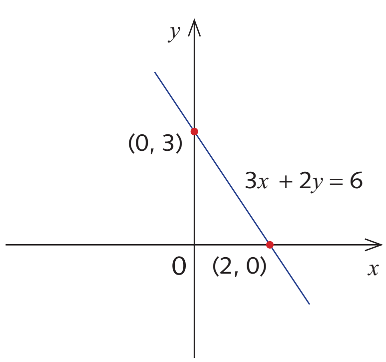
CHAPTER

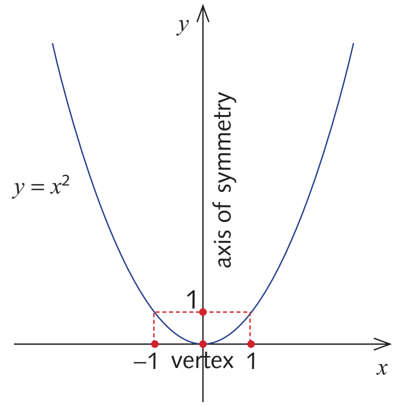
7

Number and Algebra

The parabola

In earlier years you learned how to draw curves by plotting some points from a table of values and joining them up. In particular, you should be familiar with examples such as the straight line and the parabola $y = x^2$.





In this chapter, we will learn techniques for sketching the graphs of **quadratics** such as $y = x^2 + 1$, $y = 2x^2 - 3x$ and $y = -x^2 + 4x + 6$. These graphs are also called **parabolas**.

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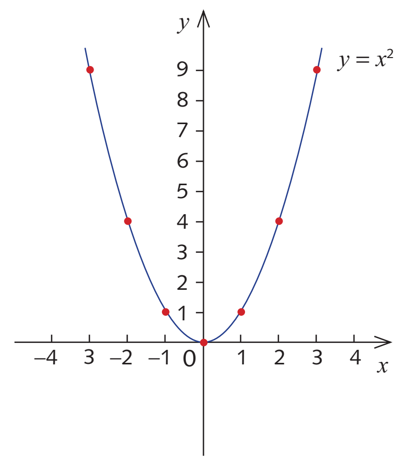
7A Parabolas congruent to $y = x^2$



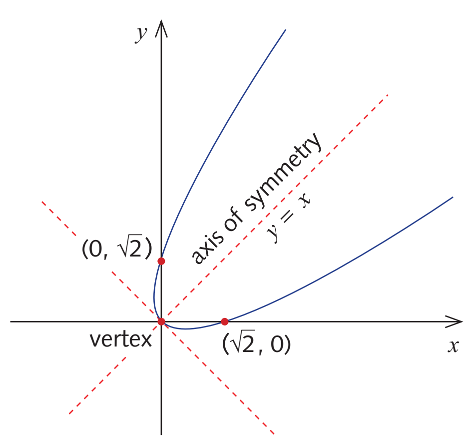
We are all familiar with the graph of $y = x^2$, which was studied in both *ICE-EM Mathematics Year 8* and *Year 9*. In this chapter we will call it the **basic parabola**.

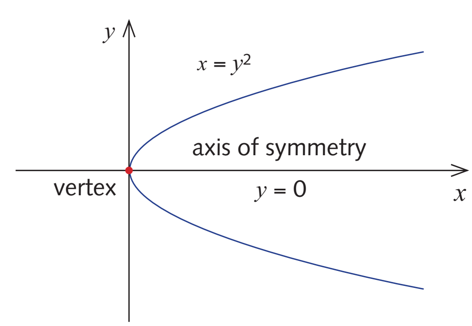
It has the following properties:

- The graph is symmetrical about the y-axis, $x = 0$. For example, the y-value at $x = 3$ is the same as the y-value at $x = -3$. In general, the y-value at $x = p$ is the same as the y-value at $x = -p$. We can visualise this as follows: if the graph were to be folded along the y-axis, the part on the left of the y-axis would land directly on top of the part on the right. The y-axis is called **the axis of symmetry** of the basic parabola.
- The minimum value of y occurs at the origin. It is called a **minimum turning point** since the y-values at points to both the left and the right of the origin are greater than the y-value at the origin. This turning point is called the **vertex** of the parabola.
- The **arms** of the parabola continue indefinitely, becoming steeper the higher they go.

Recall that two geometrical figures are said to be **congruent** if one can be transformed to the other by a sequence of translations, rotations and reflections. In this section we will look at parabolas congruent to $y = x^2$.

Imagine rotating the figure $y = x^2$ about the origin through 45° clockwise. The image you get is still called a parabola. Its vertex is $(0, 0)$ and its axis of symmetry is the line $y = x$, as shown in the left-hand figure below.





Similarly, rotating the basic parabola clockwise through 90° yields another parabola, $x = y^2$, as shown in the right-hand figure above.

However, for the rest of this chapter we shall only consider parabolas whose axis of symmetry is parallel to the y-axis; consequently, rotations will not be further discussed.

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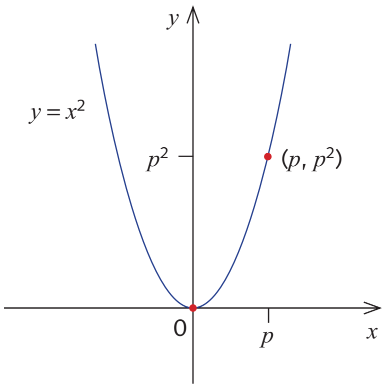
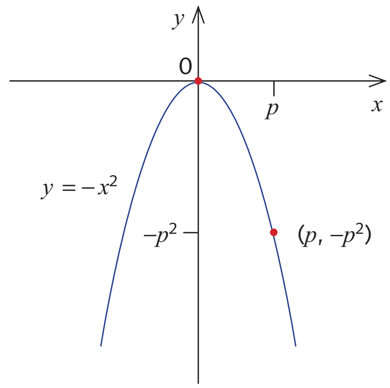
Reflection in the x-axis





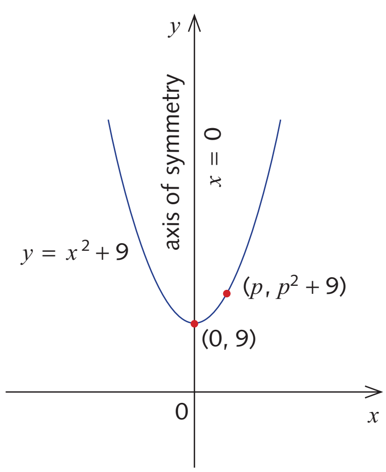
When the parabola $y = x^2$ is reflected in the x-axis, the point (3, 9) is reflected to the point (3, −9). In general, the point (p, p^2) on the parabola $y = x^2$ is reflected to the point $(p, -p^2)$. Hence, the equation satisfied by the points of the image is $y = -x^2$.

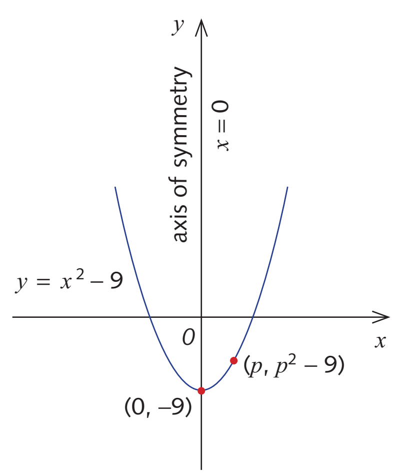
The vertex remains at the origin under the transformation. However, the y-value of the vertex now represents a **maximum turning point** since the y-values at points to the left and right of the origin are less than the y-value at the origin.

Translations of $y = x^2$

Vertical translations

We now look at what happens when we translate $y = x^2$ up or down nine units.





Translating $y = x^2$ nine units up shifts (p, p^2) to $(p, p^2 + 9)$ so the equation becomes $y = x^2 + 9$.

This is a parabola with vertex (0, 9). Similarly, $y = x^2$ becomes $y = x^2 - 9$ when translated nine units down.

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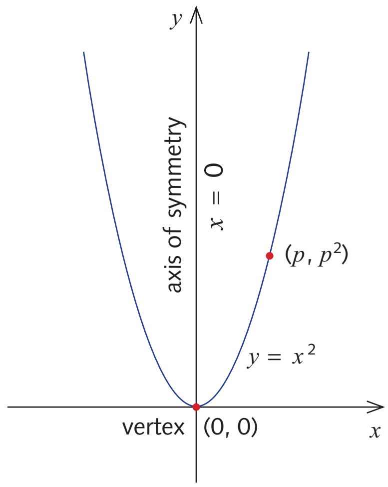
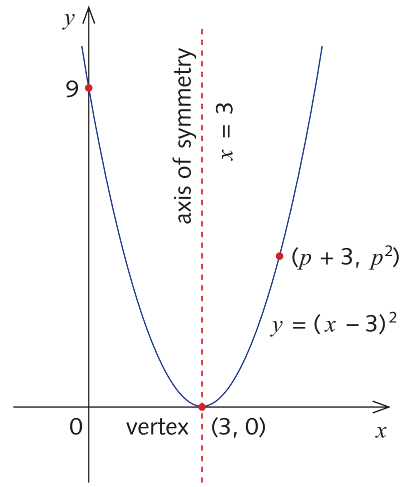
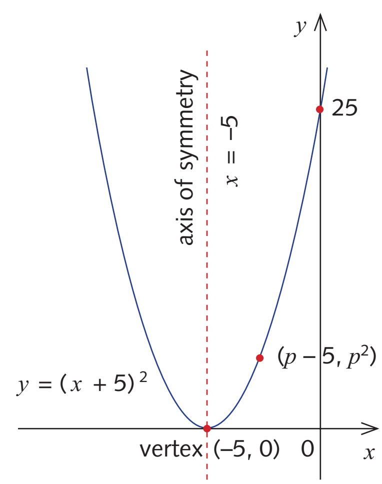
Horizontal translations

What happens when we make horizontal translations; that is, a translation to the left or to the right? This is a little trickier.





Every point on the basic parabola $y = x^2$ has coordinates (p, p^2),as in the first diagram above. If we translate the parabola three units to the right, then the vertex (0, 0) goes to (3, 0), and the axis of symmetry, $x = 0$ goes to $x = 3$. The general point (p, p^2) goes to the point $(p + 3, p^2)$. That is, each point on the image has coordinates $x = p + 3$, $y = p^2$. Eliminating p, $y = p^2 = (x - 3)^2$.

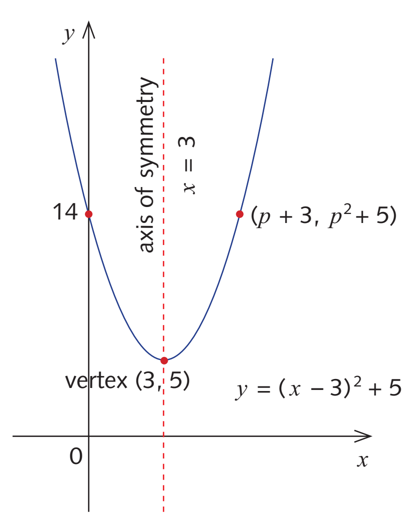


Thus, $y = (x - 3)^2$ is the equation of the parabola formed by translating $y = x^2$ three units to the right, as in the figure above on the right. The y-intercept is 9.

Similarly, translating $y = x^2$ five units to the left shifts the point (p, p^2) to the point $(p - 5, p^2)$. In the same way, we see that the image parabola has equation $y = (x + 5)^2$, as in the diagram opposite. Once again, as a check, the vertex is (–5, 0), which is the image of (0, 0) under this translation. The y-intercept is 25.

In summary, translating $y = x^2$ five units to the left gives the parabola $y = (x + 5)^2$, as in the figure to the right.

General translations



Finally, if we translate $y = x^2$ three units to the right and five units up, then $y = x^2$ becomes $y = (x - 3)^2 + 5$ or $y - 5 = (x - 3)^2$. The y-intercept is 14. Its axis of symmetry is $x = 3$ and its vertex is (3, 5), as in the diagram opposite.

Translating it three units to the right moves $y = x^2$ to $y = (x - 3)^2$ and translating this five units up takes it to $y = (x - 3)^2 + 5$.

In summary, translating $y = x^2$ three units to the right and five units up gives the parabola $y = (x - 3)^2 + 5$, as in the figure to the right.

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Example 1

Sketch each parabola and give the y-intercept, axis of symmetry and vertex.

a $y = (x - 3)^2 - 4$ **b** $y = (x + 2)^2 + 6$

Solution

a $y = (x - 3)^2 - 4$

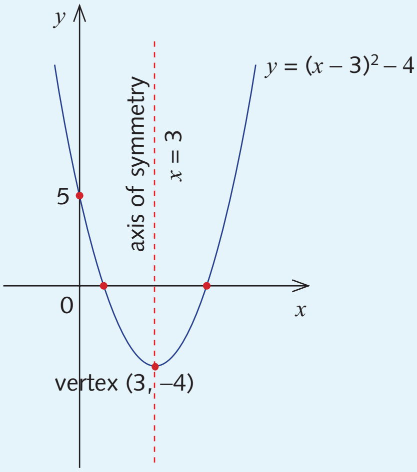
The graph of $y = (x - 3)^2 - 4$ is obtained by translating the graph of $y = x^2$ three units to the right and four units down.

Therefore, the axis of symmetry is $x = 3$.

The vertex is at $(3, -4)$.

When $x = 0$, $y = (-3)^2 - 4 = 5$

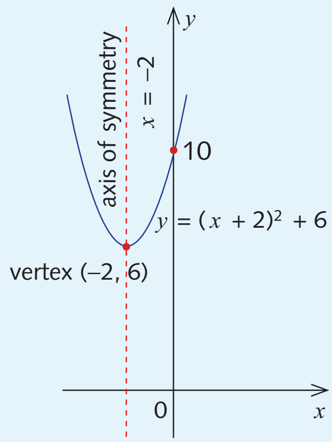
So the y-intercept is 5.



b The graph of $y = (x + 2)^2 + 6$ is obtained by translating the graph of $y = x^2$ two units to the left and six units up.
The axis of symmetry is $x = -2$ and the vertex $(-2, 6)$.

When $x = 0$, $y = (0 + 2)^2 + 6$
$= 10$

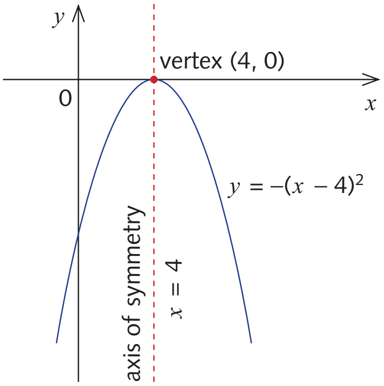
The y-intercept is 10.



Translations of $y = -x^2$

Starting with $y = -x^2$ and translating left or right and up or down, we can construct many examples.

For example, if we translate $y = -x^2$ four units to the right, we obtain the parabola $y = -(x - 4)^2$ with axis of symmetry $x = 4$ and vertex $(4, 0)$. (As a check, $y \le 0$ and only equals 0 when $x = 4$; hence, the maximum value of y occurs when $x = 4$).



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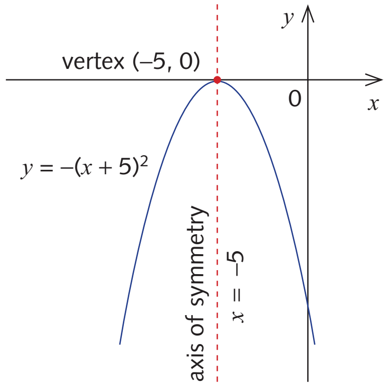
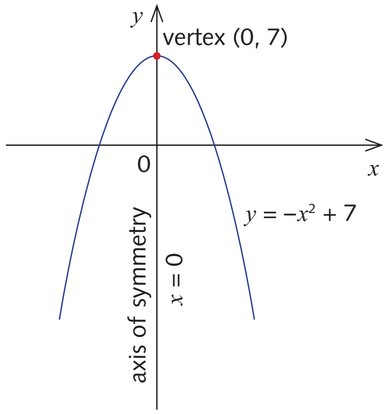
Similarly, translating $y = -x^2$ five units to the left yields $y = -(x + 5)^2$.



Next, if we translate $y = -x^2$ seven units upwards we obtain the parabola $y = -x^2 + 7$, which has axis of symmetry $x = 0$ and vertex (0, 7).



Example 2

Sketch each parabola and give the y-intercept, axis of symmetry and vertex.

a $y = -x^2 - 8$ **b** $y = -(x + 3)^2 - 4$ **c** $y = -(x - 2)^2 + 6$

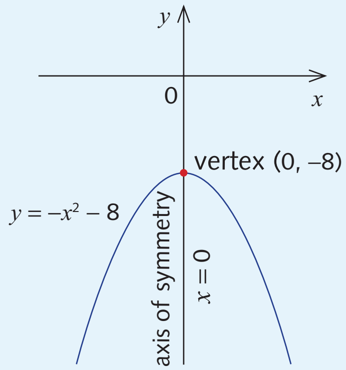
Solution

a $y = -x^2 - 8$

The graph $y = -x^2 - 8$ is obtained from the graph $y = -x^2$ by translating 8 units down.

The axis of symmetry is $x = 0$ and the vertex is $(0, -8)$.

The graph has no x-intercepts.



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b $y = -(x + 3)^2 - 4$

When $x = 0$, $y = -9 - 4$

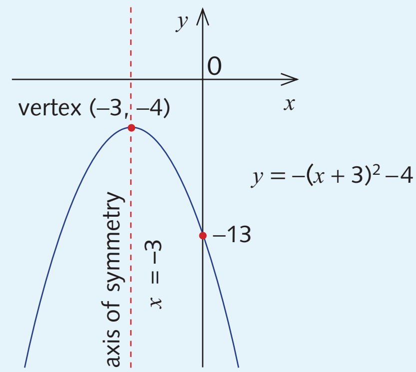
$= -13$

The y-intercept is -13.

The graph $y = -(x + 3)^2 - 4$ is obtained from the graph $y = -x^2$ by translating 3 units to the left and 4 units down.

The axis of symmetry is $x = -3$ and the vertex is $(-3, -4)$.

The graph has no x-intercepts.



c $y = -(x - 2)^2 + 6$

When $x = 0$, $y = -(0 - 2)^2 + 6$

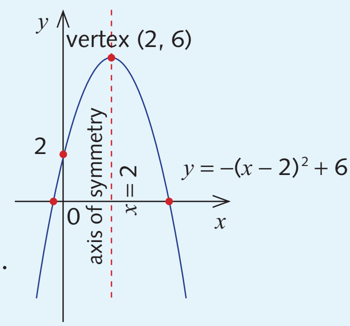
$= -4 + 6$

$= 2$

The y-intercept is 2.

The graph $y = -(x - 2)^2 + 6$ is obtained from the graph $y = -x^2$ by translating 2 units to the right and 6 units up.
The axis of symmetry is $x = 2$ and the vertex is $(2, 6)$.

The graph has two x-intercepts: $2 + \sqrt{6}$ and $2 - \sqrt{6}$.



The parabola

Properties of the parabola $y = x^2$

- The y-axis is the axis of symmetry, called simply the **axis** of the parabola.
- The minimum y-value occurs when $x = 0$.
- The origin is the vertex of the parabola.

Properties of the parabola $y = a(x - h)^2 + k$, $a = 1$ or $a = -1$

- The axis of symmetry is $x = h$, and the vertex is (h, k).
- When $a = 1$, the parabola is 'upright' and k is the minimum y-value ($y \geq k$ for all x)
- When $a = -1$, the parabola is 'upside down' and k is the maximum y-value ($y \leq k$ for all x)

Exercise 7A

1 Find the axis of symmetry and the vertex for:

a $y = (x - 4)^2$ **b** $y = x^2 - 4$ **c** $y = (x - 2)^2 + 6$

d $y = (x + 3)^2 + 7$ **e** $y = (x + 2)^2 + 3$ **f** $y = -x^2 + 9$

g $y = (x - 3)^2 - 4$ **h** $y = (x + 2)^2 - 3$ **i** $y = (x - 6)^2 + 6$

j $y = -(x + 1)^2$ **k** $y = -(x - 2)^2 + 1$ **l** $y = -(x + 3)^2 + 5$

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2 Find the y-intercept of:

a $y = (x - 2)^2 - 7$ **b** $y = (x - 7)^2 - 3$ **c** $y = (x + 1)^2 + 4$

d $y = -(x - 3)^2$ **e** $y = -(x + 2)^2 - 4$ **f** $y = -(x - 2)^2 + 6$

Example 1

3 Sketch the graphs of the following, labelling the y-intercept and vertex.

a $y = (x - 5)^2$ **b** $y = (x - 1)^2 - 3$ **c** $y = (x + 2)^2 + 3$

d $y = (x - 4)^2 - 3$ **e** $y = (x - 1)^2 + 6$ **f** $y = (x - 4)^2 - 4$

Example 2

4 Sketch the graphs of the following, labelling the y-intercept and vertex.

a $y = -x^2 - 7$ **b** $y = -x^2 + 7$ **c** $y = -(x - 3)^2 + 5$

d $y = -(x - 3)^2 - 7$ **e** $y = -(x + 4)^2$ **f** $y = -(x - 6)^2$

g $y = -(x + 4)^2 - 3$ **h** $y = -(x + 3)^2 + 11$ **i** $y = -(x - 1)^2 + 6$

5 Write the equation of the parabola obtained when the basic parabola, $y = -x^2$, is:

a translated 3 units to the right

b translated b units to the left

c translated 6 units down

d translated c units up

6 Write the equation of the parabola obtained when the basic parabola, $y = x^2$, is:

a translated 3 units up and 4 units to the left

b translated 5 units down and 6 units to the right

c translated a units to the right and b units up

d translated c units down and d units to the left

7 Consider the parabola $y = (x + 3)^2 - 8$. Sketch this parabola. What is the equation of the image if it is:

a translated 8 units up and 3 units to the right?

b translated 2 units to the left and 3 units down?

c translated a units to the right and b units up?

8 Consider the parabola $y = (x - 1)^2 + a$. Find the value of a if the y-intercept is:

a 1 **b** 3 **c** 0 **d** –7

9 Consider the parabola $y = -(x - 2)^2 + b$. Find the value of b if the y-intercept is:

a 1 **b** 3 **c** –4 **d** –7

10 Consider the basic parabola $y = x^2$. Draw a sketch of the parabola if it is:

a rotated about the origin through 135° clockwise

b rotated about the origin through 90° anticlockwise

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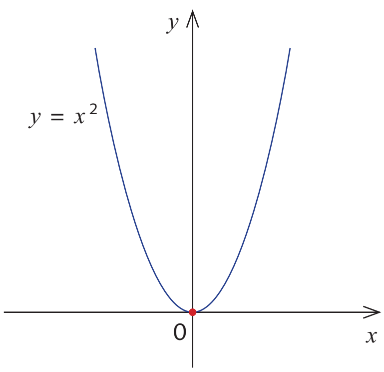
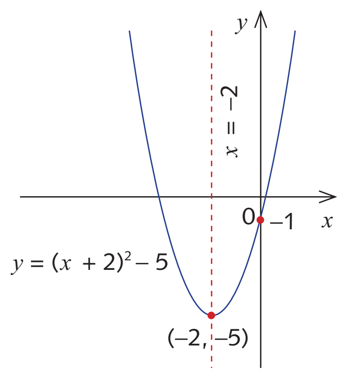
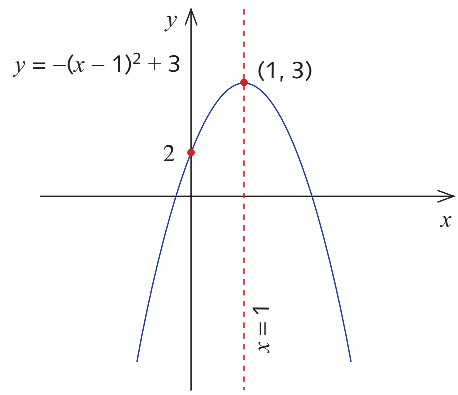
7B Sketching the graph of the quadratic $y = ax^2 + bx + c$, where $a = \pm 1$

In Section 7A we saw that $y = x^2$ becomes:

- $y = (x + 2)^2 - 5$ when translated two units to the left and five units down.
- $y = -(x - 1)^2 + 3$ when reflected in the x-axis and then translated one unit to the right and 3 units up.







By expanding brackets in each of the equations above we obtain an equation of the form $y = ax^2 + bx + c$. Any equation of this form represents a parabola. How do we find its axis of symmetry and its vertex? We do this by putting it into one of the forms above using the method of **completing the square**, which you studied in Chapter 5.

Recall that to complete the square when the coefficient of x^2 is one ($a = 1$), we add and subtract the square of half the coefficient of x. When the coefficient of x^2 is not one, that is, $a \neq 1$ or 0, we first factor a out of the expression, and then multiply through by a at the final step.

Example 3

Complete the square and hence sketch the graph of:

a $y = x^2 + 6x + 13$ **b** $y = -x^2 - 3x - 5$

Solution

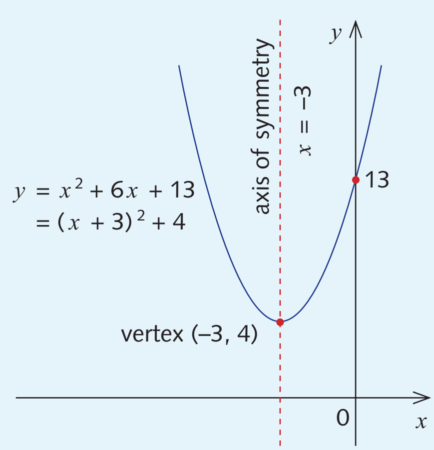
a
$$\begin{aligned} y &= x^2 + 6x + 13 \\ &= (x^2 + 6x + 3^2) + 13 - 3^2 \\ &= (x + 3)^2 + 4 \end{aligned}$$
(Add and subtract the square of half the coefficient of x.)

The axis of symmetry is $x = -3$.

The vertex is $(-3, 4)$.

When $x = 0$, $y = 13$, so the y-intercept is 13.

Note that this parabola has no x-intercepts since the minimum value of y is 4, which is positive. This is the case since $(x + 3)^2 \geq 0$.



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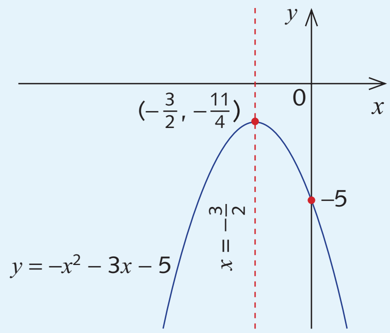
b $y = -x^2 - 3x - 5$

When $x = 0$, $y = -5$, so the y-intercept is -5.

Completing the square:



$$\begin{aligned} y &= -[x^2 + 3x + 5] \\ &= -\left[\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + 5 - \left(\frac{3}{2}\right)^2\right] \\ &= -\left[\left(x + \frac{3}{2}\right)^2 + \frac{11}{4}\right] \\ &= -\left(x + \frac{3}{2}\right)^2 - \frac{11}{4} \end{aligned}$$

So the axis of symmetry is $x = -\dfrac{3}{2}$

and the vertex is $\left(-\dfrac{3}{2}, -\dfrac{11}{4}\right)$.

Note that there are no x-intercepts, since $y \le -\dfrac{11}{4}$ for all x.

When sketching parabolas, the special features are:

- the axis of symmetry
- the vertex
- the y-intercept and the x-intercepts.

Example 4

Sketch the following parabolas. First find the y-intercept, then complete the square to find the axis of symmetry and the vertex of the parabola, then find the x-intercepts if they exist.

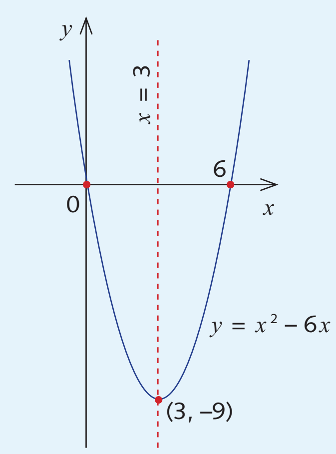
a $y = x^2 - 6x$

b $y = -x^2 - 10x$

Solution

a If $x = 0$, then $y = 0$.

Hence, the y-intercept is 0.



$$\begin{aligned} \text{Also, } y &= x^2 - 6x \\ &= (x^2 - 6x + 9) - 9 \\ &= (x - 3)^2 - 9 \end{aligned}$$

The axis of symmetry is $x = 3$ and the vertex is $(3, -9)$.

When $y = 0$, $x^2 - 6x = 0$

So $x(x - 6) = 0$,

$x = 0$ or $x = 6$

Hence, the x-intercepts are 0 and 6.

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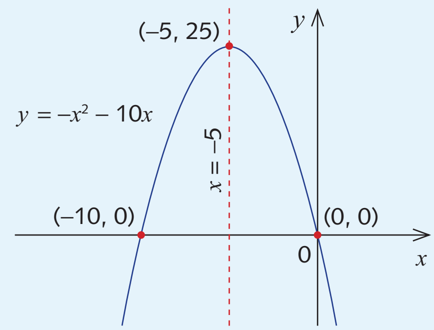
b Consider the equation $y = -x^2 - 10x$.

When $x = 0$, $y = 0$, so the y-intercept is 0.

Completing the square:

$$\begin{aligned} y &= -x^2 - 10x \\ &= -[x^2 + 10x] \\ &= -[(x^2 + 10x + 25) - 25] \\ &= -[(x + 5)^2 - 25] \\ &= -(x + 5)^2 + 25 \end{aligned}$$



Hence, the axis of symmetry is $x = -5$ and the vertex is $(-5, 25)$.

When $y = 0$, $-x^2 - 10x = 0$

so $-x(x + 10) = 0$

$x = 0$ or $x = -10$

Thus, the x-intercepts are $x = 0$ and $x = -10$.

x-intercepts

Some parabolas have x-intercepts and some do not. After completing the square to sketch a parabola, you will know whether or not it has x-intercepts. To find these, substitute $y = 0$ into the quadratic equation and solve the resulting equation for x. There are three ways to do this:

- factorise the quadratic (as in Example 4)
- complete the square
- use the quadratic formula.

These methods were discussed in detail in Chapter 5. Since the sketching technique discussed so far has included completing the square, the second method is usually used.

Example 5

Sketch the following parabolas. First find the y-intercept, then complete the square to find the axis of symmetry and the vertex of the parabola, then find the x-intercepts if they exist.

a $y = x^2 + 6x - 7$ **b** $y = -x^2 + 8x + 13$ **c** $y = x^2 + x + 1$

Solution

a $y = x^2 + 6x - 7$

When $x = 0$, $y = -7$.

The y-intercept is -7.

$$\begin{aligned} y &= x^2 + 6x - 7 \\ &= (x^2 + 6x + 9) - 9 - 7 \\ &= (x + 3)^2 - 16 \end{aligned}$$

(continued over page)

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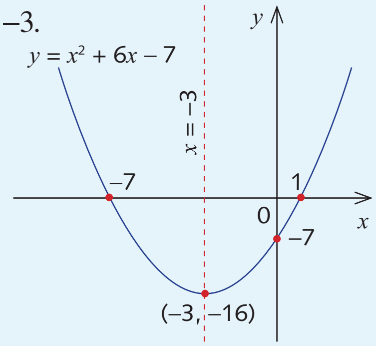
The vertex is at $(-3, -16)$ and the axis of symmetry is $x = -3$.

Substitute $y = 0$, then

$$\begin{aligned}(x+3)^2 - 16 &= 0\\ (x+3)^2 &= 16\\ x + 3 &= 4 \quad \text{or} \quad x + 3 = -4\\ x &= 1 \quad \text{or} \quad x = -7\end{aligned}$$



b The parabola is $y = -x^2 + 8x + 13$.

When $x = 0$, $y = 13$, so the y-intercept is 13.

Completing the square:

$$\begin{aligned}y &= -[x^2 - 8x - 13]\\ &= -[(x^2 - 8x + 16) - 13 - 16] \quad \text{(Complete the square inside the brackets.)}\\ &= -[(x-4)^2 - 29]\\ &= -(x-4)^2 + 29\end{aligned}$$

So the axis of symmetry is $x = 4$, and the vertex is $(4, 29)$.

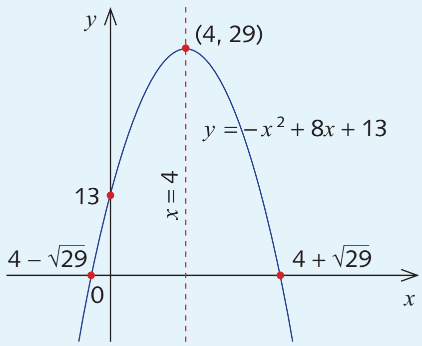
When $y = 0$,

$$\begin{aligned}-(x-4)^2 + 29 &= 0\\ (x-4)^2 &= 29\\ x - 4 &= \sqrt{29} \quad \text{or} \quad x - 4 = -\sqrt{29}\end{aligned}$$

so $x = 4 + \sqrt{29}$, which is positive

or $x = 4 - \sqrt{29}$, which is negative

Thus, the x-intercepts are $4 + \sqrt{29}$ and $4 - \sqrt{29}$.

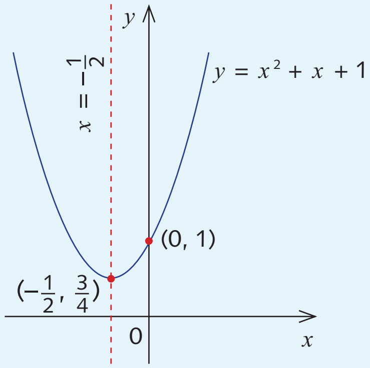


c If $x = 0$, then $y = 1$.

Thus, the y-intercept is 1.

$$\begin{aligned}\text{Also, } y &= x^2 + x + 1\\ &= \left(x^2 + x + \frac{1}{4}\right) + 1 - \frac{1}{4}\\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\end{aligned}$$

The axis of symmetry is $x = -\frac{1}{2}$ and the vertex is $\left(-\frac{1}{2}, \frac{3}{4}\right)$. This graph has no x-intercepts since the minimum value of y is $\frac{3}{4}$, which is positive.



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Graphing $y = ax^2 + bx + c$, where $a = 1$ or $a = -1$

- When $a = 1$, complete the square by adding and subtracting $\left(\frac{b}{2}\right)^2$, to write the quadratic in the form $y = (x - h)^2 + k$.
- When $a = -1$, first factor out -1 before completing the square. Then remove the outer brackets and multiply both terms by -1 to write the quadratic equation in the form $y = -(x - h)^2 + k$.
- The axis of symmetry is $x = h$.
- The coordinates of the vertex can now be read off. They are (h, k). The graph is a translation of $y = x^2$.
- When $a = 1$, k is the minimum y-value ($y \geq k$ for all x) and when $a = -1$, k is the maximum y-value ($y \leq k$ for all x)
- Find the y-intercept by substituting $x = 0$ in $y = ax^2 + bx + c$. Note: The y-intercept will always be $(0, c)$.
- Find the x-intercepts, if they exist, by substituting $y = 0$ and solving the resulting quadratic equation.

Exercise 7B

1 For each parabola:

i determine the y-intercept

ii write down the axis of symmetry and the vertex

iii sketch the parabola

iv determine from the sketch whether the parabola has no x-intercepts, one x-intercept or two x-intercepts

a $y = x^2 + 3$ **b** $y = x^2 - 7$ **c** $y = (x - 2)^2 + 4$

d $y = (x - 3)^2 - 7$ **e** $y = (x + 5)^2$ **f** $y = (x - 7)^2 - 1$

g $y = -(x - 3)^2$ **h** $y = -(x + 1)^2 + 5$ **i** $y = -(x - 2)^2 - 1$

Example 3,4

2 For each parabola:

i determine the y-intercept

ii complete the square

iii write down the axis of symmetry and the vertex

iv sketch the parabola

v determine whether the parabola has no x-intercepts, one x-intercept or two x-intercepts (Do not find the x-intercepts.)

a $y = x^2 - 6x$ **b** $y = x^2 + 6x$ **c** $y = x^2 + 2x - 4$

d $y = x^2 + 4x - 1$ **e** $y = x^2 + 6x - 3$ **f** $y = x^2 + 12x - 4$

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g $y = x^2 - 3x + 5$ **h** $y = x^2 - 5x + 2$ **i** $y = x^2 + 7x$
j $y = -x^2 - 2x$ **k** $y = -x^2 + 8x + 7$ **l** $y = -x^2 + 5x - 7$

3 Find the x-intercepts of the parabolas.

a $y = (x + 1)^2 - 4$ **b** $y = (x + 2)^2 - 9$ **c** $y = -(x - 3)^2 + 25$
d $y = (x - 4)^2 - 7$ **e** $y = (x + 3)^2 - 11$ **f** $y = -(x + 1)^2 + 4$
g $y = (x - 2)^2 - 8$ **h** $y = -(x - 5)^2 + 18$ **i** $y = (x - 3)^2 - 50$
j $y = -(x + 3)^2 + 20$ **k** $y = (x + 2)^2 - 32$ **l** $y = 16 - (x + 1)^2$

Example 5

4 Find the x-intercepts of the parabolas by completing the square.

a $y = x^2 + 2x - 4$ **b** $y = x^2 - 6x + 7$ **c** $y = x^2 - 8x + 13$
d $y = x^2 + 4x - 4$ **e** $y = x^2 + 10x - 11$ **f** $y = x^2 - 20x - 50$
g $y = -x^2 + 12x + 13$ **h** $y = -x^2 + 6x - 4$ **i** $y = -x^2 + 4x + 8$

Example 5

5 For each parabola:

i determine the y-intercept
ii complete the square
iii find the axis of symmetry and the vertex
iv determine the x-intercepts, if any, using the completed square expression
v sketch the parabola, marking all of the above features

a $y = x^2 + 4x - 5$ **b** $y = x^2 + 4x + 5$ **c** $y = x^2 + 6x + 9$
d $y = x^2 - 6x - 7$ **e** $y = x^2 + 8x - 3$ **f** $y = x^2 - x - 2$
g $y = x^2 + 5x + 10$ **h** $y = x^2 + 7x - 3$ **i** $y = x^2 - 2x + 4$
j $y = -x^2 + 4x + 3$ **k** $y = -x^2 + 12x + 4$ **l** $y = -x^2 + 2x - 2$
m $y = -x^2 + x + 1$ **n** $y = -x^2 - 5x - 1$ **o** $y = -x^2 + 11x + 20$

7C The general quadratic $y = ax^2 + bx + c$

Graphing the parabola $y = 3x^2$

Consider the parabola $y = 3x^2$, sketched overpage. Its axis of symmetry is $x = 0$ and its vertex is (0, 0). The graph of $y = x^2$ is sketched on the same set of axes.

Each point on the parabola $y = x^2$ has coordinates (p, p^2) and the matching point on the parabola $y = 3x^2$ is $(p, 3p^2)$. The parabola $y = 3x^2$ is obtained from the parabola $y = x^2$ by stretching by a factor 3 from the x-axis.

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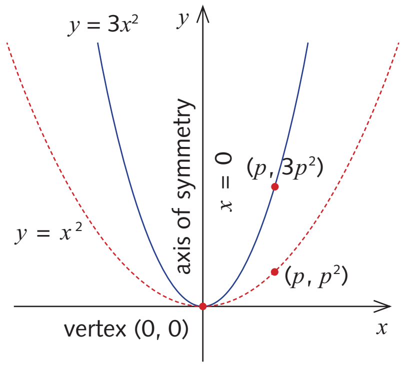
Given our investigations in the earlier sections of this chapter, we would expect that:

- translations of $y = 3x^2$ yield congruent parabolas of the form $y = 3x^2 + bx + c$
- every parabola of the form $y = 3x^2 + bx + c$ can be obtained by translating $y = 3x^2$.

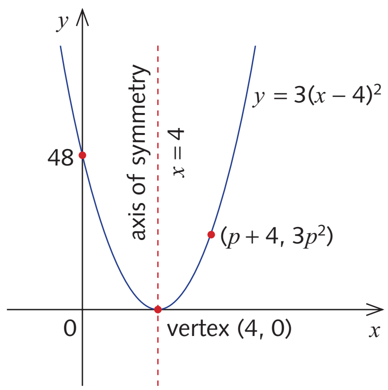
Both these statements are true.



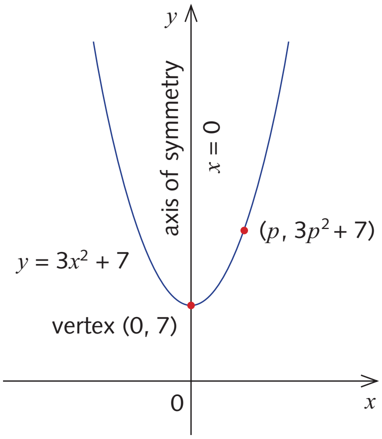
Translations of $y = 3x^2$

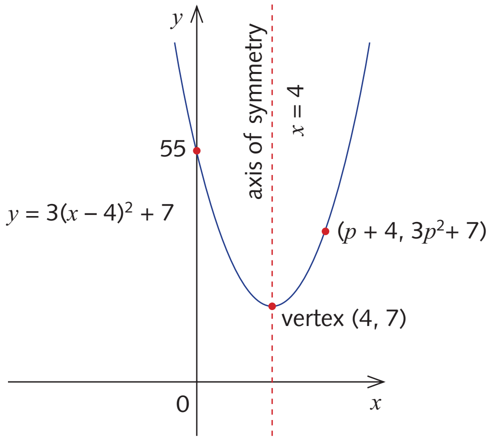
Suppose we translate $y = 3x^2$ four units to the right. The point $(p, 3p^2)$ goes to $(p + 4, 3p^2)$. Hence, we obtain the parabola $y = 3(x - 4)^2$.

The y-intercept is 48. The axis of symmetry is $x = 4$ and the vertex is $(4, 0)$.



Similarly, if we translate $y = 3x^2$ seven units up, then the image is the parabola $y = 3x^2 + 7$ (see left-hand figure below).





If we perform both translations; that is, four units to the right and then seven units up, then the parabola $y = 3x^2$ becomes the parabola $y = 3(x - 4)^2 + 7$ (see the right-hand figure above). The axis of symmetry is $x = 4$ and the vertex is $(4, 7)$.

In summary, translating $y = 3x^2$ four units to the right and seven units up gives the parabola $y = 3(x - 4)^2 + 7$, as in the figure above.

Translations of $y = ax^2$, $a \neq 0$

Clearly the above discussion holds just as well for translations of, for example, $y = 2x^2$, $y = 10x^2$ or $y = \frac{1}{2}x^2$. However, it is equally applicable to $y = -3x^2$, $y = -2x^2$ or $y = -\frac{1}{4}x^2$, where the basic parabola, $y = x^2$, has been not only stretched by a certain factor from the x-axis (3, 2 and $\frac{1}{4}$ respectively), but also reflected in the x-axis. This is shown in Example 6 overpage.

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To complete the square in $y = ax^2 + bx + c$, write:

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

and then complete the square inside the square brackets.

Example 6

Find the y-intercept, the axis of symmetry and the vertex of each parabola by completing the square. Sketch their graphs.

a $y = 2x^2 + 4x + 9$

b $y = 3x^2 + 6x - 13$

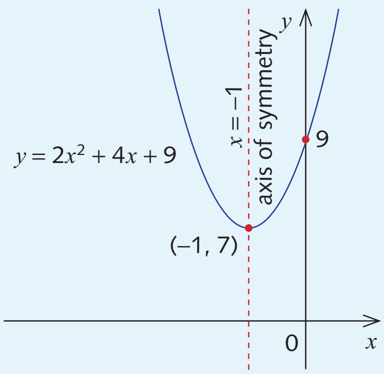
c $y = -2x^2 - 4x - 6$

d $y = -2x^2 - x + 21$

Solution

a Consider the equation $y = 2x^2 + 4x + 9$.

When $x = 0$, $y = 9$, so the y-intercept is 9.



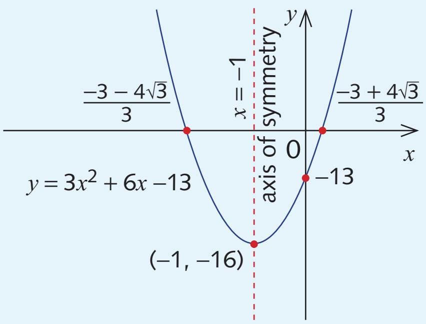
$$\begin{aligned} y &= 2\left[x^2 + 2x + \frac{9}{2}\right] \\ &= 2\left[(x^2 + 2x + 1) + \frac{9}{2} - 1\right] \\ &= 2\left[(x + 1)^2 + \frac{7}{2}\right] \\ &= 2(x + 1)^2 + 7 \end{aligned}$$

The axis of symmetry is $x = -1$ and the vertex is $(-1, 7)$.

Note: This parabola has no x-intercepts since $y \geq 7$ for all x.

b Consider the equation $y = 3x^2 + 6x - 13$.

The y-intercept is -13.



$$\begin{aligned} y &= 3\left[x^2 + 2x - \frac{13}{3}\right] \\ &= 3\left[(x^2 + 2x + 1) - \frac{13}{3} - 1\right] \\ &= 3\left[(x + 1)^2 - \frac{16}{3}\right] \\ &= 3(x + 1)^2 - 16 \end{aligned}$$

The axis of symmetry is $x = -1$ and the vertex is $(-1, -16)$.

We see from the graph that there are x-intercepts. To find them we substitute $y = 0$ and obtain:

$$\begin{aligned} 3(x + 1)^2 &= 16 \\ (x + 1)^2 &= \frac{16}{3} \\ x + 1 &= \frac{4}{\sqrt{3}} \quad \text{or} \quad x + 1 = -\frac{4}{\sqrt{3}} \end{aligned}$$

$x = \dfrac{-3 + 4\sqrt{3}}{3}$, which is positive, or $x = \dfrac{-3 - 4\sqrt{3}}{3}$, which is negative.

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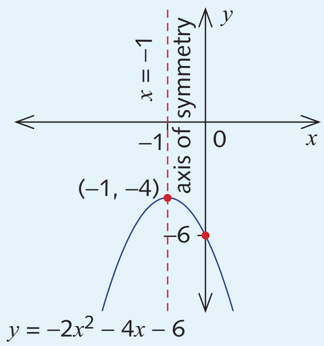
c Consider the equation $y = -2x^2 - 4x - 6$.
The y-intercept is -6.

$$\begin{aligned} y &= -2[x^2 + 2x + 3] \\ &= -2[x^2 + 2x + 1 - 1 + 3] \\ &= -2[(x+1)^2 + 2] \\ &= -2(x+1)^2 - 4 \end{aligned}$$

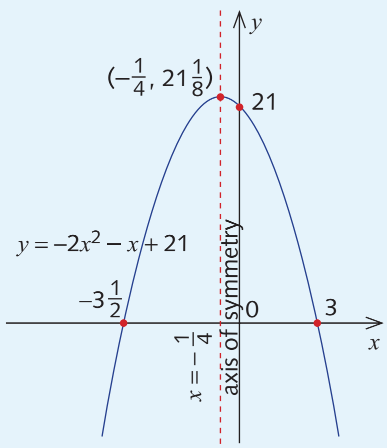
The axis of symmetry is $x = -1$ and the vertex is $(-1, -4)$.

The parabola has no x-intercepts since $y \le -4$ for all x.



d Consider the equation $y = -2x^2 - x + 21$.
The y-intercept is 21.

$$\begin{aligned} y &= -2\left[x^2 + \frac{x}{2} - \frac{21}{2}\right] \\ &= -2\left[\left(x^2 + \frac{x}{2} + \frac{1}{16}\right) - \frac{21}{2} - \frac{1}{16}\right] \\ &= -2\left[\left(x + \frac{1}{4}\right)^2 - 10\tfrac{9}{16}\right] \\ &= -2\left(x + \frac{1}{4}\right)^2 + 21\tfrac{1}{8} \end{aligned}$$



The axis of symmetry is $x = -\frac{1}{4}$ and the vertex is $\left(-\frac{1}{4}, 21\frac{1}{8}\right)$.

$$\begin{aligned} y &= -2x^2 - x + 21 \\ &= -(2x + 7)(x - 3) \end{aligned}$$

When $y = 0$, $x = 3$ or $x = -3\frac{1}{2}$. The axis of symmetry is their average, $x = -\frac{1}{4}$.

(See next section.)

A formula for the axis of symmetry

One thing is clear from some of the previous examples. In practice, completing the square can be technically difficult to carry out and therefore prone to error.

Fortunately, there is a formula for the axis of symmetry of the parabola $y = ax^2 + bx + c$. To derive the formula, we begin by completing the square in the general case:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] \quad \text{(Add and subtract the square of half the coefficient of } x.) \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \end{aligned}$$

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This expression shows that the minimum or maximum of the quadratic occurs when $x = -\dfrac{b}{2a}$.

Hence, we have shown that the axis of symmetry of the parabola $y = ax^2 + bx + c$ is $x = -\dfrac{b}{2a}$.

Once the axis of symmetry is known, then the y-coordinate of the vertex can be determined by substituting $x = -\dfrac{b}{2a}$ into the quadratic. To complete the sketch, find the y-intercept and the x-intercepts if they exist.

We have also shown that when we write the equation of the parabola in the form $y = a(x - h)^2 + k$, then (h, k) is the vertex and $x = h$ is the axis of symmetry.

The general parabola $y = ax^2 + bx + c$

- The parabola $y = ax^2 + bx + c$ is a translation of, and congruent to, the parabola $y = ax^2$.
- To complete the square, first take out a as a factor, $y = a\left[x^2 + \dfrac{b}{a}x + \dfrac{c}{a}\right]$, and then complete the square inside the brackets.
- The equation for the parabola $y = ax^2 + bx + c$ can also be written in the form $y = a(x - h)^2 + k$, where (h, k) is the vertex and $x = h$ is the axis of symmetry.
- The axis of symmetry of the parabola $y = ax^2 + bx + c$ has equation $x = -\dfrac{b}{2a}$. The y-coordinate of the vertex can be found by substitution.

Example 7

Sketch $y = 2x^2 + 8x + 19$ using the formula for the axis of symmetry.

Solution

When $x = 0$, $y = 19$.

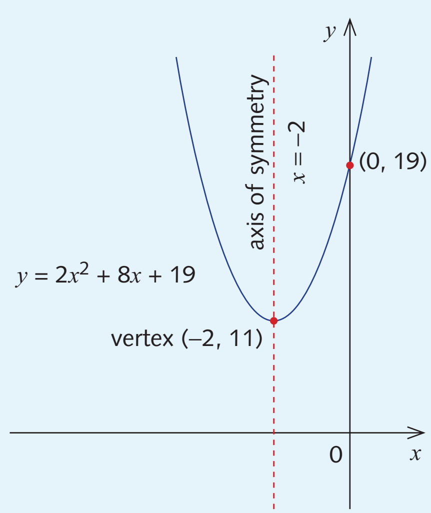
The y-intercept is 19.

The axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{8}{4} = -2$$

To find the vertex, we calculate the y-value when $x = -2$, which gives $y = 8 - 16 + 19 = 11$, so the vertex is $(-2, 11)$.

The graph has no x-intercepts.



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Finding the equation of a parabola given the vertex and one other point

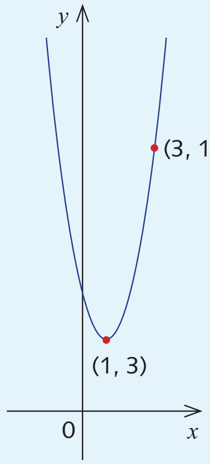
Given the vertex and one other point on a parabola, we can find the equation of the parabola. Since the vertex is (h, k), the equation is $y = a(x - h)^2 + k$. The value of a can be found by substituting in the values of the coordinates of the other point.

Example 8

A parabola has vertex at $(1, 3)$ and passes through the point $(3, 11)$. Find its equation.

Solution

The sketch shows the information that has been given. Since the vertex is at $(1, 3)$, the equation must be of the form $y = a(x - 1)^2 + 3$ for some $a \neq 0$.



Since $(3, 11)$ is on the parabola,

$$\begin{aligned} 11 &= a(3 - 1)^2 + 3 \\ 11 &= a \times 4 + 3 \\ 4a + 3 &= 11 \\ 4a &= 8 \\ a &= 2 \end{aligned}$$

Hence, the equation of the parabola is $y = 2(x - 1)^2 + 3$.

Exercise 7C

1 **a** Sketch the graphs of these parabolas on the one set of axes.

i $y = x^2$ **ii** $y = 3x^2$ **iii** $y = \frac{1}{3}x^2$

b Sketch the graphs of these parabolas on the one set of axes.

i $y = -x^2$ **ii** $y = -3x^2$ **iii** $y = \frac{1}{3}x^2$

c Sketch the graphs of these parabolas on the one set of axes.

i $y = x^2$ **ii** $y = 2x^2$ **iii** $y = 3x^2$

2 Sketch the parabolas. In each case, determine the x- and y-intercepts and the vertex of the parabola.

a $y = 2x^2 + 1$ **b** $y = 4x^2 - 1$ **c** $y = 6x^2 - 1$

d $y = -2x^2 + 8$ **e** $y = -2x^2 + 9$

3 Consider the following parabolas. Determine those that are congruent to each other using translations and reflections in the x-axis.

a $y = 2x^2$ **b** $y = 3x^2$ **c** $y = -5x^2$

d $y = 2x^2 + 7x + 9$ **e** $y = -2x^2 + 5x$ **f** $y = -3x^2 - 7x - 11$

g $y = 2x^2 - 6x$ **h** $y = 5x^2 + 6x + 13$ **i** $y = 7x^2 + 6x + 13$

j $y = x^2$ **k** $y = -x^2 - x + 1$ **l** $y = 5 - 3x^2$

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Example 6

4 Write each parabola in the form $y = a(x - h)^2 + k$.

a $y = 3x^2 + 6x + 3$
b $y = 3x^2 - 9x + 17$
c $y = 2x^2 + 10x - 13$
d $y = 2x^2 + 7x + 3$
e $y = 3x^2 + 5x + 7$
f $y = -5x^2 + 20x + 37$

Example 6

5 For each parabola:

i determine the y-intercept
ii complete the square
iii find the axis of symmetry and the vertex
iv determine any x-intercepts
v sketch the parabola

a $y = 2x^2 + 4x + 3$
b $y = -2x^2 - 4x + 7$
c $y = 3x^2 + 12x - 19$
d $y = 3x^2 + 8x - 2$
e $y = 2x^2 + x - 15$
f $y = 2x^2 - x + 5$

Example 7

6 Use the formula $x = \dfrac{-b}{2a}$ to find the axis of symmetry and vertex of the parabolas. Which of them have x-intercepts?

a $y = x^2 - x + 3$
b $y = 3x^2 + 5x - 13$
c $y = 2x^2 + x + 1$
d $y = 5x^2 + 3x + 7$
e $y = -3x^2 + 5x - 7$
f $y = -x^2 + 3x + 2$

Example 8

7 Consider the parabola $y = 2(x - 1)^2 + k$. Find the value of k if the y-intercept is:

a 8 **b** 10 **c** −6 **d** 0

8 Consider the parabola $y = a(x - 2)^2 - 6$. Find the value of a if the y-intercept is:

a 6 **b** −2 **c** −4 **d** −18

9 Consider the parabola $y = 2(x - h)^2 + 3$. Find the value of h if the y-intercept is:

a 5 **b** 21 **c** 4 **d** 9

Example 8

10 A parabola has vertex $(1, -2)$ and passes through the point $(3, 2)$. Find its equation.

11 A parabola has vertex $(-2, -1)$ and passes through the point $(1, 26)$. Find its equation.

12 A parabola has y-intercept 4 and vertex at $(1, 6)$. Find its equation.

13 Sketch the parabolas. In each case, determine the x- and y-intercepts, the vertex and the axis of symmetry.

a $y = 2(x - 1)^2 + 3$
b $y = -2(x - 1)^2 + 8$
c $y = -4(x - 2)^2 + 12$
d $y = -4(x + 3)^2 + 12$
e $y = 4(x + 2)^2 - 16$
f $y = 2(x + 2)^2 - 12$
g $y = 4 - 2(x - 3)^2$
h $y = 3(x + 1)^2 - 15$
i $y = 5(x + 4)^2 + 1$

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7D Symmetry and factorisation

In this section, we will deal with another method for sketching parabolas. It is based on the fact that if we can locate two points on the parabola that are symmetric with respect to the axis of symmetry, then the axis of symmetry and hence the vertex can be found.

Sketching parabolas in factorised form

Sometimes we are given a quadratic in factorised form. For example, $y = (x - 6)(x - 4)$ or $y = 5(x - 1)(x - 3)$. These parabolas are easy to sketch since the axis of symmetry is simply given by the *average* of the two x-intercepts.

A second easy case is when the parabola is given with the square already completed, for example, $y = -7x^2 + 1$.

Example 9

Factor if necessary, and sketch, marking the intercepts, axis of symmetry and vertex.

a $y = (x - 6)(x - 4)$ **b** $y = 6 + x - x^2$ **c** $y = 5x^2 - 20x + 15$

d $y = -3(x + 5)(x + 7)$ **e** $y = -7x^2 + 1$

Solution

a $y = (x - 6)(x - 4)$

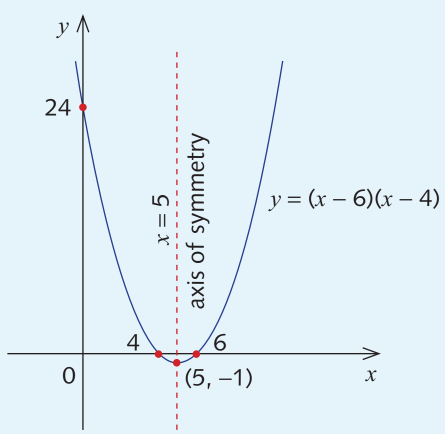
When $x = 0$, $y = 24$, so the y-intercept is 24.

When $y = 0$, $x = 4$ or $x = 6$, so the x-intercepts are 4 and 6.

Taking the average of the x-intercepts, $\frac{4 + 6}{2}$.

Therefore, $x = 5$ is the axis of symmetry.

When $x = 5$, $y = (5 - 6)(5 - 4) = -1$, so the vertex is $(5, -1)$.



b $y = -x^2 + x + 6$

This is an upside-down parabola.

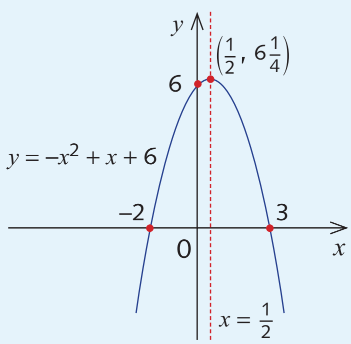
When $x = 0$, $y = 6$, so this is the y-intercept.

$$\begin{aligned} y &= -x^2 + x + 6 \\ &= -(x^2 - x - 6) \\ &= -(x - 3)(x + 2) \end{aligned}$$

So the x-intercepts are 3 and -2.

$\frac{3 + (-2)}{2} = \frac{1}{2}$, therefore the axis of symmetry is $x = \frac{1}{2}$.

When $x = \frac{1}{2}$, $y = -\frac{1}{4} + \frac{1}{2} + 6 = 6\frac{1}{4}$, so the vertex is $\left(\frac{1}{2}, 6\frac{1}{4}\right)$.



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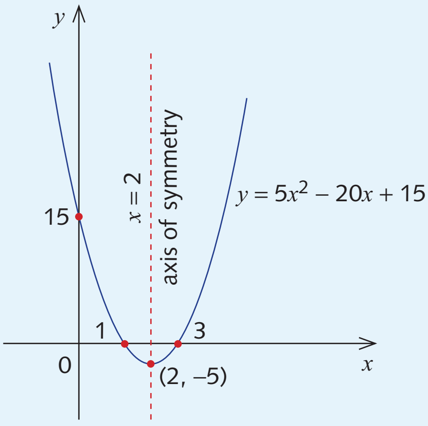
c $y = 5x^2 - 20x + 15$

When $x = 0$, $y = 15$. So the y-intercept is 15.

$$\begin{aligned} y &= 5x^2 - 20x + 15 \\ &= 5(x^2 - 4x + 3) \\ &= 5(x - 3)(x - 1) \end{aligned}$$

The two x-intercepts are $x = 1$ and $x = 3$.

Hence, the axis of symmetry is $x = 2$ and the vertex is $(2, -5)$.

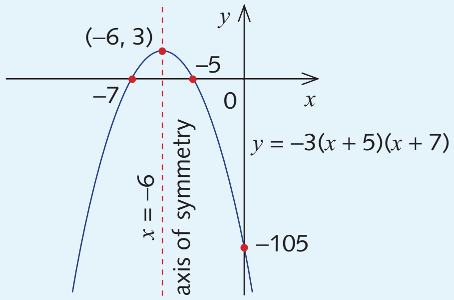


d $y = -3(x + 5)(x + 7)$

The parabola is upside-down with y-intercept -105.

The two x-intercepts are $x = -5$ and $x = -7$.

Hence, the axis of symmetry is $x = -6$ and the vertex is $(-6, 3)$.



e $y = -7x^2 + 1$

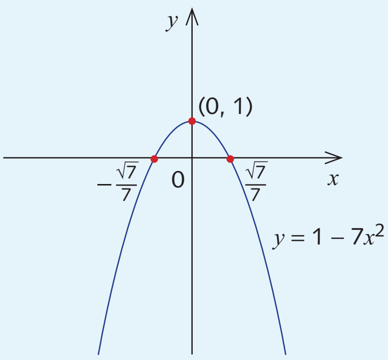
The y-intercept is 1, the axis of symmetry is $x = 0$ and the vertex is $(0, 1)$.

The parabola is upside-down.

When $y = 0$, $1 - 7x^2 = 0$

$$\begin{aligned} x^2 &= \frac{1}{7} \\ x &= \frac{1}{\sqrt{7}} \quad \text{or} \quad x = -\frac{1}{\sqrt{7}} \\ x &= \frac{\sqrt{7}}{7} \quad \text{or} \quad x = -\frac{\sqrt{7}}{7} \end{aligned}$$

So the x-intercepts are $x = -\frac{\sqrt{7}}{7}$ and $\frac{\sqrt{7}}{7}$.



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Finding the equation of a parabola from the x-intercepts

If a parabola has two known x-intercepts at $x = u$ and $x = v$, or exactly one x-intercept at $x = t$, then we know the parabola has the form $y = a(x - u)(x - v)$ or $y = a(x - t)^2$, respectively. The value of a can be determined by substitution if we know the coordinates of some other point on the parabola.

Example 10

A parabola has x-intercepts $x = 5$ and $x = -5$, and y-intercept at 10. Find its equation.

Solution

$(x - 5)$ and $(x + 5)$ are factors. Therefore, $y = a(x - 5)(x + 5)$ for some $a \neq 0$.

When $x = 0$, $y = 10$.

So $10 = a(-5)(5)$

$$a = -\frac{10}{25} = -\frac{2}{5}$$

So $y = -\frac{2}{5}(x - 5)(x + 5)$.

Exercise 7D

1 A parabola has x-intercepts of 1 and 7. What is the x-coordinate of the vertex?

2 A parabola has x-intercepts of -2 and 4. What is the x-coordinate of the vertex?

Example 9

3 Find the x-intercepts, the y-intercept and the vertex of each parabola and sketch it.

a $y = x(x - 4)$ **b** $y = (x - 3)(x - 2)$ **c** $y = 5(x + 1)(x - 3)$

d $y = 2(x + 1)(5 - x)$ **e** $y = 2(x - 5)(x - 6)$ **f** $y = 3(x - 1)(x + 2)$

g $y = 5(x - 4)(x + 2)$ **h** $y = -6(x - 4)(x + 3)$ **i** $y = 7(2x - 1)(x + 1)$

Example 9

4 Factor each quadratic and hence find the x-intercepts.

a $y = x^2 + 6x + 5$ **b** $y = x^2 + 7x + 12$ **c** $y = x^2 - 3x - 18$

d $y = x^2 + 2x - 15$ **e** $y = 2x^2 - 19x - 10$ **f** $y = 16 - x^2$

g $y = 1 - 4x^2$ **h** $y = x^2 - 3x$ **i** $y = 2x^2 + 8x$

Example 9

5 Sketch each parabola, clearly labelling the x- and y-intercepts, the axis of symmetry and the vertex.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 4x + 3$ **c** $y = x^2 + 4x - 12$

d $y = x^2 - 2x$ **e** $y = x^2 + 3x$ **f** $y = 2x^2 - 3x + 1$

g $y = 2x^2 + 7x + 6$ **h** $y = 6x^2 - 7x + 2$ **i** $y = 8x^2 + 6x + 1$

j $y = 9 - x^2$ **k** $y = 3x - 2x^2$ **l** $y = 3 + x - 2x^2$

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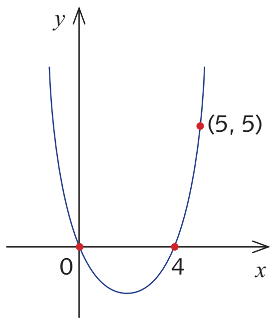
6 Find the equation of each parabola.

a



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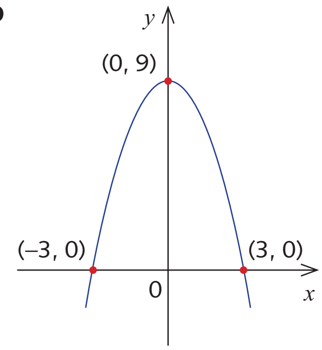

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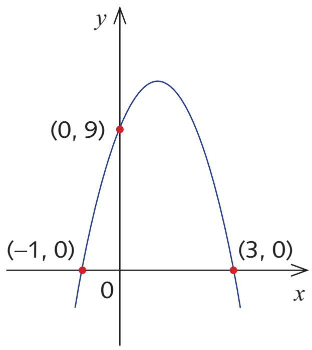
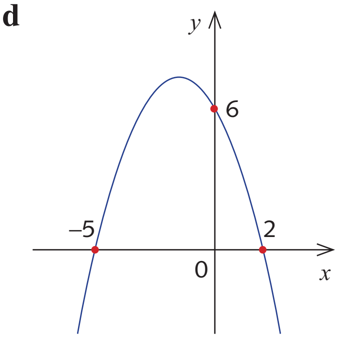


d



7 A parabola has x-intercepts of -1 and 2 and passes through the point $(4, 10)$. Find its equation.

8 A parabola has x-intercepts of -2 and 3 and y-intercept -3. Find its equation.

9 A parabola has x-intercepts of $\frac{1}{2}$ and 2 and passes through the point $(1, -3)$. Find its equation.

7E Sketching via the discriminant

In this section we will discuss how determining the value of the discriminant can assist with parabola sketching.

The discriminant $\Delta = b^2 - 4ac$ and sketching $y = ax^2 + bx + c$

We now have a few approaches to sketching parabolas when given in general form, $y = ax^2 + bx + c$.

- Complete the square to transpose the equation into the form $y = a(x - h)^2 + k$.
- Find the x-intercepts via factorisation and use the symmetry property to find the x-value of the vertex.
- Use the rule $x = -\dfrac{b}{2a}$ to find the x-value of the vertex.

In all three cases we need to label the y-intercept $(0, c)$, and label x-intercepts, if they exist.

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In Chapter 5 we discussed the following techniques for finding x-intercepts (solving equations of the form, $ax^2 + bx + c = 0$).

- Completing the square
- Factorisation
- The quadratic formula

If you are asked to sketch $y = ax^2 + bx + c$, which approach should you take? Knowing the value of the discriminant, $\Delta = b^2 - 4ac$, can assist you in approaching the sketch efficiently. See the table below.

$\Delta = b^2 - 4ac$	Number of x-intercepts
$\Delta < 0$	0
$\Delta = 0$	1
$\Delta > 0$	2

The use of the discriminant is demonstrated in the following example.

Example 11

Sketch the following by whatever means, labelling all key features.

a $y = -3x^2 + 8x - 6$

b $y = 4x^2 - 4x - 3$

c $y = 5x^2 + 3x - 12$

Solution

a $y = -3x^2 + 8x - 6$

$$\begin{aligned}\Delta = b^2 - 4ac &= 8^2 - 4(-3)(-6)\\ &= 64 - 72 = -8\end{aligned}$$

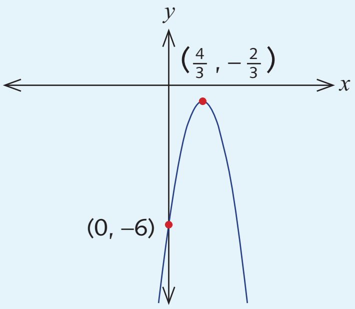
$\Delta < 0$, therefore there are no x-intercepts.

The axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{8}{2(-3)} = \frac{4}{3}$$

The y-value of the vertex is:

$$\begin{aligned}-3\left(\frac{4}{3}\right)^2 + 8\left(\frac{4}{3}\right) - 6 &= -\frac{48}{9} + \frac{32}{3} - 6\\ &= \frac{-48 + 96 - 54}{9}\\ &= -\frac{2}{3}\end{aligned}$$



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b $y = 4x^2 - 4x - 3$

$$\Delta = b^2 - 4ac = (-4)^2 - 4(4)(-3)$$
$$= 16 + 48 = 64$$

$\Delta > 0$ and is the square of a rational number, therefore there are two *rational* x-intercepts.

Find the x-intercepts via factorising:

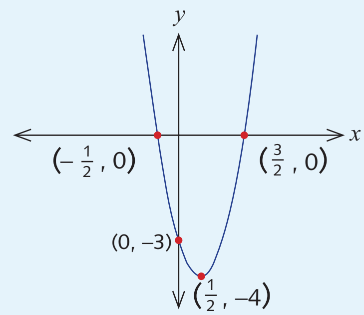
$$4x^2 + 2x - 6x - 3 = 0$$
$$2x(2x + 1) - 3(2x + 1) = 0$$
$$(2x + 1)(2x - 3) = 0$$
$$2x + 1 = 0 \quad \text{or} \quad 2x - 3 = 0$$
$$x = -\frac{1}{2} \quad \text{or} \quad x = \frac{3}{2}$$

Axis of symmetry:

$$x = \left(-\frac{1}{2} + \frac{3}{2}\right) \div 2 = \frac{1}{2}$$

y-value of vertex is:

$$4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) - 3 = 1 - 2 - 3 = -4$$



c $y = 5x^2 + 3x - 12$

$$\Delta = b^2 - 4ac = (3)^2 - 4(5)(-12)$$
$$= 9 + 240 = 249$$

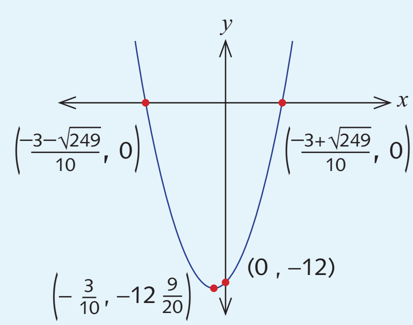
$\Delta > 0$ but is not the square of a rational number, therefore there are two *irrational* x-intercepts.

The axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{3}{2(5)} = -\frac{3}{10}$$

The y-value of the vertex is:

$$5\left(-\frac{3}{10}\right)^2 + 3\left(-\frac{3}{10}\right) - 12 = \frac{45}{100} - \frac{9}{10} - 12$$
$$= -12\tfrac{9}{20}$$



Find the x-intercepts via the quadratic formula: $a = 5$, $b = 3$, $c = -12$

$$x = \frac{-3 - \sqrt{249}}{10} \quad \text{or} \quad x = \frac{-3 + \sqrt{249}}{10}$$

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Exercise 7E

Example 11

For each of the following, determine the value of the discriminant and hence sketch the parabola using the most efficient approach. Label all intercepts and the vertex.

a $y = 4x^2 - 2x$
b $y = 3x^2 - 12x + 12$
c $y = x^2 + 4x + 2$
d $y = 2x^2 - 4x + 5$
e $y = -2x^2 + 6x + 3$
f $y = -x^2 - 4x + 12$
g $y = 3x - 4x^2$
h $y = 4x^2 + 12x + 15$
i $y = 25x^2 - 30x + 9$
j $y = -5x^2 - 4x + 10$
k $y = -4 + x - 2x^2$
l $y = 4x^2 - 4x - 15$

7F Applications involving quadratics

Many practical problems can be solved using quadratics.

For example:

$$s = 30t - 4.9t^2$$

is a formula used to calculate the height, s metres, of a cricket ball t seconds after it has been thrown in the air vertically with an initial speed of 30 m/s.

Example 12 shows a problem about a right-angled triangle that leads to a quadratic equation.

Example 12

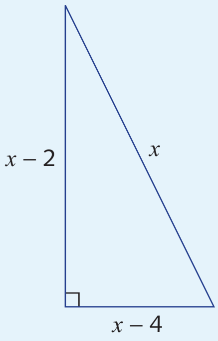
The two sides of a right-angled triangle are, respectively, 2 cm and 4 cm shorter than the hypotenuse. Find the side lengths of the triangle.

Solution

Let x cm be the length of the hypotenuse. Then the two other sides have lengths $(x - 2)$ cm and $(x - 4)$ cm.

By Pythagoras' theorem:

$$\begin{aligned} x^2 &= (x-2)^2 + (x-4)^2 \\ x^2 &= x^2 - 4x + 4 + x^2 - 8x + 16 \\ 0 &= x^2 - 12x + 20 \\ (x-10)(x-2) &= 0 \\ x &= 10 \text{ or } 2 \end{aligned}$$



But the solution $x = 2$ is impossible, since it leads to a triangle with a negative side length.

Hence, $x = 10$ and the side lengths are 6 cm, 8 cm and 10 cm.

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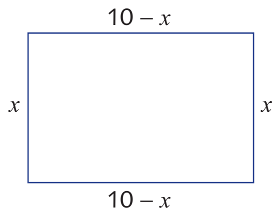
Minimum-maximum problems

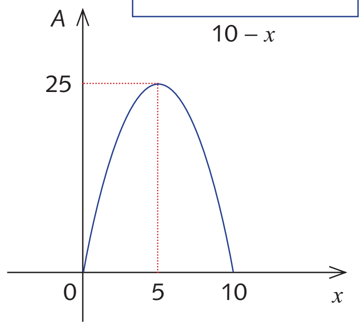
Suppose we have 20 centimetres of wire, which is to be bent to form a rectangle. The area of the rectangle will change as its dimensions change, as you can see in the table below.

Length	2	2.5	4	5	6	7
Width	8	7.5	6	5	4	3
Area	16	18.75	24	25	24	21



Suppose that we have a rectangle with perimeter 20 cm. Let x cm be the length. Then the width is $(10 - x)$ cm.

Suppose that the area is $A\,\text{cm}^2$, then $A = x(10 - x)$
$= 10x - x^2$



The graph of A against x is an upside-down parabola, but note that the x-values are restricted to between 0 and 10.

The x-intercepts are 0 and 10, so the parabola has $x = 5$ as its axis of symmetry. Hence, it has a maximum value of 25 at $x = 5$. This value of x makes the rectangle into a square.

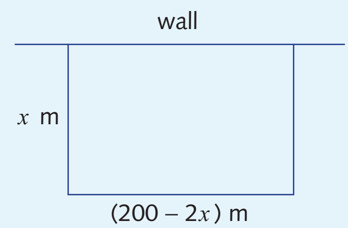
Thus, of all the rectangles with fixed perimeter, the square is the one with greatest area, as you may have noticed from the table of values.

The idea of maximising (or minimising) a quantity using parabolas has many uses.

Example 13

A farmer needs to construct a small rectangular paddock using a long wall for one side of the paddock. He has enough posts and wire to erect 200 m of fence. What are the dimensions of the paddock if the fences are to enclose the largest possible area?

Solution

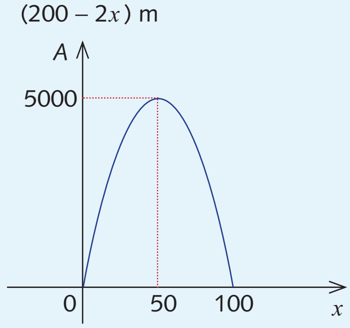


Let x m be the length of the side perpendicular to the wall.

Then the length of the side parallel to the wall is $(200 - 2x)$ m.

Let $A\ \text{m}^2$ be the area of the paddock.

Then $A = x(200 - 2x)$
$= 200x - 2x^2$



Hence, $A = 0$ when $x = 0$ or $x = 100$ (as in the sketch). Taking the average, the axis of symmetry is $x = 50$ and the maximum area occurs when $x = 50$.

Thus, the dimensions of the paddock are 50 m by 100 m, and the area is 5000 square metres.

As we saw in Example 12, the algebra can lead to a number of possible answers. Some of these may not be admissible as they do not satisfy the requirements of the question.

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Exercise 7F

The solutions to most of the practical problems should begin with a diagram.

1. A rectangular paddock is 50 m longer than it is wide. If the area of the paddock is 10 400 m^2, find its dimensions.
2. A large triangular road sign with base length equal to its height has an area of 1800 cm^2. Find the length of the base.
3. The product of two consecutive positive integers is 650. Find the numbers.
4. The product of two consecutive even numbers is 224. Find the numbers.
5. The product of two consecutive odd numbers is 195. Find the numbers.
6. One more than a certain positive number is five less than the number squared. Find the number.
7. If Tom's age is squared, it will be equal to his age in 56 years' time. How old is he now?
8. If the amount of Karlima's savings is squared and then doubled, the amount would be \$66 more than her savings now. How much has she saved?

Example 12

9. A right-angled triangle has one side 7 cm longer than the side perpendicular to it. If the hypotenuse is 17 cm, find the side lengths of the triangle.
10. A right-angled triangle has hypotenuse 9 cm longer than its shortest side. Given that the third side is 21 cm long, find the side lengths of the triangle.
11. A piece of sheet metal 50 cm by 40 cm has squares cut out of each corner so that it can be bent and formed into a lidless box with a base area of 1200 cm^2. Find the dimensions of the box.
12. The formula for finding the number of diagonals of a convex polygon with n sides is $\frac{n}{2}(n-3)$. How many sides does a polygon with 902 diagonals have? (As an interesting counting argument, prove the formula for the number of diagonals of a convex polygon.)
13. Show that the sum of the first n positive integers is $\frac{n(n+1)}{2}$. How many integers are needed to produce a sum of 136?
14. The height (h metres) of an arrow above the ground, t seconds after release from the bow, is given by $h = 23.7t - 4.9t^2$. Find the time taken for the arrow to reach a height of 27 metres, correct to two decimal places.
15. What is the minimum value of $x^2 - 6x + 2$?
16. What is the maximum value of $-x^2 + 3x - 1$?
17. What is the maximum and minimum value of $3x^2 + 7x - 2$ if:

    a $-3 \le x \le 0$? **b** $0 \le x \le 3$?
18. A piece of wire is 100 cm long. Find the dimensions of the rectangle formed by bending this wire when the area is a maximum.

Example 13

19. A farmer has a straight fence along the boundary of his property. He wishes to fence an enclosure for a bull and has enough materials to erect 300 m of fence. What would be the dimensions of the largest possible rectangular paddock, assuming that he uses the existing boundary fence as one of its sides?

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20 The height, h metres, reached by a ball after t seconds when thrown vertically upwards is given by $h = 25t - 4.9t^2$. Find, correct to three decimal places, the maximum height reached and the time the ball is in the air.

21 A rectangular piece of land of area $5000\ \text{m}^2$ is to be enclosed by a wall, and then divided into three equal regions by partition walls parallel to one of its sides. If the total length of the walls is 445 m, calculate the possible dimensions of the land.

22 A rectangle is constructed so that one vertex is at the origin, and another vertex is on the graph of $y = 3 - \frac{2x}{3}$, where $x > 0$ and $y > 0$ and adjacent sides are on the axes. What is the maximum possible area of the rectangle?

7G Quadratic inequalities

In this section, we answer such questions as:

For which values of x is $x^2 - 1 < 0$?

When is $x^2 + 8x + 7 \geq 0$?

Recall the method for solving a linear inequality.

For example:

$$3x + 7 < 5x + 11$$
$$-2x < 4$$
$$x > -2$$

Note: When dividing through by a negative number, the inequality is reversed.

When solving quadratic inequalities, a graphical technique is used. The following examples explore this technique.

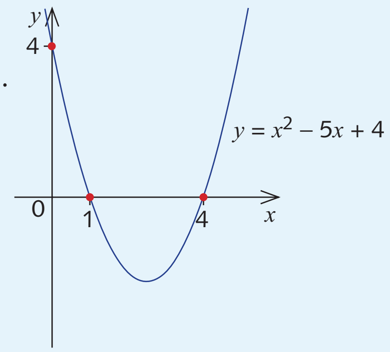
Example 14

a Solve the inequality $x^2 - 5x + 4 < 0$.

b Solve the inequality $x^2 - 5x + 4 \geq 0$.

Solution

The graph of $y = x^2 - 5x + 4 = (x - 4)(x - 1)$ is drawn.



a The y-values are negative when the graph is below the x-axis.
Thus, $x^2 - 5x + 4 < 0$ if $1 < x < 4$.

b The y-values are positive when the graph is above the x-axis.
Thus, $x^2 - 5x + 4 > 0$ if $x > 4$ or $x < 1$.

The y-values are equal to 0 when $x = 4$ or $x = 1$.

Thus, $x^2 - 5x + 4 \geq 0$ if $x \geq 4$ or $x \leq 1$.

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Example 15

a Solve the inequality $-x^2 + 5x - 6 < 0$.

b Solve the inequality $-x^2 + 5x - 6 \geq 0$.

Solution

The graph of $y = -x^2 + 5x - 6 = -(x - 2)(x - 3)$ is drawn.

a The y-values are negative when the graph is below the x-axis.

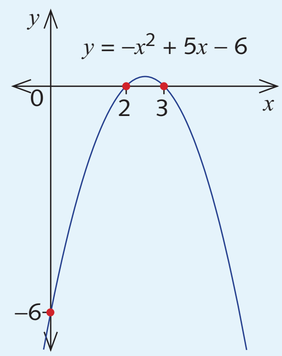
Thus, $-x^2 + 5x - 6 < 0$ if $x < 2$ or $x > 3$.

b The y-values are positive when the graph is above the x-axis.

Thus, $-x^2 + 5x - 6 > 0$ if $2 < x < 3$.

The y-values are equal to 0 when $x = 2$ or $x = 3$.

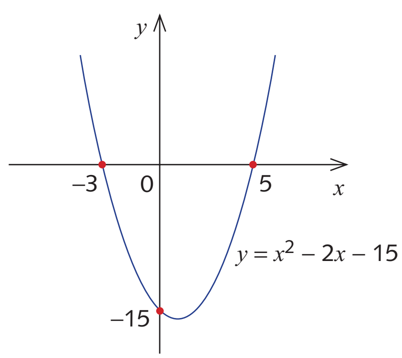
Thus, $x^2 + 5x - 6 \geq 0$ if $2 \leq x \leq 3$.



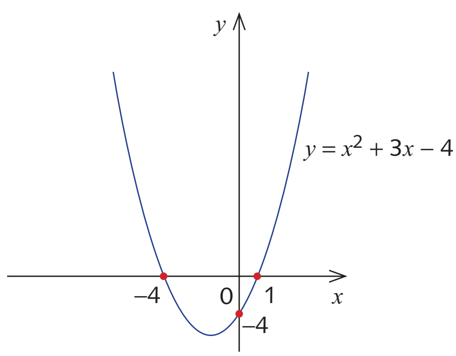
Exercise 7G

1 Use the graphs given to find the set of x-values described by each inequality.

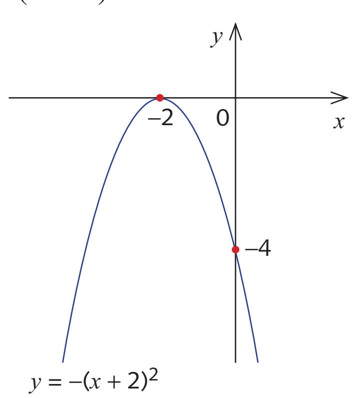
a $x^2 - 2x - 15 < 0$



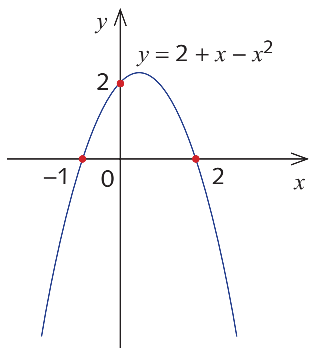
b $x^2 + 3x - 4 > 0$



c $-(x + 2)^2 \geq 0$



d $2 + x - x^2 \leq 0$



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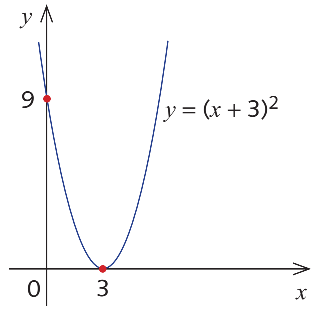
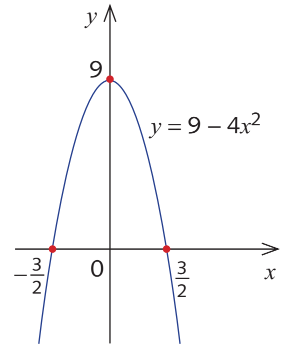
e $(x+3)^2 \geq 0$



f $9-4x^2 < 0$



2 Sketch a graph and find all values of x such that:

a $(x-3)(x+2) > 0$

b $(x+1)(x+4) \leq 0$

c $(x-5)(x-2) \geq 0$

d $x(x+3) < 0$

Example 14

3 Solve the quadratic inequalities.

a $x^2+3x-70 > 0$

b $x^2-5x-24 < 0$

c $x^2+9x+20 \geq 0$

d $x^2-7x+12 \leq 0$

Example 15

4 Solve the quadratic inequalities.

a $-x^2-3x+40 > 0$

b $-x^2+5x+24 \geq 0$

c $-x^2+12x-35 \leq 0$

d $-x^2+11x < 0$

Review exercise

1 Find the y-intercept of:

a $y = x^2+5x+2$

b $y = 5(x-3)^2-21$

c $y = 3x^2+2x$

d $y = 2-2(x+1)^2$

e $y = 5-(x-1)^2$

f $y = -3(x+2)^2-4$

2 Consider the parabola $y=(x-h)^2+5$. Find the value of h if the y-intercept is:

a 5

b 21

c 14

d 9

3 Find the x-intercepts of each parabola.

a $y = x^2+3x-4$

b $y = 2x^2+13x+6$

c $y = 8x^2-6x-9$

d $y = 8x^2-16x-10$

e $y = x^2-49$

f $y = 2x^2-10x$

4 Find the exact values of the x-intercepts of each parabola.

a $y=(x+2)^2-5$

b $y=(x-3)^2-2$

c $y=2(x+1)^2-10$

d $y=3(x-2)^2-15$

e $y=5(x-3)^2-7$

f $y=6-3(x-2)^2$

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5 Find the exact values of the x-intercepts by completing the square.

a $y = x^2 + 4x - 2$

b $y = x^2 - 6x + 1$

c $y = 2x^2 + 10x + 3$

d $y = -2x^2 - 8x + 5$

6 State whether the graph of each quadratic has a maximum or minimum turning point (vertex).

a $y = x^2 + 6x - 5$

b $y = -x^2 + 2x + 1$

c $y = 7 - 2x - 3x^2$

d $y = 3x^2 - 2x + 1$

7 Determine which pairs of parabolas are congruent.

$y = x^2$, $y = -2x^2$, $y = 3x^2$, $y = 3x^2 + 1$,

$y = 2 + 3x - 4x^2$, $y = 3 - 2x^2$, $y = x^2 - x$, $y = 1 + 4x^2$

8 State the transformations that need to be applied to the graph of $y = x^2$ to obtain the graph of:

a $y = x^2 - 1$

b $y = x^2 + 2$

c $y = 4 - x^2$

d $y = 1 - x^2$

Note: There are many possible answers to this question.

9 State the transformations that need to be applied to the graph of $y = x^2$ to obtain the graph of:

a $y = (x + 2)^2$

b $y = (x - 1)^2$

c $y = -(x + 1)^2$

d $y = (x + 1)^2 - 3$

e $y = (x - 2)^2 - 3$

f $y = 1 - (x - 3)^2$

10 Write the equation of the parabola obtained when the graph of $y = x^2$ is:

a translated 2 units to the left

b translated 3 units to the right and 1 unit up

c translated 2 units down and 5 units to the right

d translated 3 units to the left and 2 units down

11 Write the equation of the parabola obtained when the graph of $y = 3x^2$ is:

a translated 3 units to the left and 2 units up

b translated 3 units to the right and 2 units down

12 Write the equation of the parabola obtained when the graph of $y = x^2$ is:

a reflected in the x-axis and translated 1 unit to the right

b reflected in the x-axis and translated 2 units to the left

c reflected in the x-axis, then translated 1 unit to the left and 2 units down

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13 For each parabola, state the coordinates of the vertex.

a $y = (x - 1)^2 + 2$ **b** $y = (x + 2)^2 + 3$ **c** $y = (x + 4)^2 - 2$

d $y = (x - 5)^2 + 11$ **e** $y = -3(x + 2)^2 - 1$ **f** $y = 4 - 2(x - 3)^2$

14 A parabola has vertex $(1, -2)$ and passes through the point $(3, 2)$. Find its equation.

15 A parabola has x-intercepts of -5 and 3 and passes through the point $(1, -12)$. Find its equation.

16 A parabola has x-intercepts of -2 and -4 and a y-intercept of -8. Find its equation.

17 Sketch the graph of each quadratic, clearly labelling the x- and y-intercepts, the axis of symmetry and the vertex.

a $y = x^2 - 6x + 5$ **b** $y = x^2 - 4x - 12$ **c** $y = x^2 - 3x$

d $y = x^2 + 5x$ **e** $y = 16 - x^2$ **f** $y = 3x - 9x^2$

18 Sketch the graph of each quadratic, clearly labelling the x- and y-intercepts, the axis of symmetry and the vertex.

a $y = (x - 3)^2 + 4$

b $y = 3(x + 1)^2 - 6$

c $y = 5 - (x + 3)^2$

d $y = 6 - 3(x - 5)^2$

19 A parabola has vertex $(2, -4)$ and passes through the point $(1, 7)$. Find its equation.

20 A parabola has equation $y = 3(x + h)^2 + 4$ and y-intercept 7. Find the value of h.

21 Sketch the graph of each quadratic, clearly labelling the x- and y-intercepts, the axis of symmetry and the vertex.

a $y = x^2 + 2x - 5$ **b** $y = x^2 - 6x + 2$ **c** $y = -x^2 - 4x - 7$

d $y = -x^2 + 8x - 13$ **e** $y = 2x^2 + 4x + 5$ **f** $y = 7 + 6x - 2x^2$

22 In a right-angled triangle, one side is 7 cm longer than its shortest side and the hypotenuse is 8 cm longer than its shortest side. Find the side lengths of the triangle.

23 A piece of sheet metal 50 cm × 60 cm has squares cut out of each corner so that it can be bent and formed into a lidless box with a base area of 2184 cm^2. Find the length, width and height of the box.

24 A farmer has a straight fence along the boundary of his property. He wishes to fence an enclosure for a bull and has enough materials to erect 500 m of fence. What would be the dimensions of the largest possible paddock, assuming that he uses the existing boundary fence as one of its sides?

25 By considering a graph, solve:

a $(x - 5)(x + 3) < 0$

b $(x + 2)(x + 5) \leq 0$

c $x(x - 2) > 0$

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Challenge exercise

1 **a** What is the maximum value of $2x^2 + 9x - 5$ if $-2 \le x \le 0$?

b What is the minimum value of $2x^2 + 9x - 5$ if $0 \le x \le 2$?

2 Consider the quadratic inequality $x^2 + 4x + c \le 0$. For each of the following sets of values of x, find the values of c for which the given set satisfies the inequality:

a $-7 \le x \le 3$

b $x = -2$

c no x values

3 The distance between two towns is 120 km by road and 150 km by rail. A train takes 10 minutes longer than a car, whose average speed is 10 km/h less than the train's average speed.

The purpose of this problem is to find the average speed of the car.

a Let the average speed of the car be x km/h and let the time taken by the car be t hours. Show that the information in the question gives:

$$xt = 120 \qquad (1)$$

$$(x+10)\left(t+\frac{1}{6}\right) = 150 \qquad (2)$$

b Subtract (1) from (2) to obtain a linear equation linking x and t.

c Make t the subject of this linear equation, substitute it into (1) and solve for x, obtaining $x = 80$ or $x = 90$.

d Calculate the corresponding values of t and check that both pairs of solutions make sense.

The next six questions are similar to the previous question. That is, it is best to introduce two variables, eliminate one and then solve the resulting quadratic equation. Do not forget to check that the solutions are feasible; that is, that they make sense and satisfy the original problem.

4 A train could save 1 hour on a journey of 200 km by increasing its average speed by 10 km/h. What is the original speed of the train?

5 A farmer purchased a number of cattle for $3600. Five of them died, but he sold the remainder at $20 per head more than he paid for them, making a profit of $400. How many did he buy?

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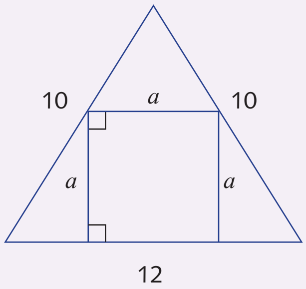
6 The distance between two towns is 80 km by road and 90 km by rail. A car takes 15 minutes longer than a train, whose average speed is 8 km/h greater than the car's average speed. Find the average speed of the car and of the train.

7 In cricket, batting average $= \dfrac{\text{total number of runs scored}}{\text{number of times out}}$. In a season, a cricketer scored 1800 runs. If he had been out on one more occasion, his average would have been three runs less. What is his average?

8 Two boys, one of whom can run 1 m/s faster than the other, compete in a 400 m race. The slower competitor is given a 20 m start and loses by 10 seconds. What was the average speed of each runner (correct to three decimal places)?

9 A and B are two towns, 120 km apart. A car starts from A to travel to B at the same time as a second car, whose speed is 20 km/h faster than the first, starts from B to travel to A. The slower car reaches B 1 hour and 48 minutes after it passes the other car. Find their speeds.

10 The diagram shows a square inscribed in an isosceles triangle with side lengths 10, 10 and 12. Find a.



11 To solve $x^2 - gx + h = 0$ graphically, let A be the point $(0, 1)$ and B the point (g, h). Draw a circle with AB as its diameter. Then the points (if any) where the circle cuts the x-axis are the roots of $x^2 - gx + h = 0$.

a Illustrate the method by graphically solving $x^2 - 5x + 6 = 0$.

b Prove that the method works.

Note: This construction is called Carlyle's method.

12 Take a piece of string of length 100 cm. Cut it into two pieces, x cm and $(100 - x)$ cm, and form the first piece into a circle and the other into a square.

a Write down a quadratic expression for the combined area enclosed by the separate pieces.

b Find the minimum possible sum of the two areas and the value of x for which it occurs.

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13 Recall that two geometric figures are by definition *congruent* if there is a sequence of translations, rotations and reflections taking one figure to the other. Also recall that two geometric figures are *similar* if we can enlarge one figure so that its enlargement is congruent to the other figure.

In this question we will show that all parabolas are similar. It is not, however, true that all parabolas are congruent.

a Explain why the ideas in Section 7B show that every parabola $y = x^2 + ax + b$ is congruent to the basic parabola $y = x^2$.

b Explain why the ideas in Section 7B show that every parabola $y = -x^2 + ax + b$ is congruent to the basic parabola $y = x^2$.

c Let $a > 0$. Explain why the ideas in Section 7C show that every parabola $y = ax^2 + bx + c$ and every parabola $y = -ax^2 + bx + c$ is congruent to the parabola $y = ax^2$.

d Every point on $y = x^2$ has coordinates (p, p^2) for some p. Find a similar expression for the points on $y = 5x^2$. Show that the transformation taking (x, y) to $(x, 5y)$ maps $y = x^2$ to $y = 5x^2$. Show that this transformation is not a similarity transformation.

e Show that there is an enlargement that takes $y = x^2$ to $y = 5x^2$.

f Show that all parabolas are similar.

14 In Section 7D we discussed methods for sketching parabolas using symmetry about the axis of symmetry. Here is another method.

For the parabola $y = ax^2 + bx + c$:

First find the two points where $y = c$ meets the parabola. These are $(0, c)$ and $\left(-\frac{b}{a}, c\right)$.

Then find the vertex, knowing that the x-coordinate of the vertex is the average of the x-coordinates 0 and $-\frac{b}{a}$. Sketch the parabola using these three points. Use this method to sketch:

a $y = x^2 + 8x + 17$

b $y = 2x^2 + 5x - 3$

c $y = dx^2 + ex - f$

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