CHAPTER

6

Measurement and Geometry

Surface area and volume

In this chapter, we will review and extend the ideas of the surface area and the volume of a solid. Problems involving calculating volume and surface area are very practical and important. Most of the chapter concerns prisms and pyramids, which are the most common polyhedra. We will also see how to find the volume and surface area of solids such as cones, cylinders and spheres.

This chapter contains a number of formulas for volumes and areas.

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6A Review of prisms and cylinders

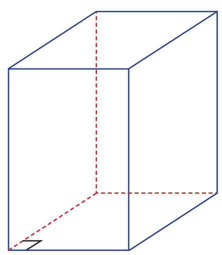
A **polyhedron** is a solid bounded by polygons called **faces**.

Two adjacent faces meet at an **edge** and two adjacent edges meet at a **vertex**.

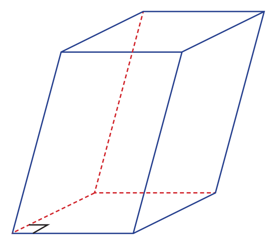
A **prism** is a polyhedron that has two congruent and parallel faces and all its remaining faces are parallelograms.

A **right prism** is a prism in which the top and bottom polygons are vertically above each other, and the vertical polygons connecting their faces are rectangles. A prism that is not a right prism is often called an **oblique prism**.

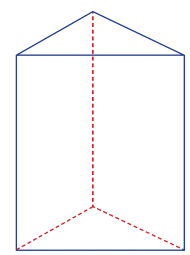
Some examples of prisms are shown below.



Right rectangular prism



Oblique rectangular prism

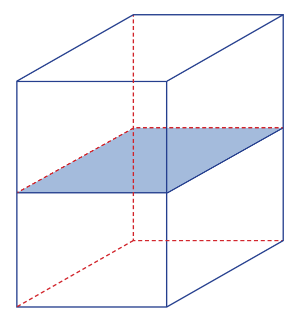


Right triangular prism

When we refer to a prism we generally mean a right prism.

A prism with a rectangular base is called a **rectangular prism**, while a **triangular prism** has a triangular base.

You will notice that if we slice a prism by a plane parallel to its base, then the cross-section is congruent to the base and so has the same area as the base.

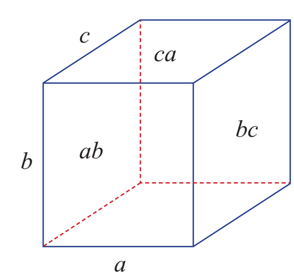


In this chapter, we will use the pronumeral S for the surface area of a solid and the pronumeral V for the volume of a solid.

Surface areas

Surface area of a prism

The surface area of a prism is the sum of the areas of its faces.



A rectangular prism with dimensions a, b and c has six faces. These occur in opposite pairs; the faces with areas ab, bc and ca each occur twice.

Thus:

$$\text{Surface area of a rectangular prism} = 2\,(ab + bc + ca)$$

We do not need to learn this formula. We can simply find the area of each face and take the sum of the areas. The same idea applies to all other types of prisms.

In the diagrams in this chapter, assume that all quadrilaterals are rectangles unless the context indicates otherwise.

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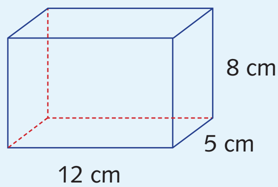
Surface area of a prism

To find the surface area of a prism, find the area of each face and calculate the sum of the areas.

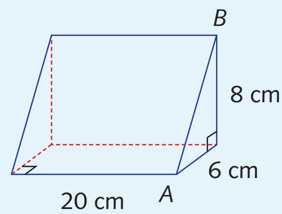
Example 1

Find the surface area of each prism.

a



b



Solution

a The prism has six faces.

$$\begin{aligned}\text{Area of top rectangle} &= 12 \times 5\\ &= 60\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of front rectangle} &= 12 \times 8\\ &= 96\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of side rectangle} &= 8 \times 5\\ &= 40\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Thus, } S &= 2 \times 60 + 2 \times 96 + 2 \times 40\\ &= 392\ \text{cm}^2\end{aligned}$$

b We need to find the length AB in the diagram. We can find this using Pythagoras' theorem.

$$\begin{aligned}AB^2 &= 6^2 + 8^2\\ &= 100\end{aligned}$$

so $AB = 10$ cm

$$\begin{aligned}\text{Area of the sloping rectangle} &= 10 \times 20\\ &= 200\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of each triangle} &= \frac{1}{2} \times 6 \times 8\\ &= 24\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of the base rectangle} &= 6 \times 20\\ &= 120\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of the back rectangle} &= 8 \times 20\\ &= 160\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Thus, } S &= 24 + 24 + 120 + 160 + 200\\ &= 528\ \text{cm}^2\end{aligned}$$

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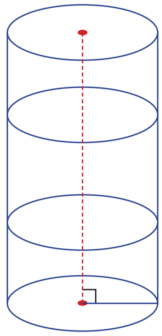
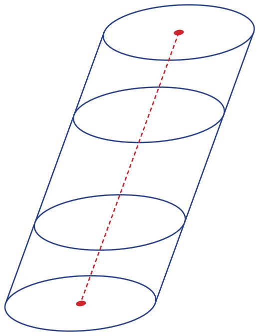
Surface area of a cylinder

A **cylinder** is a solid that has parallel circular discs of equal radius at the top and the bottom. Each cross-section parallel to the base is a circle, and the centres of these circular cross-sections lie on a straight line. If that line is perpendicular to the base, the cylinder is called a **right cylinder**. When we use the word 'cylinder' in this book, we will generally mean a right cylinder.



Right cylinder



Oblique cylinder

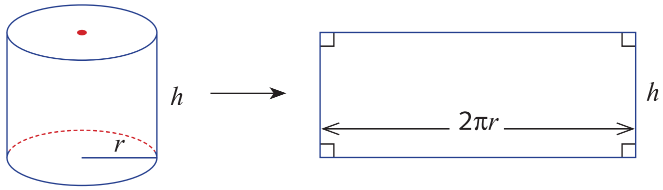
We will use a dot (•) to indicate the centre of the circular base or top.

As we did with prisms, we find the surface area of a cylinder by adding up the area of the curved section of the cylinder, and the area of the two circles.

Surface area of the curved surface

Suppose we have a cylinder with base radius r and height h. If we roll the cylinder along a flat surface through one revolution, as shown in the diagram, the curved surface traces out a rectangle. The width of the rectangle is the height of the cylinder, while the length of the rectangle is the circumference of the circle, which is $2\pi r$, so the area of the curved part is $2\pi rh$. Thus:

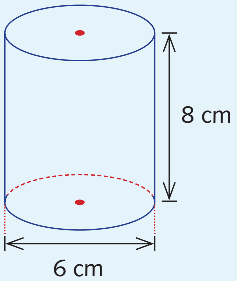
$$\text{Curved surface area of cylinder} = 2\pi rh$$



Example 2

Calculate the surface area of each solid, correct to two decimal places.

a



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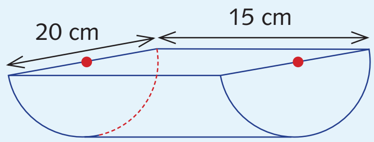

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Solution

a This is a cylinder with radius 3 cm and height 8 cm.

$$\begin{aligned}\text{Area of curved surface} &= 2\pi rh \\ &= 2 \times \pi \times 3 \times 8 \\ &= 48\pi \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of a circular end} &= \pi r^2 \\ &= \pi \times 3^2 \\ &= 9\pi \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Hence, } S &= 48\pi + 9\pi + 9\pi \\ &= 66\pi \text{ cm}^2 \\ &= 207.35 \text{ cm}^2 \quad \text{(Correct to two decimal places.)}\end{aligned}$$

b

$$\begin{aligned}\text{Area of curved section} &= \frac{1}{2} \times 2\pi rh \\ &= \pi \times 10 \times 15 \\ &= 150\pi \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of a semicircle} &= \frac{1}{2} \times \pi r^2 \\ &= \frac{1}{2}\pi \times 10^2 \\ &= 50\pi \text{ cm}^2\end{aligned}$$

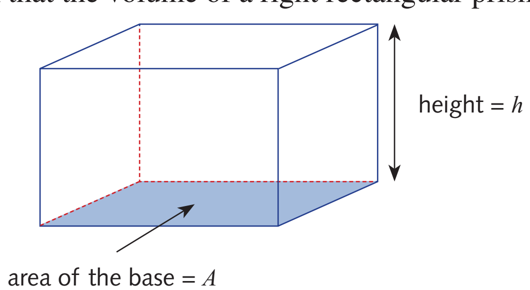
$$\begin{aligned}\text{Area of top rectangle} &= 20 \times 15 \\ &= 300 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}S &= 150\pi + 50\pi + 50\pi + 300 \\ &= (250\pi + 300) \text{ cm}^2 \\ &\approx 1085.40 \text{ cm}^2 \quad \text{(Correct to two decimal places.)}\end{aligned}$$

Volume

Volume of a rectangular prism

We have seen in earlier work that the volume of a right rectangular prism is given by:



$$\begin{aligned}\text{Volume of a right rectangular prism} &= \text{area of base} \times \text{height} \\ &= A \times h\end{aligned}$$

That is, $V = A \times h$

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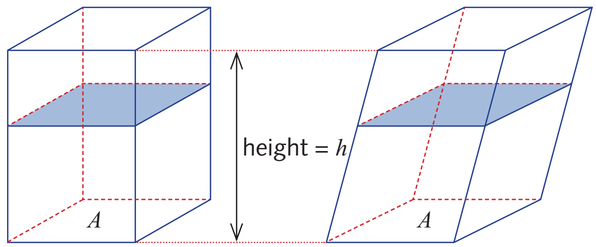
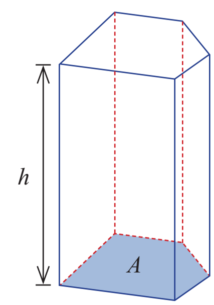
Suppose that we have two solids of the same height. If the cross-sections of the two solids, taken at the same distance above the base, have the same area, it can be shown that the solids have the same volume. This is known as **Cavalieri's principle**.



Cross-sectional areas are the same.

Cavalieri's principle allows us to say that the volume V of *any* rectangular prism, right or oblique, is given by $V = A \times h$.

Volume of a prism



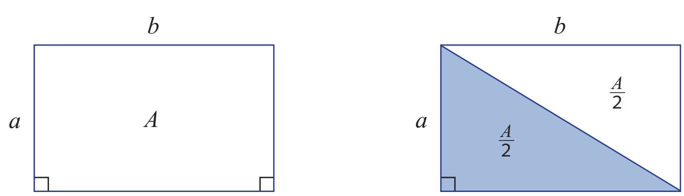
The volume V of a prism, right or oblique, is given by the formula:

$$V = Ah$$

where A is the area of the base and h is the height, as discussed previously.

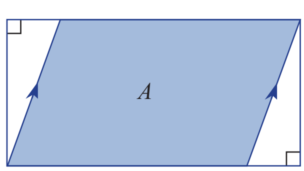
The proof is in five steps. In all of the following, the height is h.

Step 1: The base of the prism is a right-angled triangle



The right rectangular prism has volume $V = Ah$. If it is cut in half, we obtain two prisms of the same volume $\frac{Ah}{2}$ base area $\frac{A}{2}$ and height h.

Step 2: The base of the prism is a parallelogram of area A



$A = R - 2T$, where R is the area of the rectangle and T is the area of each right angled triangle.

The volume of the prism:

$$\begin{aligned} V &= hR - 2hT \\ &= h(R - 2T) \\ &= hA \end{aligned}$$

So the formula holds if the base of the prism is a parallelogram.

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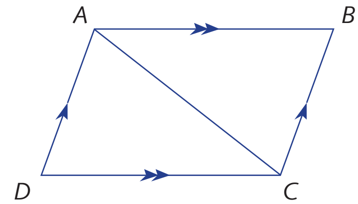
Step 3: The base is a triangle *ABC*

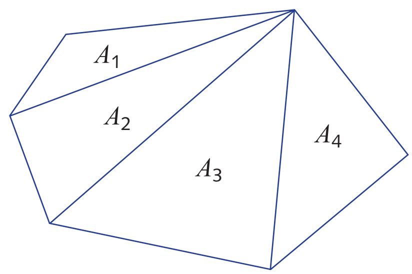
The area of the triangle *ABC* is half the area of the parallelogram *ABCD*.



The prism with triangular base *ABC* is half the volume of the prism with the parallelogram *ABCD* as the base. So the formula holds if the base of the prism is a triangle.

Step 4: The base of the prism is a polygon

As an example, the convex hexagon can be cut up into four triangles.



Clearly the volume of the prism is:

$$\begin{aligned} V &= A_1h + A_2h + A_3h + A_4h \\ &= (A_1 + A_2 + A_3 + A_4)h \\ &= Ah, \end{aligned}$$

where A is the area of the hexagon.

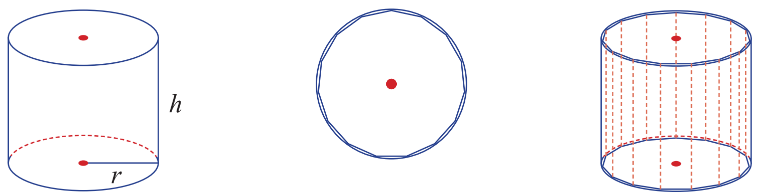
This argument applies to any right prism.

Step 5: Oblique prisms

Use Cavalieri's principle exactly as above.

Volume of a cylinder

The area of a regular polygon inscribed in a circle approximates the area of that circle. The greater the number of sides, the better the approximation. A cylinder has a circular base. Since $V = Ah$ holds for any polygon-based prism, it seems reasonable that the volume of the cylinder should be the area of its circular base multiplied by the height.



Thus, the volume of a cylinder with radius r and height h is equal to the area of the circular cross-section, πr^2, multiplied by the height, h.

$$\text{Volume of a cylinder} = \pi r^2 h$$

Cavalieri's principle shows that this also applies to an oblique cylinder.

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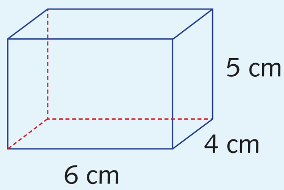
Example 3

Calculate the volume of each solid.

a



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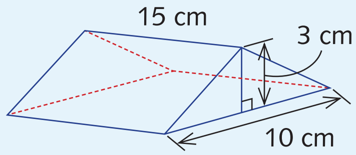
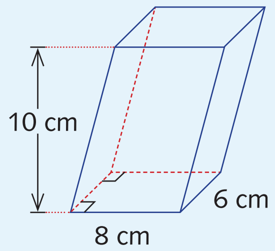
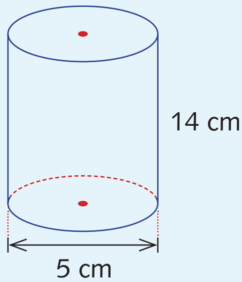


c



d



Solution

a $V = Ah$
$= 6 \times 4 \times 5$
$= 120 \text{ cm}^3$

b The solid is a triangular prism.

$V = Ah$
$= (\text{area of triangular base}) \times (\text{height of prism})$
$= \left(\frac{1}{2} \times 10 \times 3\right) \times 15$
$= 225 \text{ cm}^3$

c This is a rectangular prism of height 10 cm.

$V = Ah$
$= 8 \times 6 \times 10$
$= 480 \text{ cm}^3$

d This is a cylinder of radius 2.5 cm and height 14 cm.

$V = \pi r^2 h$
$= \pi \times 2.5^2 \times 14$
$= 87.5\pi \text{ cm}^3$

Surface area and volume

- A **prism** is a polyhedron with two parallel congruent polygonal faces and all other faces parallelograms.
- The **surface area** of a prism is the sum of the areas of its faces.
- The surface area of the curved part of a cylinder with radius r and height h is given by:

  Curved surface area $= 2\pi rh$
- The volume of a prism is given by the product of the area of the base A and the height h:

  $V = Ah$
- The volume of a cylinder with radius r and height h is given by:

  $V = \pi r^2 h$

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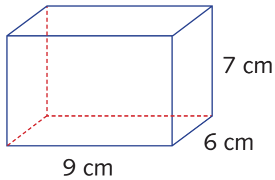
Exercise 6A

Example 1, 2

1 Calculate the surface area of each solid.

a



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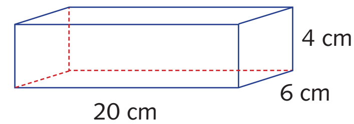

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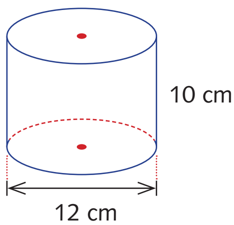

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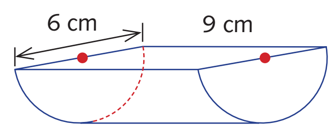

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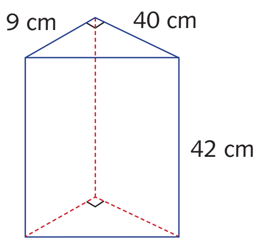

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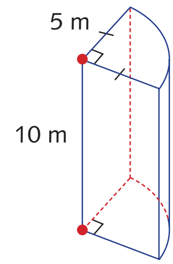

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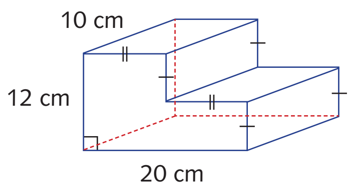

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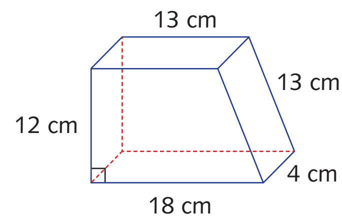

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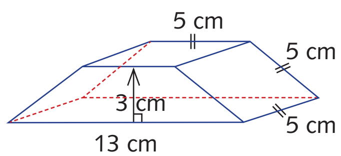

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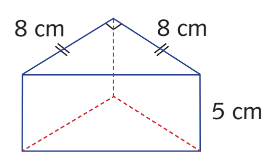

2 The base of a right prism of height 12 cm is an equilateral triangle.

a If the side length of the triangle is 6 cm, use Pythagoras' theorem to calculate the height of the triangle.

b Calculate the surface area of the prism.

3 A cube has surface area 486 m^2. Find its side length.

4 A cylinder of base radius 8 cm has curved surface area 72π cm^2. Find its height.

5 A rubbish bin is in the shape of an open cylinder.

a If the bin has radius 15 cm and height 40 cm, find its total surface area (do not include the lid).

b If the bin has radius 15 cm and curved surface area 3500 cm^2, find the height of the bin, correct to one decimal place.

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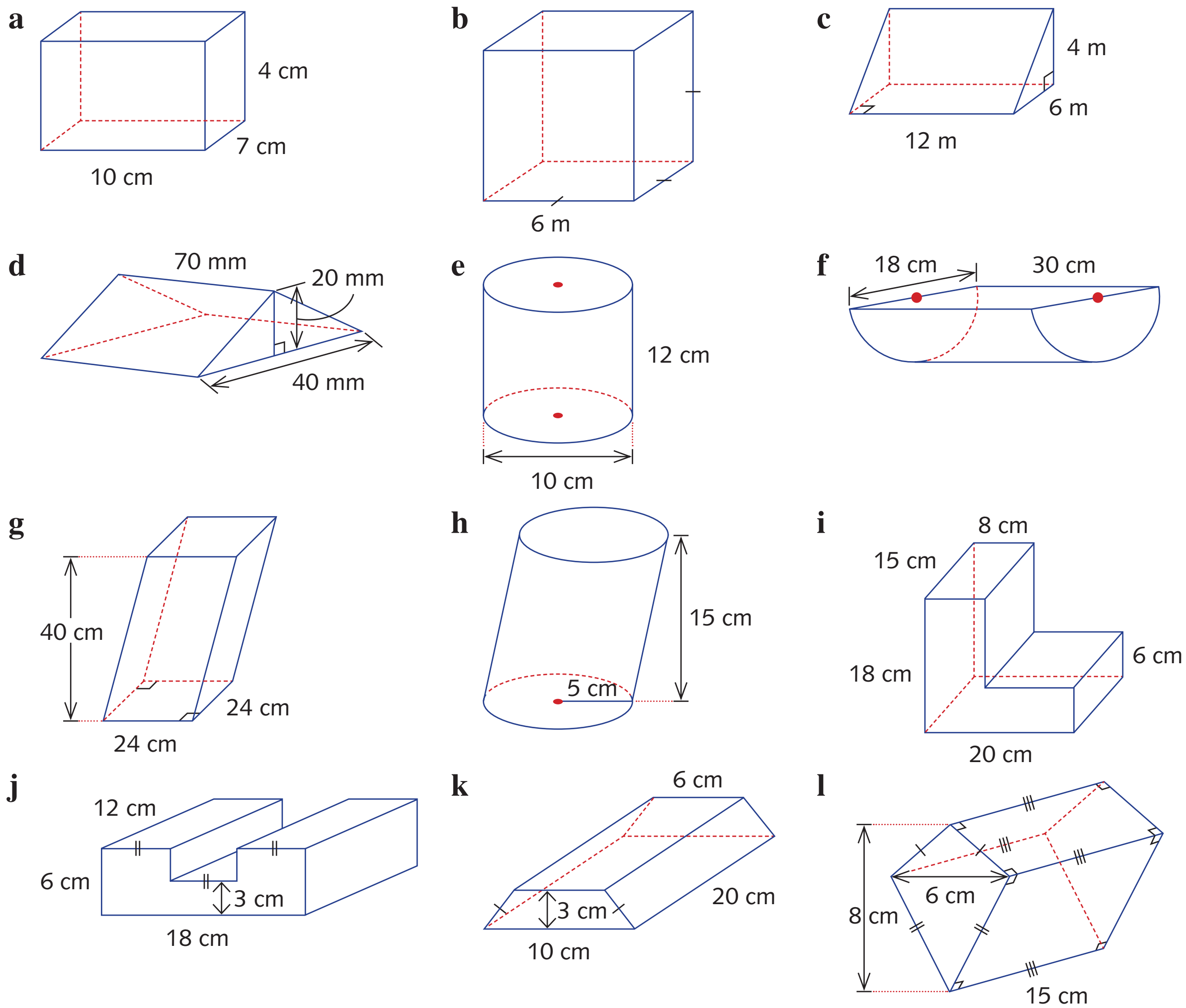
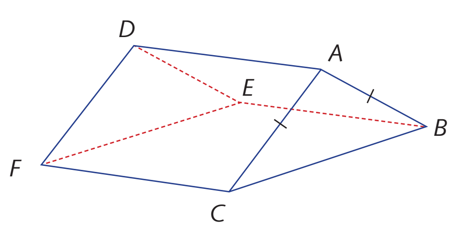
Example 3

6 Calculate the volume of each solid.



7 In the diagram below, *ABCFED* is a right triangular prism, with $AB = AC$.



a If $AB = 8$ cm and $BC = 12$ cm, find the height of ΔABC, correct to two decimal places.

b If $AD = 24$ cm, find the volume of the prism, correct to two decimal places.

8 A manufacturer of drink cans produces cylindrical cans with a volume of 1000 cm^3.

a If the radius of the can is 4 cm, find the height of the can, correct to one decimal place.

b If the height of the can is 8 cm, find the radius of the can, correct to two decimal places.

9 **a** If a cube has volume 64 cm^3, find its side length.

b If a cube has volume 400 cm^3, find its side length, correct to one decimal place.

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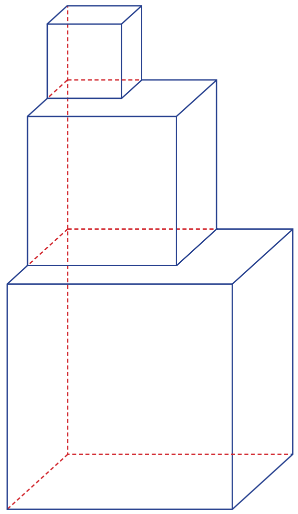
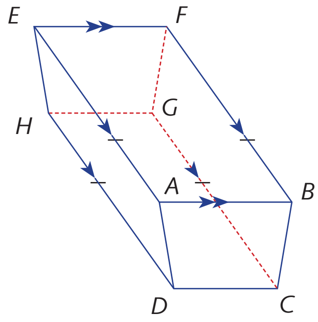
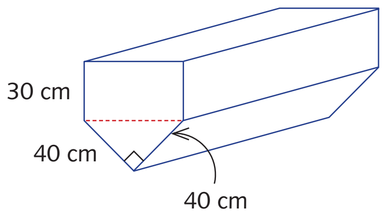
10 A modern sculpture consists of three cubes, with side lengths 0.5 m, 1 m and 1.5 m, respectively, placed on top of each other as shown in the diagram opposite. Calculate the surface area of the sculpture. Do not include the base of the large cube in your calculations.



11 The figure shows a trapezoidal prism. Its ends *ABCD* and *EFGH* are congruent trapezia, with $DC = 4$ cm and $AB = 6$ cm. If $AE = 15$ cm and the volume is 300 cm^3, find the height of the trapezium *ABCD*.



12 A farmer is making a trough that needs to be filled with water once each day. The trough is in the shape of a prism with pentagonal ends, as shown opposite. The farmer has 60 horses that each drink about 10 L of water per day. Using the fact that $1\ cm^3 = 1$ mL, what is the smallest length the trough needs to be to water the horses each day?

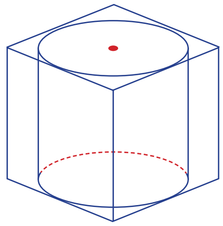


13 A cylindrical vase of radius 5 cm and height 14 cm just fits into a box. Find:

a the volume of the cylinder, correct to two decimal places

b the volume of the box

c the volume of unused space inside the box, correct to two decimal places



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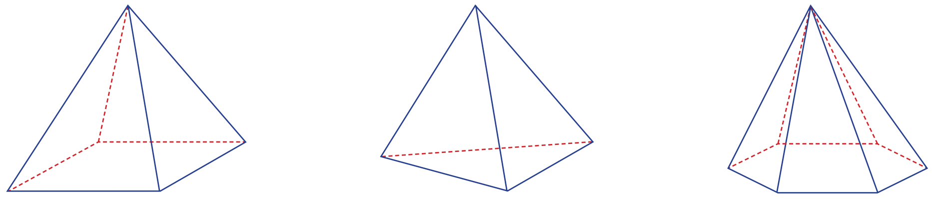
6B Pyramids

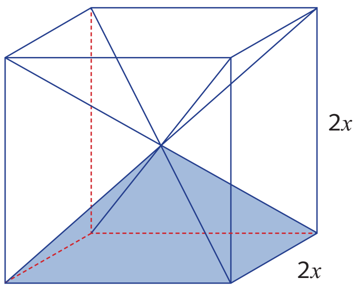
A **pyramid** is a polyhedron with a polygonal base and triangular sides that meet at a point called the **vertex**. The pyramid is named according to the shape of its base.



If we drop a perpendicular from the vertex of the pyramid to the base, then the length of the perpendicular is called the **height** of the pyramid.

A **right** pyramid is one whose vertex is directly above the centre of its base.

Volume of a pyramid



Here is a method for determining the formula for the volume of a right, square-based pyramid.

Consider a cube of side length $2x$. If we draw the long diagonals as shown, then we obtain 6 square pyramids, one of which is shaded in the diagram. Each of these pyramids has base area $2x \times 2x$ and height x. Let V be the volume of each pyramid.

The volume of the cube is $(2x)^3 = 8x^3$. Hence:

$$6 \times V = 8x^3$$

so $$V = \frac{4}{3} \times x^3$$

Now the area of the base of each pyramid is $(2x)^2 = 4x^2$ and the height of each pyramid is x, so in this case we can write:

$$\begin{aligned} V &= \frac{4}{3} \times x^3 \\ &= \frac{1}{3} \times 4x^2 \times x \end{aligned}$$

or $$V = \frac{1}{3} \times \text{area of base} \times \text{height}$$

It is possible to extend this result to any pyramid by using geometric arguments and Cavalieri's principle. We then have the following important result.

$$\text{Volume of a pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

That is, $V = \frac{1}{3}Ah$

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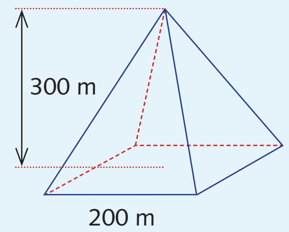
Example 4

Calculate the volume of a square pyramid with base of side length 200 m and height 300 m.



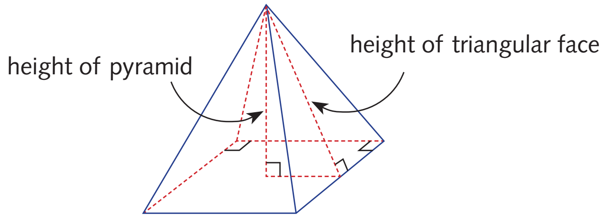
Solution

$$V = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3} \times 200 \times 200 \times 300$$
$$= 4\,000\,000 \text{ m}^3$$

The volume of the pyramid is $4\,000\,000 \text{ m}^3$.

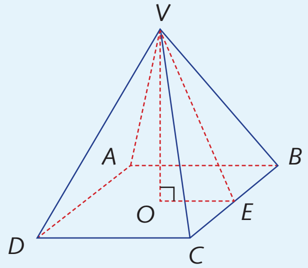
Surface area of a pyramid

To find the surface area of a pyramid, we need to find the areas of the surfaces bounding the pyramid, which will consist of a number of triangles together with the base. You will often have to use Pythagoras' theorem in calculating the surface area of a pyramid.



Example 5

VABCD is a right, square-based pyramid with vertex *V* and base *ABCD*, with *V* vertically above the centre of the square base. The height of the pyramid is 4 cm and the side length of the base is 6 cm. Find the surface area of the pyramid.



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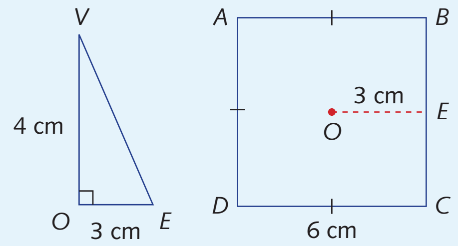
Solution

We need to find the height VE of triangle VBC, using Pythagoras' theorem.



$$\begin{aligned} VE^2 &= VO^2 + OE^2 \\ &= 4^2 + 3^2 \\ &= 25 \end{aligned}$$

Hence, $VE = 5$ cm.

$$\begin{aligned} \text{Area of } \Delta VCB &= \frac{1}{2} \times CB \times VE \\ &= \frac{1}{2} \times 6 \times 5 \\ &= 15 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of base} &= 6 \times 6 \\ &= 36 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Surface area} &= 4 \times 15 + 36 \\ &= 96 \text{ cm}^2 \end{aligned}$$

Hence, the surface area of the pyramid is 96 cm^2.

Pyramids

- A **pyramid** is a polyhedron with a polygonal base and triangular faces that meet at a point called the **vertex**.
- The pyramid is named according to the shape of its base.
- The volume of a pyramid with base area A and height h is given by:

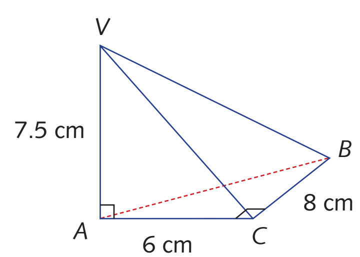
$$V = \frac{1}{3}Ah$$

Exercise 6B

In this exercise, assume all pyramids are right pyramids unless otherwise stated.

Example 4

1 In the diagram opposite, $VABC$ is a triangular pyramid. The base ΔABC is right-angled, with $AC = 6$ cm and $BC = 8$ cm. The vertex V of the pyramid is vertically above A, and $VA = 7.5$ cm. Calculate the volume of the pyramid.



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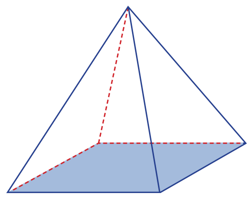
Example 4

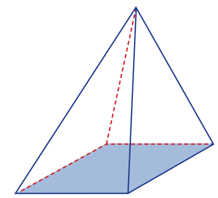
2 Calculate the volume of each of the following pyramids. The area, A cm^2, of the shaded face and the height, h cm, are given in each case.

a



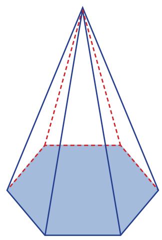
$A = 24$, $h = 10$

b



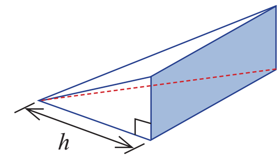
$A = 20$, $h = 15$

c



$A = 72$, $h = 9$

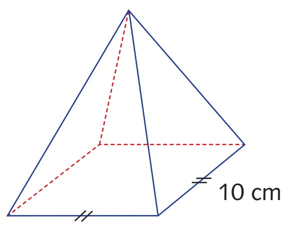
d



$A = 45$, $h = 12$

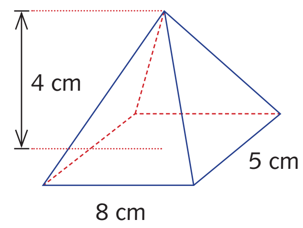
3 Calculate the volume of each pyramid.

a

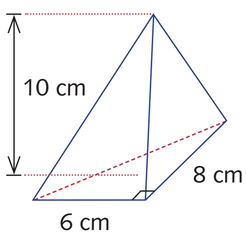


height = 12 cm

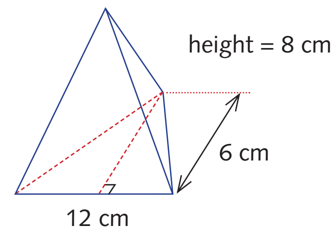
b



c



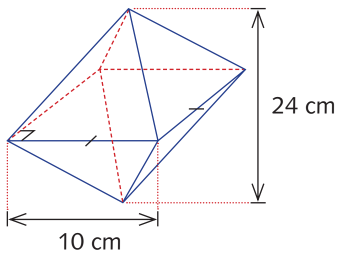
d



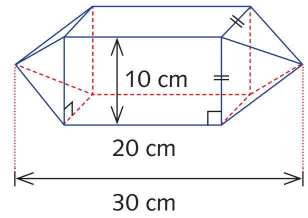
4 When it was built, the Great Pyramid of Cheops in Egypt had a height of 145.75 m and its base was a square of side length 229 m. Find its volume in cubic metres, correct to one decimal place.

5 Calculate the volume of each solid, giving the answer correct to two decimal places.

a



b



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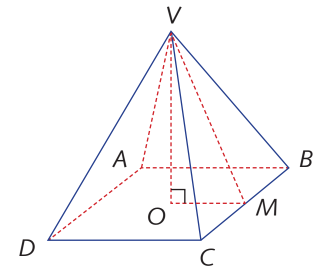
Example 5

6 In the diagram opposite, $VABCD$ is a square pyramid and O is the centre of square $ABCD$. If the height of the pyramid $VO = 12$ cm and $AB = 10$ cm, find:



a the length OM, where M is the midpoint of BC

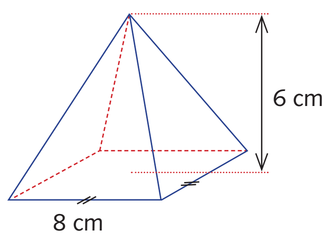
b the length VM

c the area of ΔVCB

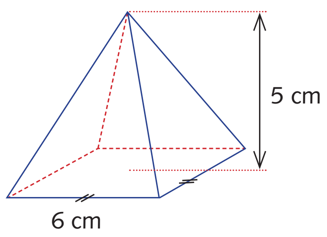
d the surface area of the pyramid

7 Use a technique similar to that in question **6** to find the surface areas of these square pyramids, assuming the vertex is above the centre of the square.

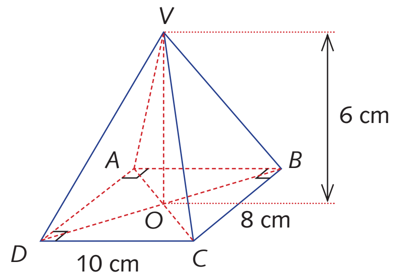
a



b



8 In the diagram opposite, $VABCD$ is a rectangular pyramid. If $VO = 6$ cm is the height of the pyramid, $CD = 10$ cm and $BC = 8$ cm, find:

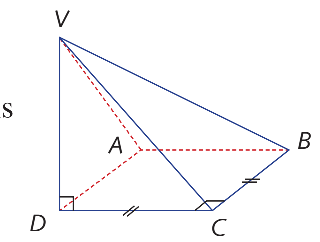


a the height of ΔVBC

b the height of ΔVDC

c the surface area of the pyramid

9 In the diagram opposite, $VABCD$ is a square pyramid, with vertex V directly above D. If $AB = 8$ cm and $VD = 6$ cm:



a name each triangle in the diagram and state what type of triangle it is

b find the length of:

i VC **ii** VB **iii** VA

c find the surface area of the pyramid

6C Cones

A **cone** is a solid that is formed by taking a circle and a point, called the **vertex**, which lies above or below the circle. We then join the vertex to each point on the circle. Of course, the cone can be in any orientation.

If the vertex is directly above or below the centre of the circular base, we call the cone a **right cone**. In this section, the only cones we consider are right cones, which we will simply call cones.

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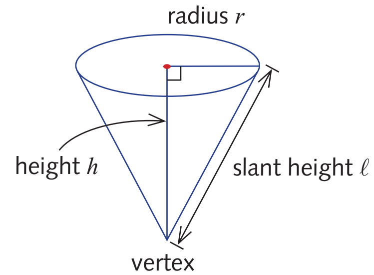
The distance from the vertex of the cone to the centre of the circular base is called the **height**, h, of the cone.

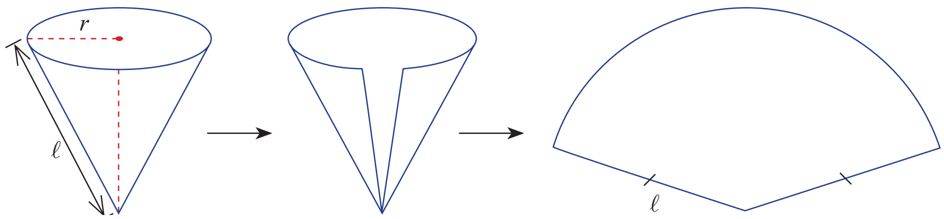
The distance from the vertex of the cone to the circumference of the circular base is called the **slant height**, ℓ, of the cone.

We will use a dot (•) to indicate the centre of the base.

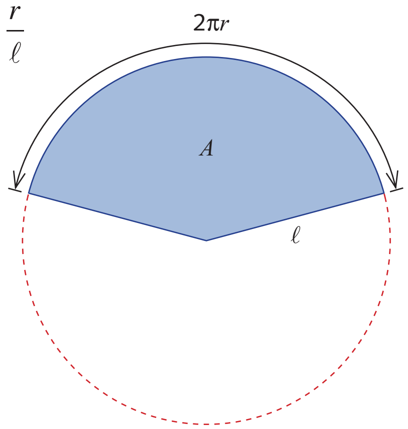


Surface area of a cone

To calculate the surface area of a cone, we need to find the area of each surface. To find the area of the curved surface, we cut and open up the curved surface to form a sector, as shown below.



- The arc length of the sector $=$ circumference of the circular base of the cone $= 2\pi r$
- The proportion of a whole circle $= \dfrac{\text{arc length}}{\text{whole circumference}} = \dfrac{2\pi r}{2\pi\ell} = \dfrac{r}{\ell}$
- Area of sector $A = \left(\dfrac{r}{\ell}\right) \times \pi\ell^2 = \pi r\ell$



Thus, the surface area of the curved part of a cone is given by:

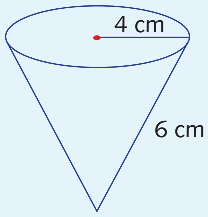
Curved surface area $= \pi r\ell$,

where r is the base radius and ℓ is the slant height.

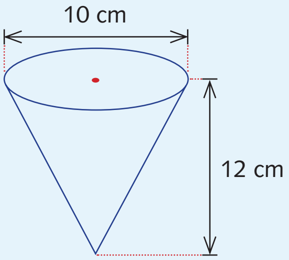
Example 6

Find the surface area of each cone. Give your answer correct to two decimal places.

a



b



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Solution

a We have $r = 4$ and $\ell = 6$.

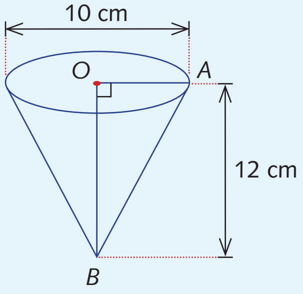
$$\begin{aligned}\text{Area of curved surface} &= \pi r\ell \\ &= \pi \times 4 \times 6 \\ &= 24\pi \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of base} &= \pi r^2 \\ &= 16\pi \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Thus, } S &= 24\pi + 16\pi \\ &= 40\pi \text{ cm}^2 \\ &\approx 125.66 \text{ cm}^2 \quad \text{(Correct to two decimal places.)}\end{aligned}$$

Hence, the surface area of the cone is approximately 125.66 cm^2.

b We first find the slant height AB.



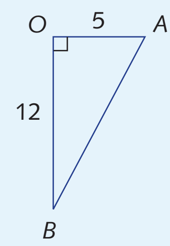
By Pythagoras' theorem:

$$\begin{aligned}AB^2 &= 5^2 + 12^2 \\ &= 169\end{aligned}$$

$AB = 169$ cm

Hence, $r = 5$ and $\ell = 13$.

$$\begin{aligned}\text{Area of curved surface} &= \pi r\ell \\ &= \pi \times 5 \times 13 \\ &= 65\pi \text{ cm}^2\end{aligned}$$



$$\begin{aligned}\text{Area of base} &= \pi r^2 \\ &= 25\pi \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Then } S &= 65\pi + 25\pi \\ &= 90\pi \text{ cm}^2 \\ &\approx 282.74 \text{ cm}^2 \quad \text{(Correct to two decimal places.)}\end{aligned}$$

Hence, the surface area of the cone is approximately 282.74 cm^2.

Example 7

The curved surface area of a cone is 44 cm^2 and the base radius is 2 cm. Find, correct to two decimal places:

a the slant height of the cone

b the height of the cone

Solution

a Curved surface area of a cone $= \pi r\ell$

Here $r = 2$, so $2\pi\ell = 44 \text{ cm}^2$.

$$\begin{aligned}\ell &= \frac{22}{\pi} \text{ cm} \\ &\approx 7.00 \text{ cm} \quad \text{(Correct to two decimal places.)}\end{aligned}$$

Hence, the slant height is approximately 7.00 cm.

(continued over page)

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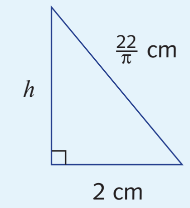
b Using Pythagoras' theorem:



$$h^2 + 2^2 = \left(\frac{22}{\pi}\right)^2$$

$$\text{so} \quad h = \sqrt{\left(\frac{22}{\pi}\right)^2 - 2^2}$$

$$\approx 6.71 \text{ cm} \qquad \text{(Correct to two decimal places.)}$$

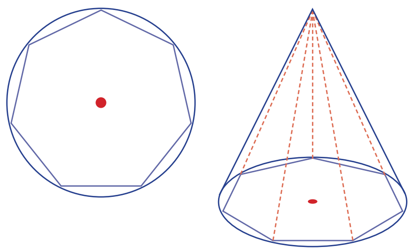
Hence, the height of the cone is approximately 6.71 cm.

Volume of a cone

The formula for the volume of a cone is the same as the formula for the volume of a pyramid, which is $\frac{1}{3} \times$ area of the base $\times$ height. For a cone, the base area is πr^2, so:

$$\text{Volume of a cone} = \frac{1}{3}\pi r^2 h,$$

where r is the radius of the base and h is the height.



To illustrate this informally, imagine constructing a polygon inside the circular base of the cone and joining the vertex of the cone to each of the vertices of the polygon.

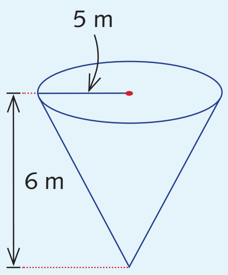
This would give us a polygonal pyramid with volume equal to:

$$\frac{1}{3} \times \text{area of the base} \times \text{height}$$

The more sides we take in the polygon, the area of the base gets closer and closer to πr^2, so the volume of the cone equals $\frac{1}{3}\pi r^2 h$.

Example 8

Calculate the volume of a cone with base radius 5 m and height 6 m.



Solution

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3} \times \pi \times 5^2 \times 6$$

$$= 50\pi \text{ m}^3$$

The volume of the cone is $50\pi \text{ m}^3$.

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Right cones

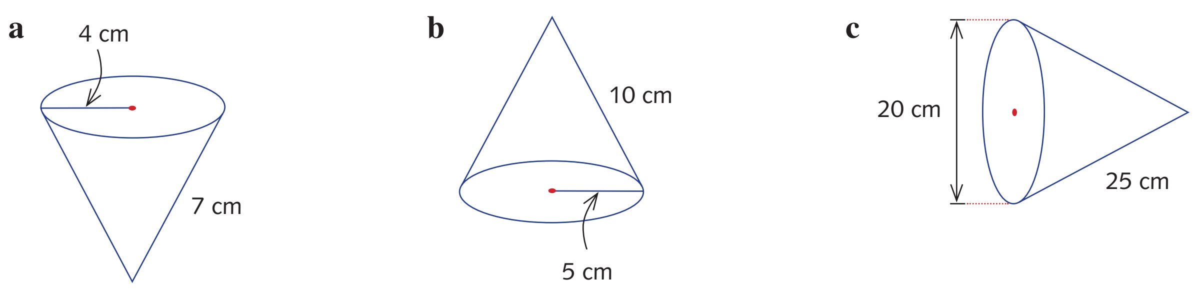
- In a **cone**, the distance between the vertex and the centre of the base is called the **height**, h, of the cone.
- The length of a straight line joining the vertex to a point on the circumference of the circle is called the **slant height**, ℓ, of the cone.
- The surface area of the curved part of a cone is given by $\pi r\ell$, where r is the base radius and ℓ is the slant height.
- The volume of a cone is given by:

  $V = \frac{1}{3}\pi r^2 h$, where r is the base radius and h is the height.

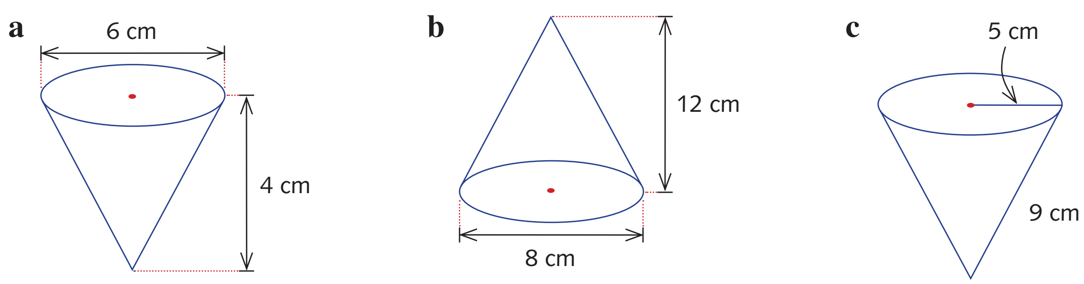
Exercise 6C

Example 6

1 Calculate the total surface area of each cone, including the base. Give your answers correct to two decimal places.



2 Calculate the total surface area of each cone.



Example 7

3 The curved surface of a cone of base radius 4 cm has surface area 40π cm^2. Find:

a the slant height of the cone

b the height of the cone, correct to four decimal places

4 A cone has base radius 10 cm and total surface area 1000 cm^2. Find, correct to two decimal places:

a the surface area of the curved part of the cone

b the slant height of the cone

c the height of the cone

5 **a** Calculate the area of the sector shown opposite. Give your answer correct to two decimal places.

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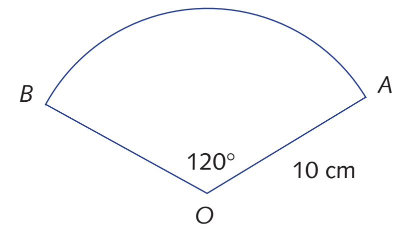
b If the radii *OA* and *OB* are joined together to form the curved surface of a cone, find, correct to two decimal places:

i the slant height of the cone

ii the base radius of the cone

iii the height of the cone



6 A cone has radius r cm and height h cm.

a If $r = 5$ and $h = 10$, find:

i the slant height of the cone

ii the surface area of the curved part of the cone

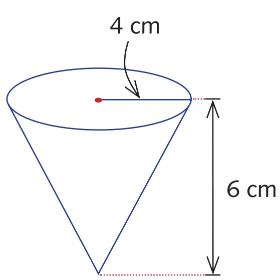
iii the angle of the sector we get if we cut the curved part of the cone

b If $r = h$, find the angle of the sector that produces the curved part of the cone.

Example 8

7 Calculate the volume of each solid. Give your answers correct to two decimal places.

a



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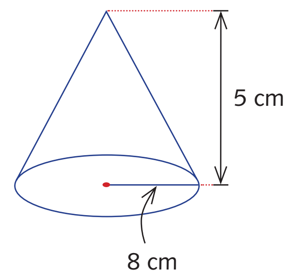
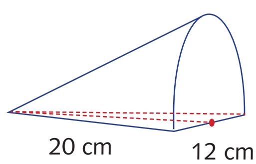
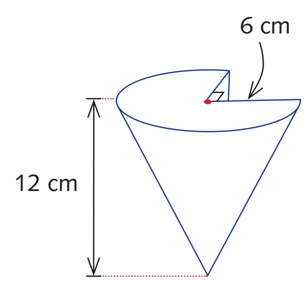


c



d

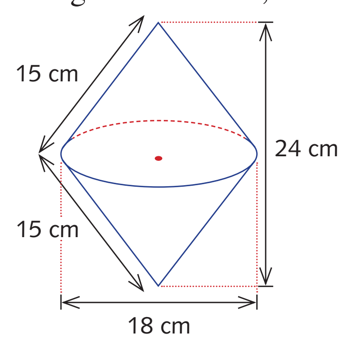


8 A cone with diameter 6 cm has a volume of 120 cm^3. Find the height of the cone, correct to the nearest millimetre.

9 For the solid shown, find:

a the volume

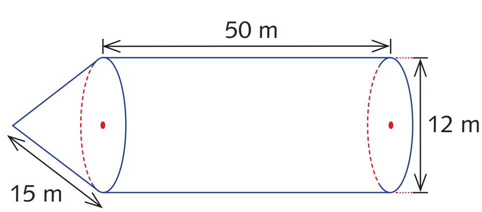
b the total surface area



10 For the solid shown, find:

a the volume

b the total surface area



11 A mound of earth is shaped like a cone. It is 6 metres high with a radius of 25 metres. Find the cost, to the nearest dollar, of moving the mound if it costs \$5 to move one cubic metre.

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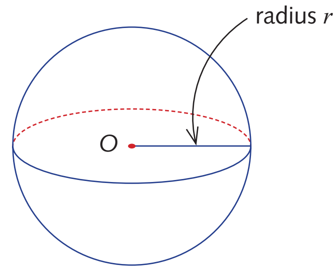
6D Spheres

The word **sphere** comes from the Greek word *sphaira*, meaning 'ball'. Every point on the surface of a sphere lies at a distance r, called the **radius** of the sphere, from a fixed point O, called the **centre** of the sphere. It is difficult to derive the formulas for the surface area and volume of a sphere. These formulas are discussed in the Challenge exercises.



We will use a dot (•) to indicate the centre of a sphere.

Surface area of a sphere

The surface area of a sphere is given by:

$$S = 4\pi r^2,$$

where r is the radius of the sphere.

Example 9

Calculate, correct to two decimal places, the surface area of a sphere:

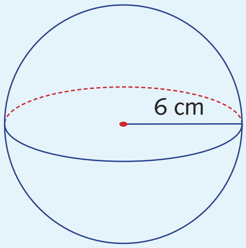
a with radius 6 cm

b with diameter 10 cm

Solution

a We have

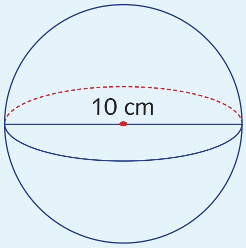
$$\begin{aligned} S &= 4\pi r^2 \\ &= 4 \times \pi \times 6^2 \\ &= 144\pi \text{ cm}^2 \\ &\approx 452.39 \text{ cm}^2 \quad \text{(Correct to two decimal places.)} \end{aligned}$$



The surface area is approximately 452.39 cm^2.

b The diameter = 10 cm, so $r = 5$ cm.

$$\begin{aligned} \text{Then } S &= 4\pi r^2 \\ &= 4 \times \pi \times 5^2 \\ &= 100\pi \text{ cm}^2 \\ &\approx 314.16 \text{ cm}^2 \quad \text{(Correct to two decimal places.)} \end{aligned}$$



The surface area is approximately 314.16 cm^2.

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Volume of a sphere

The formula for the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3,$$

where r is the radius of the sphere.

Example 10

Calculate the volume of a sphere with a diameter of 30 m.

Solution

The radius is 15 m.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times 15^3 \\ &= 4500\pi \text{ m}^3 \end{aligned}$$

Example 11

A sphere has volume 2800 cm^3. Find the radius of the sphere, correct to the nearest millimetre.

Solution

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ 2800 &= \frac{4}{3}\pi r^3 \\ \text{so} \quad r^3 &= \frac{2800 \times 3}{4\pi} \\ r &= \sqrt[3]{\frac{2800 \times 3}{4\pi}} \\ &\approx 8.7 \text{ cm} \qquad \text{(Correct to one decimal place.)} \end{aligned}$$

The radius is approximately 87 mm.

Spheres

- The surface area of a sphere of radius r is given by:

  $S = 4\pi r^2$, where r is the radius of the sphere.

- The volume of a sphere of radius r is given by:

  $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

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Exercise 6D

Example 9

1 Calculate the surface area, correct to two decimal places, of:

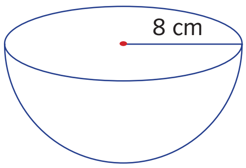
a a sphere of radius 8 cm

b a sphere of radius 15 cm

c a sphere of diameter 14 cm

d a sphere of diameter 21 cm

2 For the solid hemisphere shown opposite, find:



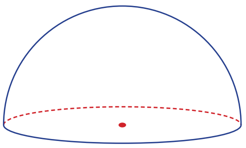
a the area of the flat surface

b the surface area of the curved part of the hemisphere

c the total surface area of the solid hemisphere

3 A sphere has a surface area of $500\ \text{cm}^2$. Find its radius, correct to four decimal places.

4 A hemispherical tent is made using $28\ \text{m}^2$ of material. Find, correct to two decimal places, the radius of the tent if:

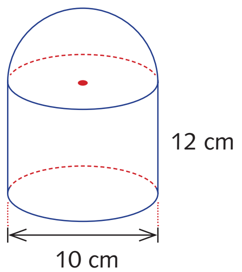


a the tent does not have a material floor

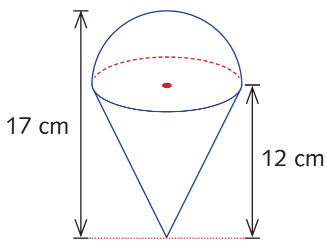
b the tent does have a material floor

5 Calculate the surface area of each object.

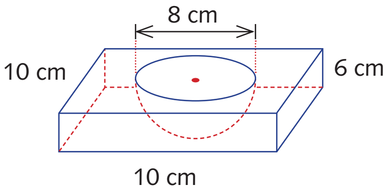
a



b



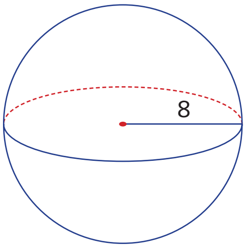
c



Example 10

6 Calculate the volume of each solid. Give your answers correct to two decimal places.

a



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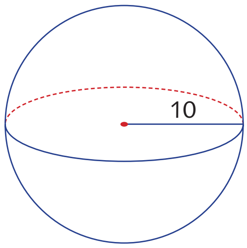

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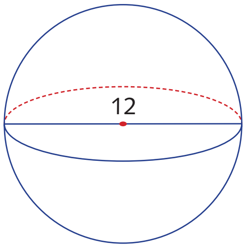

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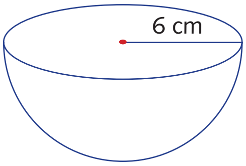

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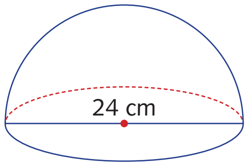
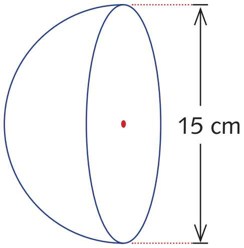


f



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Example 11

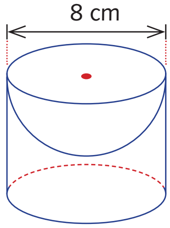
7 **a** Calculate, correct to two decimal places, the radius of a sphere with a volume of $1000\ \text{cm}^3$.

b Calculate, correct to the nearest millimetre, the diameter of a sphere with a volume of $3000\ \text{cm}^3$.

c Calculate, correct to one decimal place, the diameter of a hemispherical bowl with a volume of $250\ \text{cm}^3$.

8 Calculate the volume of each solid. Where appropriate, give your answers correct to two decimal places.

a



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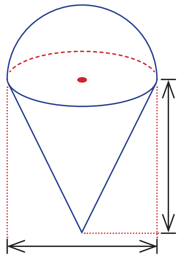
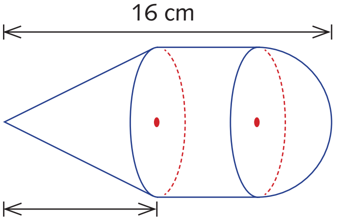


c



9 A spherical soccer ball of diameter 22 cm is packaged in a box that is in the shape of a cube with edges of length 24 cm. Find, correct to the nearest cm^3, the volume of unused space inside the box.

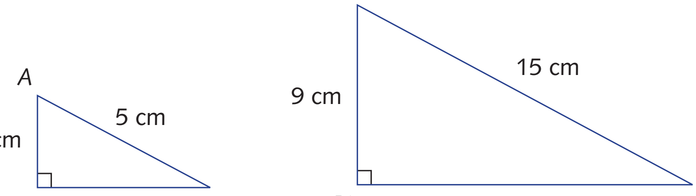
10 Tennis balls are packaged in cylindrical canisters. A tennis ball can be considered to be a sphere of diameter 70 mm. If the canister has base diameter 75 mm and height 286 mm, and each canister holds four balls, find the volume of unused space inside the canister, correct to two decimal places.

6E Enlargement

In this section, we will investigate what happens to the area and volume of figures under enlargement.

In the diagram below, $\Delta A'B'C'$ is an enlargement of ΔABC. The sides of the larger triangle are three times the lengths of those of the smaller one.

We say that the **enlargement factor** is 3.



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Plane figures and enlargements

Notice what happens when we compare the areas of the two triangles. The area of ΔABC is 6 cm^2, and the area of $\Delta A'B'C'$ is 54 cm^2. In this case, the area of the larger triangle is 9 times the area of the smaller one. Since $9 = 3^2$, we see that the area of the smaller triangle is multiplied by the **square** of the enlargement factor.

Example 12

By what factor does the area of a circle with radius 2 cm change when we enlarge the radius by a factor of 5?

Solution

A circle with radius 2 cm has area $\pi \times 2^2 = 4\pi$ cm^2.

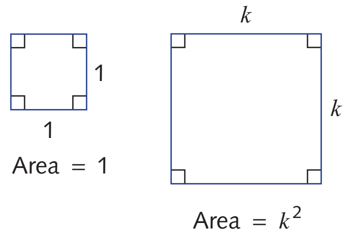
Enlarging by a factor of 5 gives a radius of 10 cm.

A circle with radius 10 cm has area $\pi \times 10^2 = 100\pi$ cm^2.

The area has been multiplied by a factor of $\dfrac{100\pi}{4\pi} = 25$.

(Notice that this is the square of the enlargement factor.)

In general, if each of the dimensions of a plane figure is enlarged by a factor of k, then the area of the figure is multiplied by a factor of k^2.



Example 13

A regular pentagon has area 45 cm^2. If the sides of the pentagon are enlarged by a factor of 8, what is the area of the resulting pentagon?

Solution

The enlargement factor is 8, so the area is multiplied by a factor of $8^2 = 64$.

Hence, the area of the resulting pentagon is $64 \times 45 = 2880$ cm^2.

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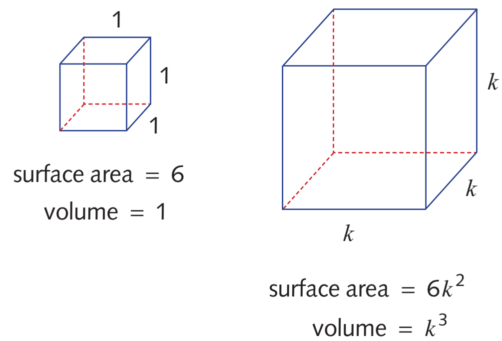
Solids and enlargements

The diagram to the right enables us to draw the following general conclusions. If the dimensions of a solid are enlarged by a factor of k, then the surface area of the figure is multiplied by a factor of k^2.

If the dimensions of a solid are enlarged by a factor of k, then the volume of the figure is multiplied by a factor of k^3.



Example 14

A spherical balloon has radius 10 cm. It is inflated so that the radius becomes 15 cm. By what factor has the following changed?

a surface area **b** volume

Solution

a Method 1 – Find the two areas

For a sphere, $A = 4\pi r^2$

Hence, surface area of the balloon $= 4 \times \pi \times 10^2 = 400\pi \text{ cm}^2$

If the radius becomes 15 cm, then the surface area becomes $4 \times \pi \times 15^2 = 900\pi \text{ cm}^2$,

so the surface area has been multiplied by a factor of $\frac{900\pi}{400\pi} = \frac{9}{4}$

Method 2 – Enlargement factor method

Enlargement factor for the radius is $\frac{15}{10} = \frac{3}{2}$.

Hence, enlargement for the surface area is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

b Method 1 – Find the two volumes

For a sphere, $V = \frac{4}{3}\pi r^3$.

Hence, volume of the balloon $= \frac{4}{3} \times \pi \times 10^3 = \frac{4000}{3}\pi \text{ cm}^3$.

If the radius becomes 15 cm, then the volume becomes $\frac{4}{3} \times \pi \times 15^3 = 4500\pi \text{ cm}^3$, so the volume has been multiplied by a factor of $4500\pi \div \frac{4000\pi}{3} = \frac{27}{8}$.

Method 2 – Enlargement factor method

The enlargement factor for the radius is $\frac{3}{2}$.

Hence, the enlargement factor for the volume is $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$.

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Scale drawings and scale models

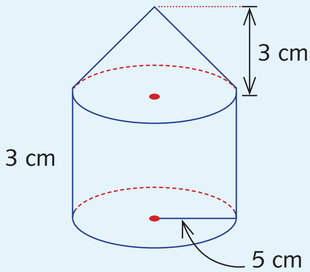
A designer, architect or engineer will often build a **scale model** of the object being designed. The real object is an enlargement of the model. A scale of, for example, 1 : 40 means that the dimensions of the real object are 40 times the dimensions of the model.

Example 15

A farmer builds a scale model of a silo, as shown opposite. The scale of the model is 1 : 100.

a Find the volume of the model.

b What is the volume of the silo, in cubic metres?



Solution

a Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 5^2 \times 3$

$= 25\pi \text{ cm}^3$

Volume of cylinder $= \pi r^2 h$

$= \pi \times 5^2 \times 3$

$= 75\pi \text{ cm}^3$

Volume of model $= 25\pi + 75\pi$

$= 100\pi \text{ cm}^3$

b The enlargement factor is 100, so the volume of the silo is $100^3 \times$ volume of the model

Hence, the volume of the silo is:

$V = (100\pi \times 100^3) \text{ cm}^3$

$= 100\pi \text{ m}^3 \qquad (100^3 \text{ cm}^3 = 1 \text{ m}^3)$

Enlargement

Let $k > 0$.

- If a plane figure is enlarged by a factor of k, then the area of the figure is multiplied by a factor of k^2.
- If a solid is enlarged by a factor of k, then the surface area of the solid is multiplied by a factor of k^2.
- If a solid is enlarged by a factor of k, then the volume of the solid is multiplied by a factor of k^3.

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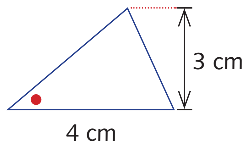
Exercise 6E

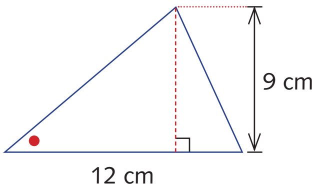
Example 12, 13

1 Find what factor the area of the smaller figure must be multiplied by to find the area of the larger figure. Work this out by:

i calculating the areas
ii using the enlargement factor method

a





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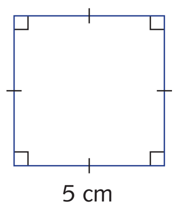

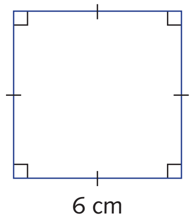

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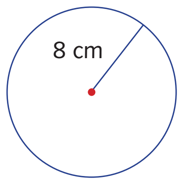
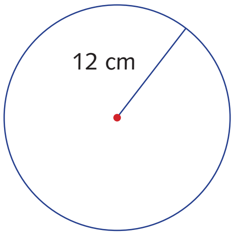


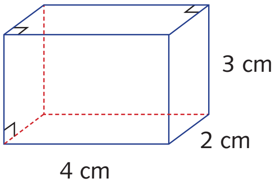


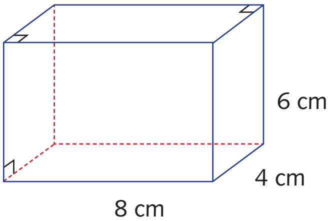
Example 14

2 **i** Find what factor the total surface area of the smaller solid must be multiplied by in order to obtain the surface area of the larger solid. In parts **a**, **c** and **e**, do this by finding the surface areas; in parts **b** and **d**, use the enlargement factor method.

ii Find what factor the volume of the smaller solid must be multiplied by in order to obtain the volume of the larger solid. In parts **b** and **d**, do this by finding the volumes; in parts **a**, **c** and **e**, use the enlargement factor method.

a





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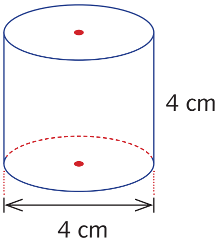

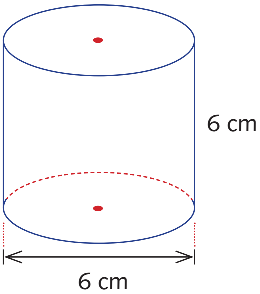

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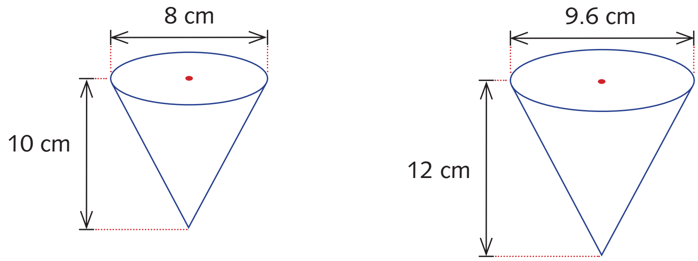
c



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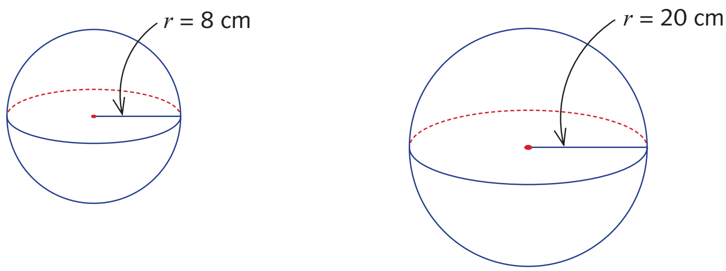


3 A cube has a volume of 343 cm^3. The cube is enlarged by a factor of 5.

a What is the volume of the resulting cube?

b Find the surface area of the resulting cube.

4 By what factor must the radius of a spherical balloon be multiplied if the volume is to be increased from 760 cm^3 to $389\,120\text{ cm}^3$?

5 A cylindrical container holds 125 cm^3 of liquid when full, and requires 240 cm^2 of material to manufacture. The company wants to increase the height and radius by the same enlargement factor to produce a cylindrical container that can hold 1000 cm^3 of liquid. How much material will be required to produce the new container?

6 A solid, A, is enlarged to form a new solid, B. If the surface area of B is twice the surface area of A, by what factor is the volume of A multiplied to give the volume of B?

Example 15

7 A model car has scale 1 : 24; that is, 1 cm on the model represents 24 cm on the actual car.

a If the model car requires 300 cm^2 of material to be made, how much material is required to make the actual car?

b If the volume of the actual car's interior is 4.5 m^3, find the volume of the model's interior, in cm^3 and correct to one decimal place.

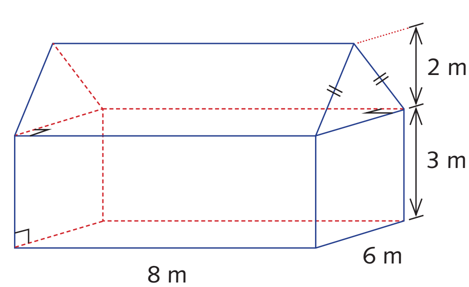
8 A barn is in the shape of a triangular prism on top of a rectangular prism, as shown below.

a Find the volume of the barn.

b If a model of scale 1 : 50 is made of the barn, find the volume of the model, in cm^3.

c Find the outside surface area of the barn (not including the floor).

d How much material, in cm^2 and correct to one decimal place, is needed to make the model?



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9 **a** The base of a triangle is increased by a factor of 7, while the height is kept the same. What happens to the area of the triangle?

b The sides of the square base of a cube are each increased in length by a factor of 6, but the height is kept the same. This produces a square prism. By what factor has the volume changed?

c The radius of a cylinder is trebled, but the height is kept the same. By what factor has the volume changed?

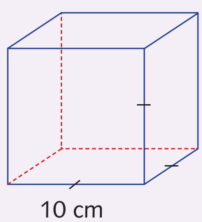
d The radius of a cone is multiplied by 9, but the height is kept the same. By what factor has the volume changed?

Review exercise

1 For each solid, calculate:

i the volume
ii the surface area

a



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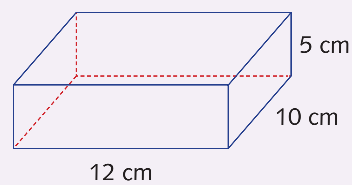

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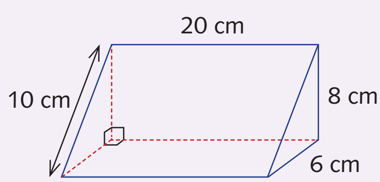

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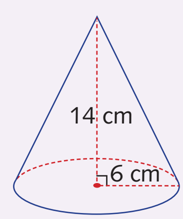

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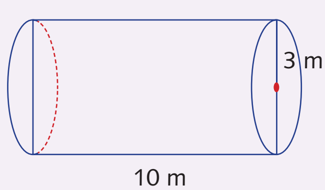

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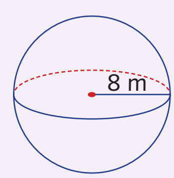
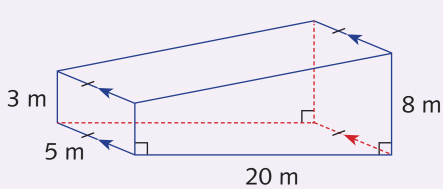


2 For each solid, calculate:

i the volume
ii the surface area

a



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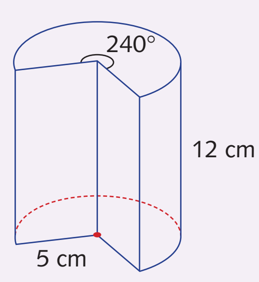
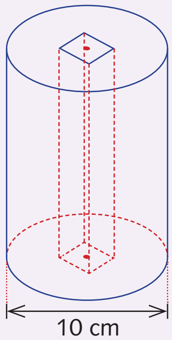
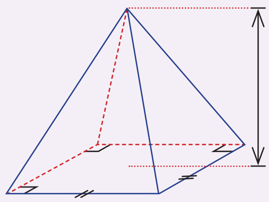


3 A solid cylinder has a shaft with a square cross-section through it, as shown in the diagram below. The volume of the solid is $(500\pi - 80)\ \text{cm}^3$. Calculate the length of a side of the square cross-section.

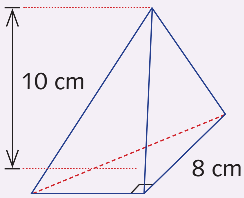


4 Calculate the volume of each solid.

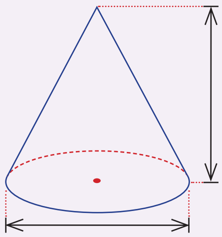
a



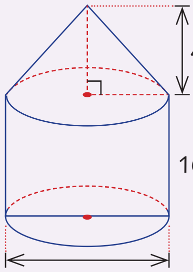
b



c



d

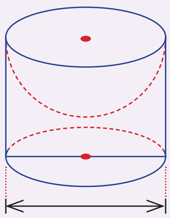


5 A triangular pyramid has all its edges equal in length. This is called a **regular tetrahedron**. Calculate the length of each edge, given that the surface area is $64\sqrt{3}\ \text{cm}^2$.

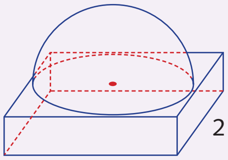
6 For each solid shown, calculate:

i the volume
ii the surface area

a



b



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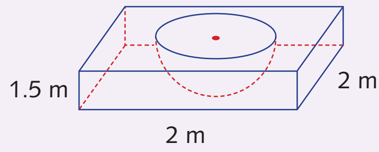
7 A hemispherical bowl is carved out of a solid block of marble, as shown.



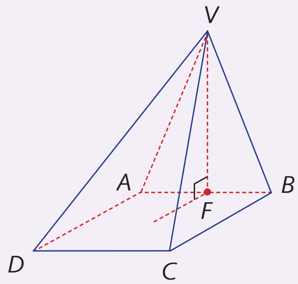
After the bowl is carved, the volume of marble remaining is $\left(6 - \frac{144}{125}\pi\right)$ m^3. Calculate the radius of the hemisphere.

8 As part of a major development, an architect designed a building and had a model made to a scale of 1 : 400; that is, 1 cm on the model is 4 m on the building.

a The external surface area of the building is 10 600 m^2. Calculate the external surface area of the model, in cm^2.

b The volume of the building is 60 000 m^3. Calculate the volume of the model, in cm^3.

9 In the pyramid below, F is the midpoint of AB. VF is vertical and is 5 m in length. $ABCD$ is horizontal and rectangular, with $AB = 6$ m and $BC = 8$ m.



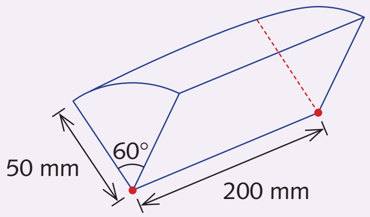
Find:

a the volume of this solid

b the surface area of this solid

10 Find the volume and surface area of the solids.

a



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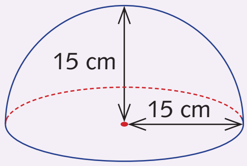

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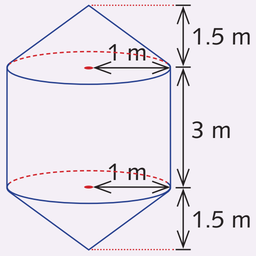

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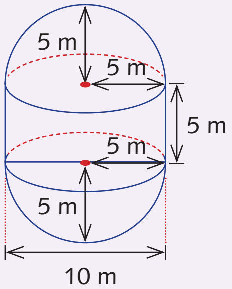

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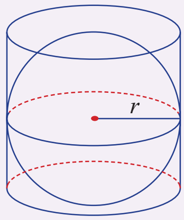
Challenge exercise

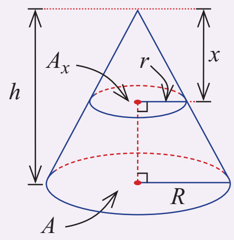
1 A **parallelepiped** is a six-faced polyhedron, each face of which is a parallelogram. A certain parallelepiped with a square base has volume 192 cm^3. Each side of its base is one-third of its height. Find the length of each side of the base.

2 The figure at the right shows a sphere of radius r fitting exactly into a cylinder. The sphere touches the cylinder at the top, bottom and curved surface. Show that the surface area of the sphere is equal to the area of the curved surface of the cylinder.



3 The surface area of a cube is x cm^2 and its volume is y cm^3. If $x = y$, find the length of the side of the cube.

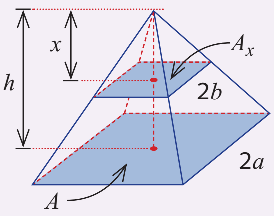
4 A cone has radius R, base area A and height h. A horizontal slice is taken at a distance x units from the vertex, as shown. Let A_x be the area of the circular slice and r be the radius.



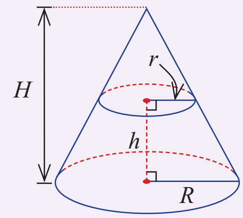
a Use similar triangles to show that $\dfrac{r}{R} = \dfrac{x}{h}$.

b Show that the ratio $A_x : A = x^2 : h^2$.

5 A square pyramid has base length $2a$, base area A and height h. A horizontal slice is taken at a distance x units from the vertex, as shown. Let A_x be the area of the square slice and $2b$ be the side length of the square slice. Use the method of question **4** to show that $A_x : A = x^2 : h^2$.



6 The portion of a right cone remaining after a smaller cone is cut off it is called a **frustum**. Suppose that the top and bottom are circles of radius r and R, respectively. Also, suppose that the height of the frustum is h and the height of the original cone is H.



a Show that the volume V of the frustum is $\dfrac{1}{3}\pi\left[H(R^2 - r^2) + r^2h\right]$.

b Use similar triangles to show that $H = \dfrac{hR}{R - r}$.

c Deduce that $V = \dfrac{1}{3}\pi h(R^2 + r^2 + rR)$.

7 **a** Use the method of question **6** to show that the volume of a truncated square pyramid with height h and square base and square top of side lengths x and y, respectively, is given by $\dfrac{1}{3}h(x^2 + y^2 + xy)$. (A truncated pyramid is formed in a similar way to a frustum.)

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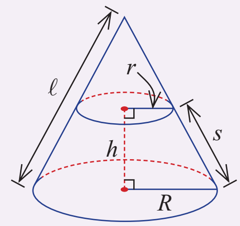
b If the top square has a side length that is half that of the bottom square, what is the ratio of the volume of the truncated pyramid to that of the whole pyramid?

8 A cone has height h and base radius r.

a Show that the surface area of the cone is $\pi r(r + \sqrt{r^2 + h^2})$.

b Suppose that the height and radius are equal. Show that the surface area is $\pi r^2(1 + \sqrt{2})$.

9 The frustum of a cone has base radii r and R, and the slant height is s.



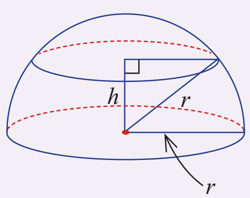
a Let the slant height of the full cone be ℓ.

Show that $\ell = \dfrac{sR}{R - r}$.

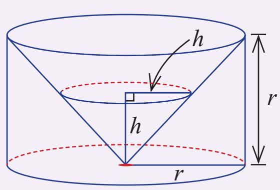
b Hence, show that the surface area of the curved section is $\pi(r + R)s$ and that the total surface area is $\pi(r^2 + R^2) + \pi(r + R)s$.

c Show that the latter can be written as $\pi(r + R)\sqrt{(R - r)^2 + h^2} + \pi(r^2 + R^2)$, where h is the height of the frustum.

10 Cavalieri's principle states that if we have two solids of the same height and the cross-sections of each solid taken at the same distance from the base have the same area, then the solids have the same volume. Take a hemisphere of radius r and look at the area of a typical cross-section at height h above the base.



Also, consider a cylinder of height r and radius r, with a cone cut out of it, also of height r and base radius r. We also take a cross-section at height h.



a Show that the radius of the circular cross-section of the sphere at height h is $\sqrt{r^2 - h^2}$.

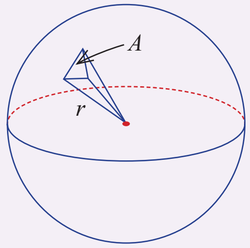
b Deduce that the area of the cross-section is $\pi(r^2 - h^2)$.

c Draw a diagram of the cross-section of the cylinder with the cone removed, and show that the area of the cross-section at height h is also $\pi(r^2 - h^2)$.

d We now conclude that the two solids have the same volume, by Cavalieri's principle. Find the volume of the cylinder minus the cone.

e Deduce the formula for the volume of the sphere.

11 Consider a sphere of radius r split up into very small pyramids, as shown. The volume of each pyramid $= \dfrac{1}{3}Ar$, where A is the area of the base of the pyramid on the surface of the sphere. The base of each pyramid is considered to be a plane surface.



Consider the sum of the volumes of these small pyramids to show that the surface area of the sphere is $4\pi r^2$.

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