 cm^2.

a The width is x cm. Find the length of the rectangle in terms of x.

b The rectangle is extended by 5 cm to form a square. Form a quadratic equation and find x.



5C Applications of quadratic equations

When we apply mathematics to real-world problems, we often obtain equations to solve. In many cases, these equations are quadratic equations. It is extremely important to keep in mind that some of the solutions we obtain to the equations may *not* be solutions to the real-world problem. For example, a quadratic equation may yield negative or fractional solutions, which may not make sense as answers to the original problem.

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Example 9

The formula for the number of diagonals of a polygon with n sides is $D = \frac{n}{2}(n - 3)$.

How many sides are there in a polygon with 35 diagonals?

Solution

$$D = 35, \text{ so } \frac{n}{2}(n-3) = 35$$

$$n(n-3) = 70 \qquad \text{(Multiply both sides by 2.)}$$

$$n^2 - 3n - 70 = 0$$

$$(n-10)(n+7) = 0$$

$$n - 10 = 0 \quad \text{or} \quad n + 7 = 0$$

$$n = 10 \quad \text{or} \quad n = -7$$

The value $n = -7$ does not make sense in this problem.

Hence, the polygon has 10 sides.

Example 10

A rectangle has one side 3 cm longer than the other. The rectangle has area 54 cm^2. What is the length of the shorter side?

Solution

Let x cm be the length of the shorter side. The other side has length $(x + 3)$ cm.

$$\text{Area} = x(x+3) = 54 \text{ cm}^2$$

$$x^2 + 3x - 54 = 0$$

$$(x-6)(x+9) = 0$$

$$x - 6 = 0 \quad \text{or} \quad x + 9 = 0$$

$$x = 6 \quad \text{or} \quad x = -9$$

Since length must be positive, the solution to the problem is $x = 6$.

Hence, the shorter side has length 6 cm.

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Exercise 5C

Use quadratic equations to solve each problem. Clearly define any pronumerals introduced into your solution.

Example 9

1 The formula for the number of diagonals of a polygon with n sides is $D = \frac{n}{2}(n - 3)$. How many sides does a polygon with 44 diagonals have?

Example 10

2 The length of a rectangle is 4 cm greater than its width. If its area is 96 cm^2, find the length of the rectangle.

3 The sum S of the first n positive integers (that is, $1 + 2 + 3 + \ldots + n$) is given by $S = \frac{n}{2}(n + 1)$. What value of n gives a sum of 136 ?

4 A number is squared and then doubled. The result is 45 more than the original number. What is the original number?

5 The difference of two numbers is 16 and the sum of their squares is 130. Find the two numbers.

6 A triangle has base length 4 cm greater than its height. If the area of the triangle is 48 cm^2, find the height of the triangle.

7 A piece of sheet metal measuring 50 cm × 40 cm has squares cut out of each corner so that it can be bent and formed into an open box (with no lid) with a base area of 1344 cm^2. Find the dimensions of the box.

8 Find two numbers such that the sum of their squares is 74 and their sum is 12.

9 A man travels 108 km at a constant speed and finds that the journey would have taken $4\frac{1}{2}$ hours less if he had travelled at a speed 2 km/h faster. What was his speed?

10 The perimeter of a rectangle is 40 cm and its area is 84 cm^2.

a If the width of the rectangle is x cm, express the length of the rectangle in terms of x.

b Find the length and width of the rectangle.

11 A rectangular swimming pool 12 m by 8 m is surrounded by a concrete path of uniform width. If the area of the path is 224 m^2, find the path's width.

12 In a right-angled triangle, one of the sides adjacent to the right angle is 4 cm longer than the other side. The area of the triangle is 48 cm^2. Find the length of each of the three sides.

13 A train travels 300 km at a constant speed. If the speed had been 5 km/h faster, the journey would have taken 2 hours less. Find the speed of the train.

14 One of the parallel sides of a trapezium is 5 cm longer than the other, and its height is half the length of the shorter parallel side. If the area is 225 cm^2, find the lengths of the parallel sides.

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5D Perfect squares and completing the square

Perfect squares

In most of the examples we have looked at so far, the quadratic equations had two solutions.

If the quadratic expression is a perfect square, then there is only one solution to the equation.

Example 11

Solve:

a $x^2 - 6x + 9 = 0$ **b** $9x^2 - 12x + 4 = 0$

Solution

a
$$\begin{aligned} x^2 - 6x + 9 &= 0 \\ (x-3)(x-3) &= 0 \\ (x-3)^2 &= 0 \\ x &= 3 \end{aligned}$$

b
$$\begin{aligned} 9x^2 - 12x + 4 &= 0 \\ 9x^2 - 6x - 6x + 4 &= 0 \\ 3x(3x-2) - 2(3x-2) &= 0 \\ (3x-2)(3x-2) &= 0 \\ (3x-2)^2 &= 0 \\ x &= \frac{2}{3} \end{aligned}$$

($9 \times 4 = 36$. Factors of 36 that sum to -12 are -6 and -6.)

Note: perfect squares can also be factored 'on inspection' using the identities $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$.

Completing the square

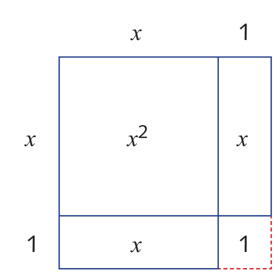
What number must be added to $x^2 + 6x$ to make a perfect square?

It is 9, which is the square of half of the coefficient of x, because $x^2 + 6x + 9 = (x+3)^2$.

The process of completing the square is an important technique that has many important applications. This section is a basic introduction to this technique.

The key step in a monic expression is to take half the coefficient of x and square it.

Now consider the quadratic expression $x^2 + 2x - 6$. Focus on $x^2 + 2x$.
(In the diagram, a 1×1 square must be added to 'complete the square'.)



We say that the related perfect square is $x^2 + 2x + 1$.

$$\begin{aligned} x^2 + 2x - 6 &= x^2 + 2x + 1 - 1 - 6 \quad \text{(Add and subtract 1.)} \\ &= (x^2 + 2x + 1) - 7 \\ &= (x+1)^2 - 7 \end{aligned}$$

This process is called **completing the square**.

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Example 12

a What number must we add to $x^2 - 12x$ to produce a perfect square?

b What number must we add to $x^2 + 3x$ to produce a perfect square?

Solution

a Half the coefficient of x is -6. Its square is 36, so $x^2 - 12x + 36 = (x - 6)^2$.

Hence, 36 must be added to produce a perfect square.

b Half the coefficient of x is $\frac{3}{2}$. Its square is $\frac{9}{4}$.

So $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$.

Hence, $\frac{9}{4}$ must be added to produce a perfect square.

Perfect squares and completing the square

- If the quadratic expression is a **perfect square**, the corresponding quadratic equation has only one solution.
- To **complete the square** for the expression $x^2 + bx$, take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square, $\left(\frac{b}{2}\right)^2$.

Example 13

Complete the square.

a $x^2 + 6x + 8$

b $x^2 + 3x - 5$

Solution

a
$$\begin{aligned} x^2 + 6x + 8 &= (x^2 + 6x + 9) - 9 + 8 \\ &= (x + 3)^2 - 1 \end{aligned}$$

b
$$\begin{aligned} x^2 + 3x - 5 &= \left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} - 5 \\ &= \left(x + \frac{3}{2}\right)^2 - \frac{29}{4} \end{aligned}$$

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Completing the square for non-monic expressions

When the co-efficient of x^2 is not 1, then this value needs to be factored out of the quadratic expression before the process of completing the square can be applied. The final step is to multiply both the perfect square and the constant term by this factored out value.

Example 14

Complete the square.

a $-x^2 + 2x + 5$ **b** $3x^2 + 12x - 1$ **c** $2x^2 - 5x + 1$

Solution

a $-x^2 + 2x + 5$

$= -(x^2 - 2x - 5)$

$= -[(x^2 - 2x + 1) - 1 - 5]$

$= -[(x - 1)^2 - 6]$

$= -(x - 1)^2 + 6$

b $3x^2 + 12x - 1$

$= 3(x^2 + 4x - \frac{1}{3})$

$= 3\left[\left(x^2 + 4x + 4\right) - 4 - \frac{1}{3}\right]$

$= 3\left[(x + 2)^2 - \frac{13}{3}\right]$

$= 3(x + 2)^2 - 13$

c $2x^2 + 5x + 1$

$= 2\left(x^2 + \frac{5x}{2} + \frac{1}{2}\right)$

$= 2\left[\left(x^2 + \frac{5x}{2} + \frac{25}{16}\right) - \frac{25}{16} + \frac{1}{2}\right]$

$= 2\left[\left(x + \frac{5}{4}\right)^2 - \frac{17}{16}\right]$

$= 2\left(x + \frac{5}{4}\right)^2 - \frac{17}{8}$

Exercise 5D

Example 11

1 Solve:

a $x^2 + 2x + 1 = 0$ **b** $x^2 + 4x + 4 = 0$ **c** $x^2 + 8x + 16 = 0$

d $x^2 - 10x + 25 = 0$ **e** $x^2 + x + \frac{1}{4} = 0$ **f** $x^2 - 3x + \frac{9}{4} = 0$

g $x^2 - \frac{4}{5}x + \frac{4}{25} = 0$ **h** $x^2 + \frac{3}{2}x + \frac{9}{16} = 0$ **i** $9x^2 - 6x + 1 = 0$

j $25x^2 + 10x + 1 = 0$ **k** $25x^2 - 20x + 4 = 0$ **l** $49x^2 + 28x + 4 = 0$

m $49x^2 - 70x + 25 = 0$ **n** $9x^2 + 30x + 25 = 0$

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2 Which of these expressions is:

a the result of a perfect square expansion?

b a difference of squares expansion?

i	$x^2 + 8x + 16$	ii	$x^2 - 16$	iii	$2x^2 + 3x + 1$
iv	$9 - y^2$	v	$25 - 10x + x^2$	vi	$x^2 + 4x + 1$
vii	$4x^2 - 25$	viii	$x^2 + 9$	ix	$4x^2 + 12x + 9$
x	$64 - 49a^2$	xi	$b^2 - 6b + 8$	xii	$36a^2 - 49b^2$

Example 12

3 What must be added to each expression to make it a perfect square?

a $x^2 + 4x$ b $x^2 + 8x$ c $x^2 - 10x$

d $x^2 - 12x$ e $x^2 + 20x$ f $x^2 + 3x$

g $x^2 + x$ h $x^2 - 7x$ i $x^2 - 11x$

Example 13

4 Complete the square:

a $x^2 + 6x + 10$ b $x^2 + 8x - 5$ c $x^2 + 12x - 10$

d $x^2 - 10x + 6$ e $x^2 - 6x - 8$ f $x^2 - 20x + 5$

g $x^2 + 3x - 2$ h $x^2 + x + 1$ i $x^2 - 5x + 6$

j $x^2 - x - 10$ k $x^2 + 3x + 7$ l $x^2 - 11x + 1$

Example 14

5 Complete the square:

a $3x^2 + 6x + 12$ b $5x^2 + 30x + 10$ c $3x^2 - 12x + 15$

d $-x^2 - 2x + 4$ e $-x^2 + 8x - 10$ f $4 - 6x - x^2$

g $3x^2 - 6x - 1$ h $2x^2 - 12x + 33$ i $4x^2 - 48x + 99$

j $2x^2 + 3x + 2$ k $4x^2 - x - 4$ l $3x^2 - 8x + 9$

m $5x^2 - x + 1$ n $2x^2 - 5x - 7$ o $4 - x - 3x^2$

5E Solving quadratic equations by completing the square

In all our examples so far, the quadratic expression factorised nicely and gave us integer or rational solutions. This is not always the case. For example, $x^2 - 7 = 0$ has solutions $x = \sqrt{7}$ and $x = -\sqrt{7}$.

Quadratic equations with integer coefficients

Quadratic equations with integer coefficients can have:

- integer or rational solutions, for example, $x^2 - 1 = 0$
- solutions involving surds, for example, $x^2 - 7 = 0$
- no solution, for example, $x^2 + 1 = 0$.

The method of completing the square enables us to deal with all quadratic equations.

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Historically, quadratic equations were solved by completing the square. This method always works, even when we cannot easily factorise the quadratic expression. A typical example is $x^2 + 2x - 9 = 0$.

Here are the steps for solving the quadratic $x^2 + 2x - 9 = 0$.

We first complete the square on the left-hand side. Half the coefficient of x is 1; its square is 1.

$(x^2 + 2x + 1) - 1 - 9 = 0$ (Add and subtract the square of half the coefficient of x.)

$$(x + 1)^2 - 10 = 0$$

$$(x + 1)^2 = 10$$

$x + 1 = \sqrt{10}$ or $x + 1 = -\sqrt{10}$

Finally, $x = -1 + \sqrt{10}$ or $x = -1 - \sqrt{10}$

These two numbers are the solutions to the original equation. Note that checking by substitution is hard. It is more efficient to check each step in the calculation.

Example 15

Solve $x^2 - 6x - 2 = 0$.

Solution

Method 1

$$x^2 - 6x - 2 = 0$$

$(x^2 - 6x + 9) - 9 - 2 = 0$ (Complete the square.)

$$(x - 3)^2 = 11$$

$x - 3 = \sqrt{11}$ or $x - 3 = -\sqrt{11}$

Hence, $x = 3 + \sqrt{11}$ or $x = 3 - \sqrt{11}$.

Method 2

$$x^2 - 6x - 2 = 0$$

$$x^2 - 6x = 2$$

$$x^2 - 6x + 9 = 2 + 9$$

$$(x - 3)^2 = 11$$

$x - 3 = \sqrt{11}$ or $x - 3 = -\sqrt{11}$

Hence, $x = 3 + \sqrt{11}$ or $x = 3 - \sqrt{11}$.

Method 1 and Method 2 are essentially the same. Adding and subtracting a number on one side of an equation has the same effect as adding that number to both sides of the equation. When solving quadratic equations, we will generally use Method 2.

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Example 16

Solve:

a $x^2 + 8x + 6 = 0$ **b** $x^2 - 7x - 3 = 0$

Solution

a
$$x^2 + 8x + 6 = 0$$
$$x^2 + 8x = -6$$
$$x^2 + 8x + 16 = -6 + 16$$
$$(x + 4)^2 = 10$$
$$x + 4 = \sqrt{10} \quad \text{or} \quad x + 4 = -\sqrt{10}$$
Hence, $x = -4 + \sqrt{10}$ or $x = -4 - \sqrt{10}$.

b
$$x^2 - 7x - 3 = 0$$
$$x^2 - 7x = 3$$
$$x^2 - 7x + \frac{49}{4} = 3 + \frac{49}{4} \quad \text{(Complete the square.)}$$
$$\left(x - \frac{7}{2}\right)^2 = \frac{61}{4}$$
$$x - \frac{7}{2} = \frac{\sqrt{61}}{2} \quad \text{or} \quad x - \frac{7}{2} = -\frac{\sqrt{61}}{2}$$
$$x = \frac{7 + \sqrt{61}}{2} \quad \text{or} \quad x = \frac{7 - \sqrt{61}}{2}$$

Example 17

Solve $3x^2 + 5x - 1 = 0$.

Solution

$$3x^2 + 5x - 1 = 0$$
$$3x^2 + 5x = 1$$
$$x^2 + \frac{5x}{3} = \frac{1}{3} \quad \text{(Divide all terms by the coefficient of } x^2\text{.)}$$
$$x^2 + \frac{5x}{3} + \frac{25}{36} = \frac{1}{3} + \frac{25}{36}$$
$$\left(x + \frac{5}{6}\right)^2 = \frac{37}{36}$$
$$x + \frac{5}{6} = \frac{\sqrt{37}}{6} \quad \text{or} \quad x + \frac{5}{6} = -\frac{\sqrt{37}}{6} \quad \left(\sqrt{\frac{37}{36}} = \frac{\sqrt{37}}{6}\right)$$
$$x = \frac{-5 + \sqrt{37}}{6} \quad \text{or} \quad x = \frac{-5 - \sqrt{37}}{6}$$

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There are quadratic equations that cannot be solved. Consider, for example, $x^2 - 6x + 12 = 0$.

$$\begin{aligned} x^2 - 6x + 12 &= 0 \\ x^2 - 6x + 9 - 9 + 12 &= 0 \\ (x-3)^2 + 3 &= 0 \\ (x-3)^2 &= -3 \end{aligned}$$

Since $(x-3)^2 \geq 0$ for all values of x, there is no solution to the equation $(x-3)^2 = -3$.

Solution of quadratic equations by completing the square

- To solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square, we:
  - move the constant, c, to the right-hand side
  - divide all terms by the coefficient of x^2, a
  - add the square of half the coefficient of x, $\left(\frac{b}{2a}\right)^2$, to both sides of the equation
  - solve for x.
- Fractions and square roots often occur in this procedure.
- We can also show that a quadratic equation has no solution using this procedure.

Exercise 5E

1 Solve:

a $x^2 - 5 = 0$ **b** $x^2 - 11 = 0$ **c** $2x^2 - 6 = 0$

d $4x^2 - 8 = 0$ **e** $50 - 5x^2 = 0$ **f** $40 - 8x^2 = 0$

Example 15, 16a

2 Solve each equation by completing the square.

a $x^2 + 2x - 1 = 0$ **b** $x^2 + 4x + 1 = 0$ **c** $x^2 - 12x + 23 = 0$

d $x^2 + 6x + 7 = 0$ **e** $x^2 - 8x - 1 = 0$ **f** $x^2 + 8x - 4 = 0$

g $x^2 + 10x + 1 = 0$ **h** $x^2 + 12x - 5 = 0$ **i** $x^2 - 10x - 50 = 0$

j $x^2 + 20x + 5 = 0$ **k** $x^2 - 100x - 80 = 0$ **l** $x^2 - 50x + 10 = 0$

Example 16b

3 Solve each equation by completing the square.

a $x^2 + x - 1 = 0$ **b** $x^2 - 3x + 1 = 0$ **c** $x^2 - 5x - 1 = 0$

d $x^2 + 3x - 2 = 0$ **e** $x^2 + 5x - 4 = 0$ **f** $x^2 - 3x - 5 = 0$

g $x^2 - 7x - 100 = 0$ **h** $x^2 - 3x - 6 = 0$ **i** $x^2 - 9x - 5 = 0$

j $x^2 - x - 5 = 0$ **k** $x^2 - 3x + 1 = 0$ **l** $x^2 - 5x + 3 = 0$

Example 17

4 Solve each equation by completing the square.

a $3x^2 - 12x + 3 = 0$ **b** $3x^2 + 6x - 12 = 0$ **c** $-x^2 - 2x + 4 = 0$

d $-x^2 + 8x - 10 = 0$ **e** $-x^2 - 6x + 12 = 0$ **f** $3x^2 + 24x - 12 = 0$

g $2x^2 - 3x - 2 = 0$ **h** $3x^2 - 8x - 6 = 0$ **i** $4x^2 - x - 4 = 0$

j $5x^2 + x - 1 = 0$ **k** $2x^2 - 5x - 3 = 0$ **l** $3x^2 + 10x - 15 = 0$

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m $\frac{1}{6}x^2 + \frac{1}{3}x - 1 = 0$ **n** $\frac{3}{4}x^2 - \frac{3}{2}x - 3 = 0$ **o** $\sqrt{2}x^2 - 4x - 2\sqrt{2} = 0$

5 Solve the equations. Some of them will factorise by trial and error, for some you will have to complete the square, and some will have no solution.

a $x^2 + 6x - 8 = 0$ **b** $x^2 - 3x - 10 = 0$ **c** $x^2 + 6x - 7 = 0$

d $x^2 - 4x - 3 = 0$ **e** $x^2 + x - 6 = 0$ **f** $x^2 - x - 3 = 0$

g $2x^2 + 5x + 2 = 0$ **h** $3x^2 - 2x - 1 = 0$ **i** $x^2 + 2x - 5 = 0$

j $x^2 + 6x - 5 = 0$ **k** $x^2 + 4x + 6 = 0$ **l** $x^2 - 6x + 10 = 0$

m $4x^2 - 25 = 0$ **n** $9x^2 - 1 = 0$ **o** $2x^2 + 4x - 70 = 0$

p $3x^2 - 3x - 36 = 0$ **q** $4x^2 - 5 = 0$ **r** $9x^2 + 7 = 0$

s $6x^2 + x - 12 = 0$ **t** $12x^2 + 23x + 5 = 0$ **u** $x^2 + 6x + 9 = 0$

v $3x^2 + 6x + 2 = 0$ **w** $2x^2 - 8x + 5 = 0$ **x** $5x^2 + 2x - 5 = 0$

y $12x^2 + 5x - 2 = 0$ **z** $4x^2 - x + 4 = 0$

6 Solve:

a $x(x + 2) = 5$ **b** $x(x - 2) = 1$ **c** $x = \frac{7}{x} - 4$

d $x + \frac{4}{x} = -6$ **e** $\frac{x + 1}{x} = x$ **f** $\frac{x + 3}{x} = \frac{2x}{3}$

5F The quadratic formula

The method of completing the square always works. From this it is possible to develop a general formula for the solutions, if they exist, of a quadratic equation in terms of the coefficients in the given equation. This formula is known as the **quadratic formula**. If you are interested in computer programming, you may like to write a program that inputs the coefficients of a quadratic and uses the formula to find the solutions.

To derive the formula, we start with a general quadratic equation of the form:

$$ax^2 + bx + c = 0, \quad \text{where } a \neq 0$$

And begin to solve for x.

$$\begin{aligned} ax^2 + bx &= -c \\ x^2 + \frac{b}{a}x &= \frac{-c}{a} \\ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\ &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

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If $b^2 - 4ac$ is negative, the equation has no solution.

If $b^2 - 4ac$ is positive or zero, then we can solve for x and obtain:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad -\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Summarising the result:

When solving $ax^2 + bx + c = 0$, first calculate $b^2 - 4ac$.

- If $b^2 - 4ac$ is negative, then there is no solution.
- If $b^2 - 4ac$ is positive, then $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.
- If $b^2 - 4ac = 0$, then there is one solution: $x = \dfrac{-b}{2a}$.

You do not need to remember the details of the derivation of this formula, but you should memorise the formula.

Example 18

Use the quadratic formula to solve:

a $x^2 - 7x + 12 = 0$ **b** $x^2 + 3x - 1 = 0$ **c** $x^2 - 10x - 3 = 0$

Solution

a Here $a = 1$, $b = -7$, $c = 12$,

so $b^2 - 4ac = (-7)^2 - 4(1)(12)$

$= 49 - 48 = 1$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{7 + \sqrt{1}}{2} \quad \text{or} \quad x = \frac{7 - \sqrt{1}}{2}$$

$$x = 4 \quad \text{or} \quad x = 3$$

Note that this equation is much easier to solve by factorising.

(continued over page)

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b Here $a = 1,\ b = 3,\ c = -1,$

so $b^2 - 4ac = (3)^2 - 4(1)(-1)$

$= 9 + 4 = 13$

$x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-3 + \sqrt{13}}{2}$ or $x = \dfrac{-3 + \sqrt{13}}{2}$

c Here $a = 1,\ b = -10,\ c = -3,$

so $b^2 - 4ac = (-10)^2 - 4(1)(-3)$

$= 100 + 12 = 112$

$x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{10 + \sqrt{112}}{2}$ or $x = \dfrac{10 - \sqrt{112}}{2}$

$x = \dfrac{10 + 4\sqrt{7}}{2}$ or $x = \dfrac{10 - 4\sqrt{7}}{2}$ (Simplify the surd.)

$x = \dfrac{2(5 + 2\sqrt{7})}{2}$ or $x = \dfrac{2(5 - 2\sqrt{7})}{2}$

$x = 5 + 2\sqrt{7}$ or $x = 5 - 2\sqrt{7}$ (Cancel common factors.)

Example 19

Use the quadratic formula to solve $x^2 - 3x - 5 = 0$, giving your answers correct to two decimal places.

Solution

Here $a = 1, b = -3, c = -5,$

so $b^2 - 4ac = (-3)^2 - 4(1)(-5)$

$= 9 + 20 = 29$

$x = \dfrac{3 + \sqrt{29}}{2}$ or $x = \dfrac{3 - \sqrt{29}}{2}$

$x \approx 4.19$ or $x \approx -1.19$ (Correct to two decimal places.)

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Solving quadratic equations – a summary

We now have three methods for solving a quadratic equation:

- completing the square
- factorisation
- the quadratic formula.

- It is a good idea to calculate $b^2 - 4ac$ first to check that it is positive or zero, otherwise there will be no solution.
- Only use the quadratic formula or complete the square if you cannot see how to factorise the quadratic expression.
- When you use the quadratic formula, take care to simplify the surd and cancel any common factors.
- A quadratic equation for which the coefficient of x^2 is 1 and in which the coefficient of x is even can be solved more quickly and efficiently by completing the square than by using the quadratic formula. You should be in the habit of using both methods and, for a given situation, choosing the one you think will be the faster.

Example 20

Solve each quadratic equation, using any method.

a $x^2 - 9x + 14 = 0$ **b** $x^2 - 8x - 1 = 0$ **c** $3x^2 - 7x + 1 = 0$

Solution

a This quadratic equation factorises easily.

$$x^2 - 9x + 14 = 0$$
$$(x - 2)(x - 7) = 0$$
$$x = 2 \quad \text{or} \quad x = 7$$

b This quadratic equation does not factorise easily, but the coefficient of x is even.

$$x^2 - 8x - 1 = 0$$
$$x^2 - 8x = 1$$
$$x^2 - 8x + 16 = 1 + 16$$
$$(x - 4)^2 = 17$$
$$x = 4 + \sqrt{17} \quad \text{or} \quad x = 4 - \sqrt{17}$$

c The quadratic formula is best here.

$$3x^2 - 7x + 1 = 0$$

Now $a = 3$, $b = -7$, $c = 1$,

so $b^2 - 4ac = 49 - 12$
$= 37$

$$x = \frac{7 + \sqrt{37}}{6} \quad \text{or} \quad x = \frac{7 - \sqrt{37}}{6}$$

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The Discriminant

The **discriminant** is the name given to $b^2 - 4ac$, the part of the quadratic formula under the square root sign. It is often denoted by the capital Greek letter delta, Δ (i.e. $\Delta = b^2 - 4ac$).

As noted, the discriminant can be used to determine the number of solutions in the quadratic equation. However, when a, b and c are rational numbers, and $\Delta > 0$, it also determines the *nature* of the solution. If the discriminant is the square of a rational number (for example, 64, 1, 25, $\frac{1}{4}$ or $\frac{49}{36}$), then the solutions are rational numbers. Otherwise, the solutions contain a surd.

Example 21

Determine the number of solutions in the following quadratic equations. Where solutions exist, state their nature.

a $3x^2 + 8x + 1 = 0$

b $4x^2 + 7x + 5 = 0$

c $9x^2 - 60x + 100 = 0$

d $8x^2 - 2x - 3 = 0$

Solution

a Here $a = 3$, $b = 8$, $c = 1$,

so $b^2 - 4ac = 64 - 12$

$= 52$

$\Delta = 52$, so $\Delta > 0$ and not the square of a rational number.

Therefore, the equation has two irrational solutions.

b Here $a = 4$, $b = 7$, $c = 5$,

so $b^2 - 4ac = 49 - 80$

$= -31$

$\Delta = -31$, so $\Delta < 0$.

Therefore, the equation has no solutions.

c Here $a = 9$, $b = -60$, $c = 100$,

so $b^2 - 4ac = 3600 - 3600$

$= 0$

$\Delta = 0$

Therefore, the equation has one rational solution.

(*Note*: The value of the single solution is easily determined using the formula,

$x = -\frac{b}{2a} = \frac{60}{18} = \frac{10}{3}$.)

d Here $a = 8$, $b = -2$, $c = -3$,

so $b^2 - 4ac = 4 + 96$

$= 100$

$\Delta = 100$, so $\Delta > 0$ and the square of a rational number.

Therefore, the equation has two rational solutions.

Note: If the discriminant is positive and the square of a rational number, then the expression can usually be factorised easily.

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The quadratic formula for $ax^2 + bx + c = 0$

- First calculate $\Delta = b^2 - 4ac$.
- If Δ is negative, then the equation $ax^2 + bx + c = 0$ has no solution.
- The solution of $ax^2 + bx + c = 0$, with $a \neq 0$, is given by:
- $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

  provided that Δ is positive or zero.

Exercise 5F

Example 18

1 Use the quadratic formula to solve each quadratic equation. Give your answers in simplest surd form.

a $x^2 - 8x + 1 = 0$ **b** $x^2 - 2x - 8 = 0$ **c** $x^2 - 3x - 1 = 0$

d $x^2 - 4x - 12 = 0$ **e** $x^2 + 5x + 2 = 0$ **f** $x^2 + 9x + 3 = 0$

g $x^2 - 8x + 2 = 0$ **h** $x^2 + 2x - 4 = 0$ **i** $x^2 + 12x + 3 = 0$

2 Use the quadratic formula to solve each quadratic equation. Give your answers in simplest surd form.

a $3x^2 + 2x - 7 = 0$ **b** $5x^2 + 3x - 1 = 0$ **c** $4x^2 - 6x + 1 = 0$

d $7x^2 - 9x + 2 = 0$ **e** $5x^2 + 3x - 2 = 0$ **f** $7x^2 - x - 1 = 0$

g $2x^2 + 12x - 1 = 0$ **h** $3x^2 - 20x - 2 = 0$ **i** $3x^2 - 4x - 5 = 0$

Example 19

3 Use the quadratic formula to solve each quadratic equation, giving your answers to two decimal places where appropriate.

a $5x^2 - 7x - 1 = 0$ **b** $x^2 - 8x + 1 = 0$ **c** $x^2 - 3x - 10 = 0$

d $x^2 + 15x + 3 = 0$ **e** $2x^2 - 10x + 12 = 0$ **f** $5x^2 - 15x - 7 = 0$

g $2x^2 - 5x - 2 = 0$ **h** $5x^2 - 3x - 1 = 0$ **i** $2x^2 - 7x + 1 = 0$

Example 21

4 The quadratic formula states that the solutions of a quadratic equation are given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

a What can you conclude about the number of solutions of a quadratic equation if:

i $b^2 - 4ac < 0$? **ii** $b^2 - 4ac = 0$? **iii** $b^2 - 4ac > 0$?

b Determine the number of solutions of each quadratic equation, and where they exist, state their nature. You do not need to find the solutions.

i $x^2 + 8x - 5 = 0$ **ii** $3x^2 - 7x + 2 = 0$ **iii** $x^2 + 6x + 9 = 0$

iv $4x^2 - 4x + 1 = 0$ **v** $x^2 + 7x + 13 = 0$ **vi** $2x^2 + 11x + 17 = 0$

vii $3x^2 - 4x - 2 = 0$ **viii** $-2x^2 + 3x + 7 = 0$

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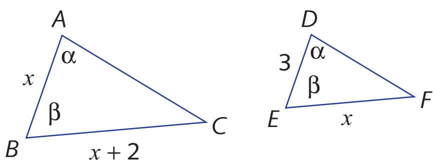
5 In each problem, introduce one pronumeral, and then construct and solve a quadratic equation. Remember to check that your solutions make sense.

a What positive real number is one more than its reciprocal?

b A rectangle has length 5 cm greater than its width. If the area of the rectangle is 30 cm^2, find the width of the rectangle (correct to two decimal places).

c Consider the triangles ABC and DEF, which have side lengths and angles as marked.



Use similar triangles to find the value of x in surd form.

6 The interval AB is extended to point P so that $AB \times AP = BP^2$.

If $AB = 8$ cm, find the lengths of AP and BP.

7 A farmer sells sheep at \$75 a head. The sheep cost \$$x$ each. The farmer finds she has made $x\%$ profit on the sale of the sheep. Find x.

8 Find two numbers whose difference is 5 and the sum of whose squares is 100.

9 An investor invests \$10 000 at $x\%$ p.a. compound interest for 2 years. He finds that he receives \$20 more in interest than if he had invested it at a simple interest rate of $x\%$ p.a. Find x.

10 A car travels 500 km at a constant speed. If it had travelled at a speed 10 km/h less, it would have taken 1 hour more to travel the distance. Find the speed of the car.

11 A rectangular field is 405 m^2 in area, and its perimeter is 200 m. Find the length of its sides.

Example 20

12 Solve each quadratic equation, using any method.

a $x^2 - 11x + 28 = 0$ b $x^2 - 12x - 4 = 0$ c $3x^2 + 2x - 6 = 0$

Review exercise

1 Solve each quadratic equation by factorising.

a $x^2 - 3x - 18 = 0$

b $3x^2 + 5x - 2 = 0$

c $2x^2 + x - 1 = 0$

d $6x^2 + 7x = 3$

e $x^2 = \frac{1}{2}(5x + 12)$

f $x + \frac{2}{x} = -\frac{9}{2}$

g $x(x - 2) = 8$

h $(x - 1)(x + 1) = 2(x + 1)^2$

i $2x^2 - 3x = 5$

j $2 - 7x + \frac{5}{x} = 0$

k $2x^2 = 11x - 5$

l $2x^2 + 11x + 5 = 0$

m $4x^2 - 10x - 6 = 0$

n $6x^2 = 20x - 6$

o $18x^2 - 12x + 2 = 0$

p $9x^2 - 42x + 49 = 0$

q $6x - \frac{49}{x} + 7 = 0$

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2 Solve each quadratic equation by completing the square.

a $x^2 - 8x + 15 = 0$ **b** $t^2 - 11t + 30 = 0$

c $x^2 - 4x + 1 = 0$ **d** $x^2 + 2x - 1 = 0$

e $y^2 + y = 3$ **f** $v^2 - 20v = 7$

g $x^2 - 3x = 7$ **h** $z^2 - 2z = 3$

i $2z^2 + 4z = 64$ **j** $2x^2 - 5x + 2 = 0$

k $3x^2 + x - 3 = 0$ **l** $4x^2 - 3x - 2 = 0$

3 Use the quadratic formula to find exact solutions to each quadratic equation.

a $x^2 - 2x - 24 = 0$ **b** $2x^2 + 3x - 2 = 0$

c $x^2 + x - 1 = 0$ **d** $2x^2 + 5x + 1 = 0$

e $2x^2 + 2x - 3 = 0$ **f** $3x^2 - x - 1 = 0$

4 Solve:

a $x^2 - 2x - 1 = 0$ **b** $2x^2 + 5x = 4$

c $4x^2 - x - 1 = 0$ **d** $\frac{1}{x} = \frac{x-1}{4}$

e $\frac{2x-1}{5} = \frac{1}{3x+2}$ **f** $\frac{2x+1}{5} = \frac{-x}{3x-2}$

5 Solve:

a $x^2 - 7x + 9 = 0$ **b** $2x^2 + 7x = 3$

c $10x^2 = 2x + 5$ **d** $5x^2 - 8x + 2 = 0$

e $4x^2 - 6x = 3$ **f** $2x^2 - 9x = 4$

g $2x^2 - 5x = 1$ **h** $3x^2 + 4x = 2$

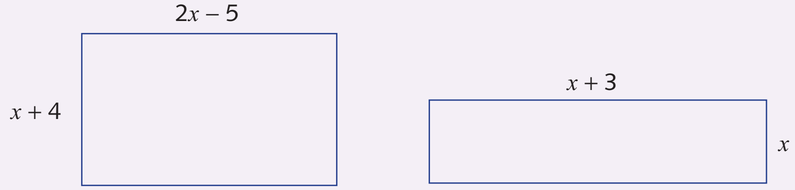
6 For each problem, introduce a pronumeral and construct a quadratic equation to solve it.

a The difference of two numbers is 16 and the sum of their squares is 130. What are the numbers?

b The perimeter of a rectangle is 50 cm and its area is 144 cm^2. Find its length and width.

c Two numbers differ by 2, but the difference of their reciprocals is $\frac{2}{15}$. What are the numbers?

7 The rectangles shown have equal area. Find the value of x.



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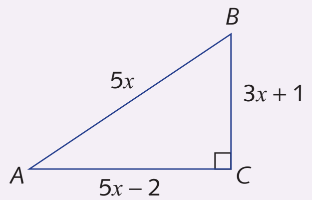
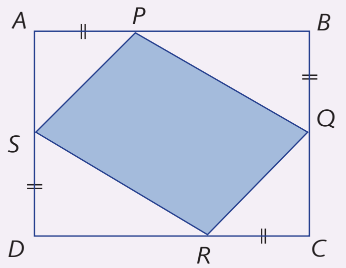
8 A man travels 196 km by train and returns in a car that travels 21 km/h faster. If the total journey takes 11 hours, find the speeds of the train and the car.

9 A wire 80 cm in length is cut into two parts and each part is bent to form a square. If the sum of the areas of the squares is 300 cm^2, find the lengths of the sides of the two squares.

10 The lengths of the sides of a right-angled triangle are $(3x + 1)$ cm, $5x$ cm and $(5x - 2)$ cm. Find the area of the triangle.



11 In the diagram below, $ABCD$ is a rectangle in which $AB = 16$ cm and $BC = 12$ cm and $AP = BQ = CR = DS$. The area of the shaded figure $PQRS$ is 112 cm^2. Find the length of AP.



Challenge exercise

1 Solve the equation $x^4 - 13x^2 + 36 = 0$ by treating it as a quadratic in x^2.

2 Solve the equation $(x^2 - 2x)^2 - 11(x^2 - 2x) + 24 = 0$.

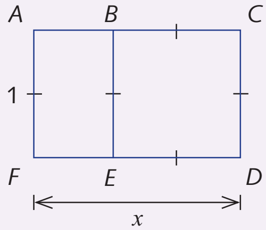
3 A **golden rectangle** is a rectangle such as $ACDF$ below, with sides of length 1 and x, and with the property that if a 1×1 square ($BCDE$) is removed, the resulting rectangle ($ABEF$) is similar to the original one. (That is, $ACDF$ is an enlargement of $ABEF$.)

a Show that $\frac{x-1}{1} = \frac{1}{x}$.

b Solve this equation to show that $x = \frac{1+\sqrt{5}}{2}$.

(This number is known as the **golden ratio**.)

c Check that your answer satisfies the equation in part **a**.



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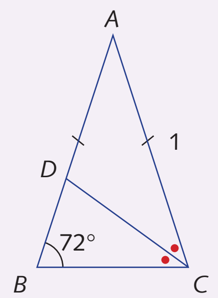
4 Find a monic quadratic equation that has roots equal to $2 - \sqrt{3}$ and $2 + \sqrt{3}$.

5 What is the average of the solutions and what is the product of the solutions of $2x^2 + 14x + 17 = 0$?

6 If x is a solution of $x + \frac{1}{x} = 3$, find $x^2 + \frac{1}{x^2}$.

7 Solve $\sqrt{7x} - \sqrt{3x} = 4$.

8 Take an isosceles triangle ABC with a base angle of 72° and $AB = AC = 1$. Bisect one of the base angles, for example, C (as shown in the diagram) and join CD.



a Prove that triangle ABC is similar to triangle CDB.

b Let $BC = x$. Prove that $x^2 + x - 1 = 0$ and hence solve for x.

c Drop a perpendicular from A to BC and show that $\cos 72° = \frac{\sqrt{5} - 1}{4}$.

d Find the cosine of 18° in simplest surd form.

9 A number, x, is defined by:

$$x = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \dots}}}}$$

where the dots indicate that the pattern continues forever.

a As the pattern repeats indefinitely, we can write $x = \cfrac{1}{2 + \cfrac{1}{3 + x}}$. Simplify this to give a quadratic equation for x.

b Solve the quadratic equation.

c Find the number $y = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$

10 Solve the equation $\frac{x - a}{x + a} = \frac{x + a}{2x - a}$ for x, where a is a constant.

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