CHAPTER

9

Number and Algebra

Indices, exponentials and logarithms – part 1

You often hear people talk about 'exponential growth' or 'exponential decay', generally in connection with business, investment, ecology and science. This chapter will explain what these terms mean.

In *ICE-EM Mathematics Year 9*, you learned how to graph parabolas such as $y = x^2$ and $y = 3x^2 - 4$. In this chapter, you will learn what the exponential and logarithm functions are, and how to draw their graphs.

ISBN 978-1-108-40434-1

Cambridge University Press
Updated June 2022

9A Review of powers and integer indices

In *ICE-EM Mathematics Year 9*, you learned that a number such as 2 could be raised to any integer power, so that:

$$2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad \ldots$$

and

$$2^{-1} = \frac{1}{2}, \quad 2^{-2} = \frac{1}{4}, \quad 2^{-3} = \frac{1}{8}, \quad \ldots$$

In the statement $2^5 = 32$, we call 2^5 a **power**, we call 2 the **base** and we call 5 the **index** or the **exponent**.

In general, if a is any number and n is a positive integer, we define a^n to be the product of n factors of a, and we define:

$$a^{-n} \text{ to be } \frac{1}{a^n},$$

provided a is non-zero.

Also, we define:

$$a^0 = 1$$

All of the index laws follow directly from these definitions. It is important to be able to recall and use these laws. In this chapter, we will use the index laws repeatedly.

Index laws

Recall that if m and n are integers and a and b are any non-zero numbers:

Index law 1 $\quad a^m a^n = a^{m+n}$

Index law 2 $\quad \dfrac{a^m}{a^n} = a^{m-n}$

Index law 3 $\quad (a^m)^n = a^{mn}$

Index law 4 $\quad (ab)^n = a^n b^n$

Index law 5 $\quad \left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Example 1

a Evaluate:

i 7^2 **ii** 2^8

b Write each number in index form with a prime-number base.

i 128 **ii** 343 **iii** $\dfrac{1}{25}$

Solution

a **i** $7^2 = 49$ **ii** $2^8 = 256$

b **i** $128 = 2^7$ **ii** $343 = 7^3$ **iii** $\dfrac{1}{25} = 5^{-2}$

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Example 2

a Simplify each expression.

i $x^7 \times x^2 \times x^3$ **ii** $x^2z^3 \times x^7z^2$ **iii** $2a^2b \times 7a^3b^2$

b Simplify each quotient.

i $\dfrac{a^3b^7}{ab^2}$ **ii** $\dfrac{60a^3b^2}{5a^2b}$

c Simplify $(a^2)^3 \times a^4$.

Solution

a **i** $x^7 \times x^2 \times x^3 = x^{12}$ **ii** $x^2z^3 \times x^7z^2 = x^9z^5$ **iii** $2a^2b \times 7a^3b^2 = 14a^5b^3$

b **i** $\dfrac{a^3b^7}{ab^2} = a^2b^5$ **ii** $\dfrac{60a^3b^2}{5a^2b} = 12ab$

c $$\begin{aligned}(a^2)^3 \times a^4 &= a^6 \times a^4\\ &= a^{10}\end{aligned}$$

Example 3

Simplify these expressions.

a $(x^2y^3)^4$ **b** $(2m^2)^3 \times (3m)^3$ **c** $\dfrac{(a^2b^3)^4}{(ab^2)^3}$

Solution

a $(x^2y^3)^4 = x^8y^{12}$

b $$\begin{aligned}(2m^2)^3 \times (3m)^3 &= 8m^6 \times 27m^3\\ &= 216m^9\end{aligned}$$

c $$\begin{aligned}\frac{(a^2b^3)^4}{(ab^2)^3} &= \frac{a^8b^{12}}{a^3b^6}\\ &= a^5b^6\end{aligned}$$

Here are two useful facts:

- $\left(\dfrac{a}{b}\right)^{-1} = \dfrac{b}{a}$, since $\dfrac{a}{b} \times \dfrac{b}{a} = 1$
- Similarly, $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

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Example 4

Evaluate:

a $\left(\frac{4}{7}\right)^{-1}$ **b** 4^{-3} **c** 10^{-3} **d** $5a^0$ **e** $\left(\frac{2}{3}\right)^{-3}$

Solution

a $\left(\frac{4}{7}\right)^{-1} = \frac{7}{4}$

b $4^{-3} = \frac{1}{4^3}$
$= \frac{1}{64}$

c $10^{-3} = \frac{1}{10^3}$
$= \frac{1}{1000}$

d $5a^0 = 5 \times 1$
$= 5$

e $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3$
$= \frac{27}{8}$

Example 5

Simplify these products, expressing each pronumeral in the answer with a positive index.

a $a^{-4} \times a^{-6}$ **b** $2a^4 \times 5a^{-6}$ **c** $(m^{-3}n^{-5})^4 \times (m^{-7}n^3)^{-5}$

Solution

a $a^{-4} \times a^{-6} = a^{-10}$
$= \frac{1}{a^{10}}$

b $2a^4 \times 5a^{-6} = 10a^{-2}$
$= \frac{10}{a^2}$

c $(m^{-3}n^{-5})^4 \times (m^{-7}n^3)^{-5} = m^{-12}n^{-20} \times m^{35}n^{-15}$
$= \frac{m^{23}}{n^{35}}$

Exercise 9A

Example 1a

1 Evaluate:

a 4^2 **b** 5^3 **c** 2^6 **d** 3^3 **e** 10^4 **f** 6^3

Example 1bi, ii

2 Write each number in index form with a prime number base.

a 8 **b** 64 **c** 81 **d** 32 **e** 625 **f** 243

Example 1biii

3 Write each number in index form with a prime number base.

a $\frac{1}{13}$ **b** $\frac{1}{49}$ **c** $\frac{1}{13^3}$ **d** $\frac{1}{1024}$ **e** $\frac{9}{729}$ **f** $(121)^{-5}$

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Example 2a

4 Simplify each expression.

a $a^4 \times a^6 \times a^5$ **b** $a^7 \times a^3 \times a$ **c** $m^4 \times m^3 \times m^8$

d $p^4 \times p^5 \times p^2$ **e** $a^2b \times a^4b^6$ **f** $m^4n^2 \times m^5n^4$

g $2a^4b^3 \times 4ab^2$ **h** $3x^3y \times 5x^2y^3$ **i** $3x^3y^7 \times 5x^5y^2$

Example 2b

5 Simplify each quotient, expressing each pronumeral in the answer with a positive index.

a $\frac{x^2y^3}{xy^4}$ **b** $\frac{30x^2y^4}{20xy^2}$ **c** $\frac{a^6m^4}{a^2m}$

d $\frac{25x^4y^6}{20x^3y^5}$ **e** $\frac{8ab^3c^4}{12ab^5c^2}$ **f** $\frac{16x^3y^6z^8}{4y^7z^8}$

Example 3a

6 Simplify:

a $(a^2b^4)^3$ **b** $(x^3y^5)^7$ **c** $(ab^2c^3d^4)^5$

d $(2a^3b)^2$ **e** $(3a^2b^4)^2$ **f** $(4a^3b^2)^3$

Example 3b, c

7 Simplify each expression, writing each pronumeral in the answer with a positive index.

a $(3m^3)^2 \times 2m^6$ **b** $(2p^5)^2 \div (4p^6)$ **c** $\frac{(a^2b)^3}{ab^4} \times \frac{a^2b^5}{ab}$

d $\frac{m^4n^2}{mn^3} \div \frac{(mn^2)^3}{m^5n^8}$ **e** $\frac{a^4b^6}{(ab^2)^2} \times \frac{a^5b}{a^2b}$ **f** $\frac{(p^4q)^2}{pq^3} \div \frac{pq}{(p^2q)^3}$

Example 4

8 Evaluate:

a 2^{-1} **b** 2^{-2} **c** 3^{-1} **d** 3^{-2} **e** 10^{-3}

f $\left(\frac{7}{8}\right)^{-1}$ **g** $\left(\frac{15}{14}\right)^{-1}$ **h** $\left(\frac{3}{5}\right)^{-2}$ **i** $\left(\frac{2}{3}\right)^{-3}$ **j** $\left(\frac{5}{11}\right)^{-3}$

k 3^{-4} **l** $5 + a^0$ **m** $\frac{4a^0}{(5b)^0}$ **n** $(2 + a)^0$ **o** $(4^3)^0$

9 Simplify each expression, writing each pronumeral in the answer with a positive index.

a $(2x^2y)^{-1}$ **b** $(3x^2y^2)^{-2}$ **c** $(4xy^{-1})^{-3}$

d $(2x^2y^{-2})^{-3}$ **e** $(3x^{-2}y^{-2})^{-3}$ **f** $(2x^5y^5)^2$

Example 5

10 Simplify each expression, writing each pronumeral in the answer with a positive index.

a $m^6 \times m^{-2} \times m^{-5}$ **b** $2a^{-1}b^3 \times 4a^{-3}b^{-6}$ **c** $5p^2q^{-1} \times 3pq^{-4}$

d $\frac{15p^4q^{-2}}{10p^{-7}q^4}$ **e** $\frac{5x^{-2}y^{-3}}{10x^4y^{-4}}$ **f** $(2x^{-1})^{-4}$

g $(2a^{-1}b^3)^{-2} \times 4(a^2b)^{-3}$ **h** $(m^{-2}n^3)^4 \times (m^{-5}n^2)^{-3}$ **i** $\left(\frac{m^2n^{-1}}{p^4}\right)^{-2}$

j $\left(\frac{a^{-1}b^4}{c^{-1}}\right)^{-2}$ **k** $\frac{2a^{-1}b^2}{a^3b^{-2}} \times \frac{4a^6b^{-1}}{6ab^{-2}}$ **l** $\frac{m^2n^{-3}}{m^4n^2} \times \frac{(mn^2)^{-3}}{m^4n^6}$

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m $\dfrac{x^4y^{-1}}{(x^2y)^{-3}} \div \dfrac{xy^2}{(xy)^{-2}}$ **n** $\dfrac{a^{-6}b^4}{(a^2b)^{-3}} \div \dfrac{(a^2b)^{-1}}{ab^3}$ **o** $\dfrac{(2a^4b^{-2})^3}{c^2} \times \dfrac{(2^2a^{-3}b^2)^{-1}}{c}$

p $\dfrac{(m^2n^3)^2}{p^{-3}} \times (mnp^{-2})^{-3}$ **q** $\dfrac{(a^2)^3}{b^3} \div \left(\dfrac{a}{b^2}\right)^{-2}$ **r** $\dfrac{(2a^4)^2}{b^7} \div \dfrac{(a^2)^{-3}}{2b}$

11 Calculate:

a $\dfrac{3^{-1} + 3^{-2}}{3 + 3^2}$ **b** $\dfrac{2^{-2} + 2^{-4}}{2^2 + 2^4}$ **c** $\dfrac{2^{-2} - 2^{-4}}{2^2 - 2^4}$

12 Calculate $\dfrac{2^{-1} + 2^{-2} + 2^{-3}}{2 + 2^3 + 2^4}$.

13 If $x = 1$, find the value of $3^x + 3^{1-x} + 3^{x-2}$.

14 Simplify each expression, writing each pronumeral in the answer with a positive index.

a $\dfrac{x - y^{-1}}{x^{-1} - y}$ **b** $(x^{-1} + y^{-1})(x^{-1} - y^{-1})$ **c** $\dfrac{3xy}{x^{-1} + y^{-1}}$

d $\dfrac{x^{-1} + y^{-1}}{x^{-2} + y^{-2}}$ **e** $(x^{-2} + y)^{-2}$ **f** $(x^{-2} + y^{-1})^{-1}$

9B Scientific notation and significant figures

Many mathematical problems have exact answers, such as $\dfrac{13}{7}$, $\sqrt{2} + \sqrt{3}$ or 400π. However, in the real world, very large numbers and very small numbers are common and, nearly always, these can only be determined approximately. To express large and small numbers conveniently, we use **scientific notation**, also known as **standard form**.

In science, whenever we measure something it is an approximation. Scientific notation and **significant figures** are useful in expressing these numbers. To deal with approximations we use significant figures.

Scientific notation or standard form

By definition, a positive number is in **scientific notation** if it is written as:

$a \times 10^b$, where $1 \le a < 10$ and b is an integer

This notation is also called the **standard form** for a number. In contrast, for example, 2345.6789, is called the **decimal notation** for that number.

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Example 6

Write each number in scientific notation.

a 2100 **b** 0.0062 **c** 764 000 000 **d** 0.000 000 2345

Solution

a $2100 = 2.1 \times 10^3$

b $0.0062 = 6.2 \times 10^{-3}$

c $764\,000\,000 = 7.64 \times 10^8$

d $0.000\,000\,2345 = 2.345 \times 10^{-7}$

Note: If the number is greater than 1, then the exponent of 10 is positive or zero when the number is written in scientific notation. If the number is positive and less than 1, then the exponent is negative.

When the number is written in scientific notation, the exponent records how many places the decimal point has to be moved to the left or right to produce the decimal notation.

Example 7

Write each number in decimal notation.

a 7.2×10^3 **b** 5.832×10^{-2} **c** 3.61×10^5

Solution

a $7.2 \times 10^3 = 7200$ **b** $5.832 \times 10^{-2} = 0.058\,32$ **c** $3.61 \times 10^5 = 361\,000$

Example 8

Evaluate each expression without using a calculator. Give your answers in scientific notation.

a $(4 \times 10^4) \times (2.1 \times 10^3)$

b $\dfrac{6.3 \times 10^5}{7 \times 10^6}$

c $(1.5 \times 10^5)^2 \times (9.0 \times 10^{-12})$

Solution

a
$$\begin{aligned}(4 \times 10^4) \times (2.1 \times 10^3) &= 4 \times 2.1 \times 10^4 \times 10^3 \\ &= 8.4 \times 10^7\end{aligned}$$

b
$$\begin{aligned}\frac{6.3 \times 10^5}{7 \times 10^6} &= 6.3 \div 7 \times 10^5 \div 10^6 \\ &= 0.9 \times 10^{-1} \\ &= 9.0 \times 10^{-2}\end{aligned}$$

c
$$\begin{aligned}(1.5 \times 10^5)^2 \times (9.0 \times 10^{-12}) &= 1.5^2 \times 9.0 \times 10^{10} \times 10^{-12} \\ &= 2.25 \times 9.0 \times 10^{-2} \\ &= 20.25 \times 10^{-2} \\ &= 2.025 \times 10^{-1}\end{aligned}$$

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Significant figures

Every time we record a physical measurement, we write down an approximation to the 'true value'. For example, we may say that a standard A4 sheet of paper is 30 cm by 21 cm. This has a conventional meaning and says that the actual length is between 29.5 cm and 30.5 cm. If we measure the sheet of paper more accurately, we could say that it is $29.7 \text{ cm} \times 21.0 \text{ cm}$. This means that we believe that the actual length is between 29.65 cm and 29.75 cm. Similarly, if we say a girl's height is 156 cm to the nearest centimetre, this means that her actual height is between 155.5 and 156.5 cm.

In this situation, we say that a measurement recorded as 156 cm is **correct to three significant figures**. Similarly, when we say that the width of the paper is 21 cm, this is correct to two significant figures.

Using approximations of π as another example, we say that 3.14 is π correct to three significant figures and 3.141 59 is π correct to six significant figures. When we **round** a number, we record it correct to a certain number of significant figures.

The rules for rounding require you to first identify the last significant digit. Then:

- if the next digit is 0, 1, 2, 3 or 4, round down
- if the next digit is 5, 6, 7, 8 or 9, round up.

So $\pi = 3.141\,592\,654\ldots$ is rounded to 3, 3.1, 3.14, 3.142, 3.1416, 3.141 59, 3.141 593 and so on, depending on the number of significant figures required.

We use the symbol $\approx$ to mean that two numbers are approximately equal to each other.

Significant figures and scientific notation

Recording a number in scientific notation makes it clear how many significant figures have been recorded. For example, it is unclear whether 800 is written to 1, 2 or 3 significant figures. However, when written in scientific notation as 8.00×10^2, 8.0×10^2 or 8×10^2, it is clear how many significant figures are recorded.

Example 9

State the number of significant figures to which each of these numbers is recorded.

a 7.321×10^8　**b** 7.200×10^9　**c** 2.0×10^{-5}

d -5.6789×10^{-9}　**e** 213 205　**f** $-0.001\,240$

Solution

a 7.321×10^8 has 4 significant figures.
b 7.200×10^9 has 4 significant figures.
c 2.0×10^{-5} has 2 significant figures.
d -5.6789×10^{-9} has 5 significant figures.
e $213\,205 = 2.132\,05 \times 10^5$ has 6 significant figures.
f $-0.001\,240 = -1.240 \times 10^{-3}$ has 4 significant figures.

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Example 10

Write each of the following numbers correct to the number of significant figures specified in the brackets.

a 214 (2) **b** 0.000 6786 (3) **c** 13.999 99 (6)

d −137.4895 (5) **e** 0.000 532 (2) **f** 132.007 31 (6)

Solution

a
$$\begin{aligned} 214 &= 2.14 \times 10^2 \\ &= 2.1 \times 10^2 \\ &\approx 210 \end{aligned}$$
(Correct to 2 significant figures.)

b
$$\begin{aligned} 0.000\,6786 &= 6.786 \times 10^{-4} \\ &\approx 0.000\,679 \end{aligned}$$
(Correct to 3 significant figures.)

c
$$\begin{aligned} 13.999\,99 &= 1.399\,999 \times 10 \\ &\approx 14.000 \end{aligned}$$
(Correct to 6 significant figures.)

d
$$\begin{aligned} -137.4895 &= -1.374\,895 \times 10^2 \\ &\approx -137.49 \end{aligned}$$
(Correct to 5 significant figures.)

e
$$\begin{aligned} 0.000\,532 &= 5.32 \times 10^{-4} \\ &\approx 0.000\,53 \end{aligned}$$
(Correct to 2 significant figures.)

f
$$\begin{aligned} 132.007\,31 &= 1.320\,0731 \times 10^2 \\ &\approx 132.007 \end{aligned}$$
(Correct to 6 significant figures.)

Scientific notation and significant figures

- **Scientific notation**, or **standard form**, is a convenient way to represent very large and very small numbers.
- To represent a number in scientific notation, insert a decimal point after the first non-zero digit and multiply by an appropriate power of 10. For example: $75\,684\,000\,000\,000 = 7.5684 \times 10^{13}$ and $0.000\,000\,000\,38 = 3.8 \times 10^{-10}$
- The term for a number expressed without a multiple of a power of 10 is **decimal notation** or decimal form.
- A number may be expressed with different numbers of **significant figures**. For example:

  3.1 has 2 significant figures, 3.14 has 3 significant figures,
  3.141 has 4 significant figures
- To write a number to a specified number of significant figures, first write the number in scientific notation and then round it correct to the required number of significant figures.
- To **round** a number to a required number of significant figures, first write the number in scientific notation and identify the last significant digit. Then:
  - if the next digit is 0, 1, 2, 3 or 4, round down
  - if the next digit is 5, 6, 7, 8 or 9, round up.

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Exercise 9B

Scientific notation

Example 6

1 Write each number in scientific notation.

a 63 **b** 0.4

c 0.62 **d** 7400

e 21 000 000 **f** 0.000 26

g –0.086 **h** 2 000 000 000 000

i 0.000 091 345 **j** 57 320

k 0.003 012 **l** 0.100 0510

2 **a** At the beginning of 2011, the population of Australia was estimated to be approximately 22.5 million. Write this number in scientific notation.

b The wavelength of red light is 6700Å, where $1\text{Å} = 10^{-10}$ m. Write this wavelength of red light in metres, using scientific notation.

c The Sun is approximately 150 billion metres from the Earth. Using scientific notation, write this distance in metres.

Example 7

3 Write each number in decimal notation.

a 6.4×10^3 **b** 9.2×10^4 **c** 4.8×10^{-2} **d** 8.7×10^{-3}

e 7.412×10^6 **f** -4.02×10^2 **g** -4.657×10^{-3} **h** 47.26×10^0

Example 8

4 Simplify each number, writing your answer in scientific notation.

a $(2 \times 10^3) \times (4 \times 10^2)$ **b** $(5 \times 10^3) \times (2 \times 10^2)$

c $(6 \times 10^4) \times (2.1 \times 10^3)$ **d** $(4 \times 10^3) \times (5.1 \times 10^2)$

e $(4 \times 10^{-3}) \times (5 \times 10^{-2})$ **f** $(2 \times 10^{-3})^2$

g $(1.1 \times 10^{-8})^2$ **h** $\dfrac{(2 \times 10^{-8})^3}{4 \times 10^{-3}}$

i $(5 \times 10^4) \div (2 \times 10^3)$ **j** $(1.2 \times 10^6) \div (4 \times 10^7)$

k $\dfrac{(2 \times 10^5)(4 \times 10^4)}{1.6 \times 10^3}$ **l** $\dfrac{(2 \times 10^{-1})^5}{(4 \times 10^{-2})^3}$

5 Using your calculator where necessary, write each number in scientific notation.

a $(2.7 \times 10^6) \times (3.8 \times 10^2)$ **b** $(5.3 \times 10^4) \times (1.1 \times 10^{-3})$

c $\dfrac{9.6 \times 10^{14}}{1.6 \times 10^{21}}$ **d** $\sqrt{9.61 \times 10^{12}} \times 1.4 \times 10^3$

e $\dfrac{8.4 \times 10^4}{\sqrt{4.9 \times 10^5}}$ **f** $\sqrt[3]{64 \times 10^9} \times \sqrt[5]{1024 \times 10^{-10}}$

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6 At the beginning of 2011, the population of Australia was estimated to be approximately 22.5 million. If the population stayed the same for the next year, and each person in Australia produced an average of 0.712 kg of waste each day, how many tonnes of waste would be produced by Australians in the following year? (1 tonne = 1000 kg, 1 year = 365 days.) Express your answer in scientific notation.

7 A light year is the distance light travels in a year. Light travels at approximately 3×10^5 km/s.

a How wide is our galaxy (in kilometres) if it is approximately 230 000 light years across?

b How far from us (in kilometres) is the farthest galaxy detected by optical telescopes if it is approximately 13×10^9 light years from us?

c How long does it take light to travel from the Sun to the Earth if the distance between the Sun and the Earth is 1.4951×10^8 km?

8 The mass of a hydrogen atom is approximately 1.674×10^{-27} kg and the mass of an electron is approximately 9.1×10^{-31} kg. How many electrons, correct to the nearest whole number, will it take to equal the mass of a single hydrogen atom?

Significant figures

Example 9, 10

9 Write each of these numbers in scientific notation, correct to the number of significant figures indicated in the brackets.

a 576.63 (4)	**b** 472.61 (3)	**c** 472.61 (2)
d 472.61 (1)	**e** 0.051 237 (4)	**f** 0.051 237 (3)
g 0.051 237 (2)	**h** 0.051 237 (1)	**i** 1603.29 (4)
j 1603.29 (3)	**k** 1603.29 (2)	**l** 1603.29 (1)
m 2.9935×10^{27} (4)	**n** 2.9935×10^{27} (3)	**o** 2.9935×10^{27} (2)
p 2.9935×10^{27} (1)	**q** 573 007 (3)	**r** 0.006 534 (1)

Example 10

10 Write each of these numbers in decimal notation, correct to three significant figures.

a 5.6023	**b** 537.97	**c** 9673.47	**d** 732 412
e 0.003 511	**f** 0.014 187	**g** 372.2	**h** 478 000

11 A cylindrical wire in an electrical circuit has radius 3.41×10^{-4} m and length 8.02×10^{-2} m. Calculate its volume in m^3, correct to three significant figures, giving the answer in scientific notation.

12 The formula for kinetic energy is $E = \frac{1}{2}mv^2$.

a Find the value of E correct to three significant figures, when $m = 9.21 \times 10^{-11}$ and $v = 3.00 \times 10^7$.

b Find the value of v correct to four significant figures, when $E = 2.834 \times 10^{-10}$ and $m = 6.418 \times 10^{-29}$.

13 For each measurement, identify the range within which the true value lies.

a 15 cm	**b** 2.00×10^3 kg	**c** 18.67 m	**d** 4.8745×10^7 mL

ICE-EM Mathematics 10 3ed ISBN 978-1-108-40434-1 Updated June 2022

9C Powers with rational indices

We begin by considering what we mean by powers such as $3^{\frac{1}{2}}$, $2^{\frac{1}{3}}$ and $\pi^{\frac{1}{10}}$, in which the exponent is the reciprocal of a positive integer.

Recall that if a is positive, $\sqrt{a}$ is the positive number whose square is a. That is:

$$\left(\sqrt{a}\right)^2 = a = a^1$$

For this reason, we introduce an alternative notation for $\sqrt{a}$: we write it as $a^{\frac{1}{2}}$. We do this because then we preserve the third index law:

$$\left(a^{\frac{1}{2}}\right)^2 = a^{2\times\frac{1}{2}} = a^1$$

Keep in mind that $a^{\frac{1}{2}}$ is nothing more than an alternative notation for $\sqrt{a}$.

Similarly, every positive number a has a cube root, $\sqrt[3]{a}$. It is the positive number whose cube is a; that is, $(\sqrt[3]{a})^3 = a = a^1$.

We define $a^{\frac{1}{3}}$ to be $\sqrt[3]{a}$. The third index law continues to hold.

$$\left(a^{\frac{1}{3}}\right)^3 = a^{3\times\frac{1}{3}} = a^1$$

The same can be done for $\sqrt[4]{a}$, $\sqrt[5]{a}$ and so on. The alternative notations are:

$\sqrt[4]{a} = a^{\frac{1}{4}}$, $\sqrt[5]{a} = a^{\frac{1}{5}}$ and so on.

n^{th} root

Let a be positive or zero and let n be a positive integer. Define $a^{\frac{1}{n}}$ to be the n^{th} root of a.

That is, $a^{\frac{1}{n}}$ is the positive number whose n^{th} power is a.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

For example, $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$.

Using a calculator, it is easy to obtain approximations for square roots, cube roots or any higher-order root.

$$\sqrt{10} \approx 3.1623,\ \sqrt[3]{10} \approx 2.1544,\ \sqrt[4]{10} \approx 1.7783,\ \sqrt[5]{10} \approx 1.5849, \ldots$$

Using our new notation, here are some other numerical approximations, all recorded correct to five significant figures.

$$2^{\frac{1}{5}} \approx 1.1487,\ 10^{\frac{1}{8}} \approx 1.3335,\ 0.2^{\frac{1}{4}} \approx 0.668\,74,\ 3.2^{\frac{1}{6}} \approx 1.2139$$

Use your calculator to check these calculations.

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Cambridge University Press Updated June 2022

Example 11

Without using your calculator, evaluate:

a $8^{\frac{1}{3}}$ **b** $1024^{\frac{1}{2}}$ **c** $1024^{\frac{1}{5}}$ **d** $\left(\frac{1}{729}\right)^{\frac{1}{2}}$ **e** $\left(\frac{1}{729}\right)^{\frac{1}{6}}$

Solution

a $2^3 = 8$, so $8^{\frac{1}{3}} = 2$

b $1024 = 2^{10}$, so $1024^{\frac{1}{2}} = (2^{10})^{\frac{1}{2}} = 2^5 = 32$

c Similarly, $1024^{\frac{1}{5}} = 2^2 = 4$

d $729 = 3^6$, so $\left(\frac{1}{729}\right)^{\frac{1}{2}} = \left(\frac{1}{3^6}\right)^{\frac{1}{2}} = \frac{1}{3^3} = \frac{1}{27}$

e Similarly, $\left(\frac{1}{729}\right)^{\frac{1}{6}} = \frac{1}{3}$

We now come to the main definition. If a is a positive number, p is an integer and q is a positive integer, then we define:

$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p$ which means $\left(\sqrt[q]{a}\right)^p$. This is the p^{th} power of the q^{th} root of a.

For example:

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$$

Throughout the rest of this chapter, we will avoid using the radical symbol $\sqrt{\ }$ wherever possible. We begin with some simple calculations and then investigate how the index laws behave when we have rational powers of numbers.

Example 12

Without using your calculator, find:

a $8^{\frac{4}{3}}$ **b** $81^{\frac{5}{4}}$ **c** $100\,000^{\frac{3}{5}}$ **d** $0.01^{\frac{3}{2}}$

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Solution

a Using the definition,

$$8^{\frac{4}{3}} = \left(8^{\frac{1}{3}}\right)^4$$

$$= 2^4 \qquad (\text{since } 2^3 = 8)$$

$$= 16$$

b Since $81 = 3^4$, we have

$$81^{\frac{5}{4}} = \left(81^{\frac{1}{4}}\right)^5$$

$$= 3^5$$

$$= 243$$

c Since $100\,000 = 10^5$,

$$100\,000^{\frac{3}{5}} = 10^3$$

$$= 1000$$

d $0.01^{\frac{3}{2}} = (10^{-2})^{\frac{3}{2}}$

$$= 10^{-3}$$

$$= 0.001$$

Example 13

a Write each number in the form $\sqrt[n]{a}$.

i $7^{\frac{1}{3}}$ **ii** $11^{\frac{1}{5}}$

b Write each number in index form.

i $\sqrt[3]{17}$ **ii** $\left(\sqrt[5]{13}\right)^2$ **iii** $7\sqrt{7}$ **iv** $6^2 \times \sqrt[5]{6}$

Solution

a i $7^{\frac{1}{3}} = \sqrt[3]{7}$

ii $11^{\frac{1}{5}} = \sqrt[5]{11}$

b i $\sqrt[3]{17} = 17^{\frac{1}{3}}$

ii $\left(\sqrt[5]{13}\right)^2 = \left(13^{\frac{1}{5}}\right)^2 = 13^{\frac{2}{5}}$

iii $7\sqrt{7} = 7 \times 7^{\frac{1}{2}}$

$$= 7^{\frac{3}{2}}$$

iv $6^2 \times \sqrt[5]{6} = 6^2 \times 6^{\frac{1}{5}}$

$$= 6^{\frac{11}{5}}$$

Example 14

Calculate the exact value of each number.

a $16^{-\frac{1}{2}}$ **b** $125^{-\frac{1}{3}}$ **c** $32^{-\frac{1}{5}}$

Solution

a $16^{-\frac{1}{2}} = \left(\frac{1}{16}\right)^{\frac{1}{2}}$

$$= \frac{1}{4}$$

b $125^{-\frac{1}{3}} = \left(\frac{1}{125}\right)^{\frac{1}{3}}$

$$= \frac{1}{5}$$

c $32^{-\frac{1}{5}} = \left(\frac{1}{32}\right)^{\frac{1}{5}}$

$$= \frac{1}{2}$$

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Index laws for rational indices

The five index laws introduced previously for integer indices are equally valid for rational indices. This follows from the definition of a^x, where x is rational.

We will leave a discussion of the proofs of these laws to the Challenge exercises.

Using index law 3, we note that, for rational x and y:

$$(a^x)^y = a^{xy} = a^{yx} = (a^y)^x$$

Hence:

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = (a^p)^{\frac{1}{q}}$$

This means that when we evaluate $a^{\frac{p}{q}}$, it does not matter if we take the q^{th} root first and the p^{th} power second, or the p^{th} power first and the q^{th} power second. For example:

$$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$$

and also:

$$4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = 64^{\frac{1}{2}} = 8$$

Example 15

Simplify:

a $3^{\frac{1}{3}} \times 3^{\frac{1}{2}}$ **b** $5^{\frac{1}{2}} \div 5^{\frac{2}{3}}$ **c** $27^{\frac{4}{3}}$ **d** $16^{\frac{3}{4}}$ **e** $\left(\frac{16}{25}\right)^{-\frac{3}{2}}$

Solution

a
$$\begin{aligned} 3^{\frac{1}{3}} \times 3^{\frac{1}{2}} &= 3^{\frac{1}{3}+\frac{1}{2}} \\ &= 3^{\frac{5}{6}} \end{aligned}$$

b
$$\begin{aligned} 5^{\frac{1}{2}} \div 5^{\frac{2}{3}} &= 5^{\frac{1}{2}-\frac{2}{3}} \\ &= 5^{-\frac{1}{6}} \end{aligned}$$

c
$$\begin{aligned} 27^{\frac{4}{3}} &= \left(27^{\frac{1}{3}}\right)^4 \\ &= 3^4 \\ &= 81 \end{aligned}$$

d
$$\begin{aligned} 16^{\frac{3}{4}} &= \left(16^{\frac{1}{4}}\right)^3 \\ &= 2^3 \\ &= 8 \end{aligned}$$

e
$$\begin{aligned} \left(\frac{16}{25}\right)^{-\frac{3}{2}} &= \left(\frac{25}{16}\right)^{\frac{3}{2}} \\ &= \left(\left(\frac{25}{16}\right)^{\frac{1}{2}}\right)^3 \\ &= \left(\frac{5}{4}\right)^3 \\ &= \frac{125}{64} \end{aligned}$$

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Example 16

Simplify each of these expressions, writing your answers with positive indices.

a $a^{\frac{2}{3}} \times a^{\frac{1}{2}}$ **b** $m^{\frac{1}{2}} \div m^{\frac{3}{5}}$ **c** $(32m^{\frac{3}{4}})^{\frac{2}{5}}$

Solution

a $a^{\frac{2}{3}} \times a^{\frac{1}{2}} = a^{\frac{2}{3}+\frac{1}{2}}$

$= a^{\frac{7}{6}}$

b $m^{\frac{1}{2}} \div m^{\frac{3}{5}} = m^{\frac{1}{2}-\frac{3}{5}}$

$= m^{-\frac{1}{10}}$

$= \dfrac{1}{m^{\frac{1}{10}}}$

c $\left(32m^{\frac{3}{4}}\right)^{\frac{2}{5}} = 32^{\frac{2}{5}} m^{\frac{3}{4}\times\frac{2}{5}}$

$= (2^5)^{\frac{2}{5}} m^{\frac{3}{10}}$

$= 4m^{\frac{3}{10}}$

Index laws for rational indices

If a and b are positive numbers and x and y are rational numbers, then:

Index law 1 $a^x a^y = a^{x+y}$

Index law 2 $\dfrac{a^x}{a^y} = a^{x-y}$

Index law 3 $(a^x)^y = a^{xy}$

Index law 4 $(ab)^x = a^x b^x$

Index law 5 $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Exercise 9C

Example 11a, b, c

1 Calculate the exact value of each number.

a $4^{\frac{1}{2}}$ **b** $49^{\frac{1}{2}}$ **c** $27^{\frac{1}{3}}$ **d** $32^{\frac{1}{5}}$ **e** $1000^{\frac{1}{3}}$ **f** $625^{\frac{1}{4}}$

Example 11d, e

2 Calculate the exact value of each number.

a $\left(\dfrac{1}{125}\right)^{\frac{1}{3}}$ **b** $\left(\dfrac{1}{64}\right)^{\frac{1}{2}}$ **c** $\left(\dfrac{1}{64}\right)^{\frac{1}{6}}$ **d** $\left(\dfrac{1}{10\,000}\right)^{\frac{1}{2}}$ **e** $\left(\dfrac{1}{10\,000}\right)^{\frac{1}{4}}$

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Example 12

3 Calculate the exact value of each number.

a $27^{\frac{2}{3}}$ **b** $64^{\frac{2}{3}}$ **c** $81^{\frac{3}{4}}$ **d** $32^{\frac{3}{5}}$ **e** $9^{\frac{5}{2}}$

f $100^{\frac{5}{2}}$ **g** $\left(\frac{1}{8}\right)^{\frac{2}{3}}$ **h** $\left(\frac{8}{27}\right)^{\frac{4}{3}}$ **i** $121^{\frac{3}{2}}$ **j** $343^{\frac{2}{3}}$

k $\left(\frac{4}{25}\right)^{\frac{3}{2}}$ **l** $\left(\frac{25}{36}\right)^{\frac{3}{2}}$ **m** $\left(\frac{1}{10\,000}\right)^{\frac{3}{4}}$ **n** $\left(\frac{49}{100}\right)^{\frac{1}{2}}$ **o** $\left(\frac{32}{243}\right)^{\frac{4}{5}}$

Example 13a

4 Write each number in the form $\sqrt[n]{a}$.

a $2^{\frac{1}{3}}$ **b** $3^{\frac{1}{4}}$ **c** $20^{\frac{1}{5}}$ **d** $10^{\frac{1}{4}}$ **e** $9^{\frac{1}{5}}$

Example 13b

5 Write each number in index form.

a $\sqrt[3]{4}$ **b** $\sqrt[7]{13}$ **c** $\sqrt[4]{5}$ **d** $\sqrt[3]{11}$

e $\left(\sqrt[3]{5}\right)^2$ **f** $\left(\sqrt{7}\right)^3$ **g** $\left(\sqrt[5]{11}\right)^3$ **h** $\left(\sqrt[7]{10}\right)^2$

Example 13b

6 Write each number in index form.

a $5\sqrt{5}$ **b** $5 \times \sqrt[3]{5}$ **c** $6 \times \sqrt[4]{6}$

d $7 \times \sqrt[5]{7}$ **e** $5^2 \times \sqrt[3]{5}$ **f** $11^2 \times \sqrt{11}$

Example 14

7 Calculate the exact value of each number.

a $9^{-\frac{1}{2}}$ **b** $16^{-\frac{1}{4}}$ **c** $121^{-\frac{1}{2}}$

d $100^{-\frac{1}{2}}$ **e** $1\,000\,000^{-\frac{1}{2}}$ **f** $1331^{-\frac{1}{3}}$

Example 15c, d, e

8 Calculate the exact value of each number.

a $16^{-\frac{3}{4}}$ **b** $100^{-\frac{3}{2}}$ **c** $\left(\frac{1}{8}\right)^{-\frac{2}{3}}$ **d** $125^{-\frac{4}{3}}$ **e** $1000^{-\frac{2}{3}}$ **f** $32^{-\frac{3}{5}}$

Example 15a

9 Simplify, expressing each answer with a positive index.

a $2^{\frac{2}{3}} \times 2^{\frac{1}{3}}$ **b** $3^{\frac{4}{5}} \times 3^{\frac{1}{3}}$ **c** $7^{\frac{1}{5}} \times 7^{\frac{2}{5}}$

d $3^{\frac{1}{4}} \times 3^{\frac{1}{3}}$ **e** $10^{\frac{1}{2}} \times 10$ **f** $10^{\frac{2}{3}} \times 10^{\frac{1}{4}}$

g $3^{\frac{2}{3}} \times 3^{-\frac{1}{5}}$ **h** $2^{\frac{1}{5}} \times 2^{-\frac{1}{4}}$ **i** $5^{\frac{2}{5}} \times 5^{-\frac{7}{10}}$

Example 15b

10 Simplify, expressing each answer with a positive index.

a $2^{\frac{3}{5}} \div 2^{\frac{1}{5}}$ **b** $7^{\frac{11}{3}} \div 7^{\frac{8}{3}}$ **c** $8^{\frac{1}{4}} \div 8^{\frac{1}{7}}$

d $7^{\frac{2}{3}} \div 7^{\frac{1}{2}}$ **e** $8^{\frac{8}{9}} \div 8^{\frac{5}{9}}$ **f** $10^{\frac{3}{7}} \div 10^{\frac{5}{7}}$

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11 Use your calculator to find the value of each of these numbers, correct to five significant figures.

a $10^{\frac{3}{5}}$ **b** $24^{\frac{2}{3}}$ **c** $86^{\frac{4}{7}}$ **d** $127^{\frac{3}{11}}$

e $19.6^{\frac{3}{4}}$ **f** $1.8^{\frac{3}{2}}$ **g** $(\pi + \pi^2)^3$ **h** $(\sqrt{3} + \pi)^{\frac{4}{7}}$

Example 16

12 Simplify each expression. In your answers, use only positive indices.

a $m^{\frac{2}{3}} \times m^{\frac{1}{4}}$ **b** $a^{\frac{2}{5}} \times a^{\frac{1}{3}}$ **c** $x^{\frac{1}{2}}y^{\frac{1}{3}} \times x^{\frac{1}{4}}y^{\frac{2}{5}}$

d $a^{\frac{4}{5}}b^{\frac{1}{3}} \times a^{\frac{3}{10}}b^{\frac{1}{2}}$ **e** $a^{\frac{4}{5}} \div a^{\frac{3}{10}}$ **f** $m^{\frac{2}{3}} \div m^{\frac{5}{6}}$

g $b^{\frac{4}{7}} \div b^{\frac{1}{3}}$ **h** $\left(2m^{\frac{4}{5}}\right)^2$ **i** $\left(3m^{\frac{1}{2}}\right)^{-3}$

j $\left(5a^{-\frac{1}{2}}\right)^{-4}$ **k** $\left(4m^{-\frac{2}{3}}\right)^2 \times 5m^{\frac{3}{4}}$ **l** $\left(2m^{-\frac{3}{4}}\right)^{-2} \times 4m^{\frac{2}{3}}$

m $\left(8m^6\right)^{\frac{1}{3}} \times \left(16m^2\right)^{\frac{1}{4}}$ **n** $\left(27m^{-6}\right)^{\frac{1}{3}} \times \left(64m^2\right)^{-\frac{1}{2}}$

13 Evaluate each number, giving the answers correct to four significant figures.

a $6^{1.2}$ **b** $18.5^{2.1}$ **c** $0.84^{-0.7}$ **d** $1.59^{-0.1}$ **e** $12.6^{-1.8}$ **f** $5.9^{-3.7}$

14 Simplify each expression, giving your answers with positive indices.

a $a^{1.6} \times a^{3.2}$ **b** $m^{4.7} \times m^{1.3}$

c $p^{8.2} \div p^{4.6}$ **d** $b^{4.1} \div b^{2.85}$

e $(2p^{1.3})^2$ **f** $(4p^{2.1})^3$

g $4a^{1.3}b^{0.6} \div (8a^2b^{-1})$ **h** $12m^{-1.2}n^{3.5} \div \left(18(mn^{-1.5})^3\right)$

i $\dfrac{a^{1.2}b^{4.3}}{(ab^{-1})^{1.2}} \times \dfrac{ab^{0.6}}{a^{1.8}b}$ **j** $\dfrac{m^{0.9}n}{(mn^{1.5})^2} \times \dfrac{1}{mn^{3.8}}$

9D Graphs of exponential functions

In the previous section, we saw how to define 2^x for all rational numbers x. There are a number of ways of defining 2^x for all real numbers x, but it is not possible to deal with them in this book. The calculator gives approximations to 2^x and we will use these values. Consider the following list of approximate values of powers of 2.

$2^1 = 2$ $2^{1.1} \approx 2.1435$ $2^{1.2} \approx 2.2974$

$2^{1.3} \approx 2.4623$ $2^{1.4} \approx 2.6390$ $2^{1.5} \approx 2.8284$ $(2^{1.5} = 2\sqrt{2})$

This list of values suggests that 2^x increases as x increases. This is in fact the case.

Throughout the rest of this chapter, you will often need to use your calculator to calculate values of exponential functions.

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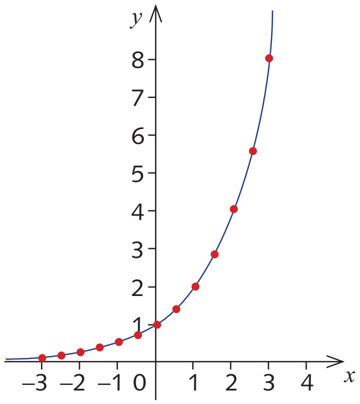
Consider the function $y = 2^x$. A table of approximate values, correct to three decimal places, follows.

x	–3	–2.5	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2	2.5	3
y	0.125	0.177	0.25	0.354	0.5	0.707	1	1.414	2	2.828	4	5.657	8

By plotting these values and connecting them up with a smooth curve, we obtain the graph of $y = 2^x$.



Key features:

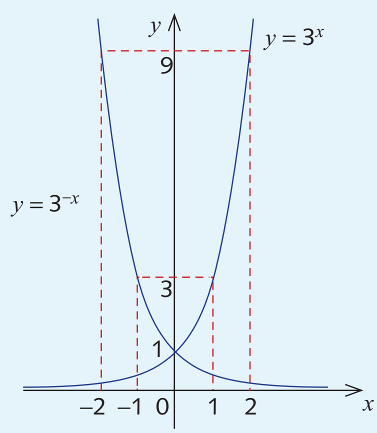
- $y = 2^x$ is an **increasing** function; that is, 2^x increases as x increases.
- A y-intercept occurs at $(0, 1)$ but there is no x-intercept.
- As x moves away from 0 in the negative direction (to the left), the value of 2^x gets close to 0, but it never equals 0. Why? We say that the x-axis is an **asymptote** for the graph of $y = 2^x$.

Example 17

Produce a table of values for the functions $y = 3^x$ and $y = 3^{-x}$. Draw the graphs on the same set of axes.

Solution

x	–3	–2	–1	0	1	2	3
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
3^{-x}	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



Note: $y = 3^x$ is an **increasing** function; that is, 3^x increases as x increases; and that $y = 3^{-x}$ is a **decreasing** function; that is, 3^{-x} decreases as x increases.

The two graphs in Example 17 are reflections of each other in the y-axis.

Note: $\frac{1}{3} = 3^{-1}$; hence, $\left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x}$.

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Multiplication by a constant

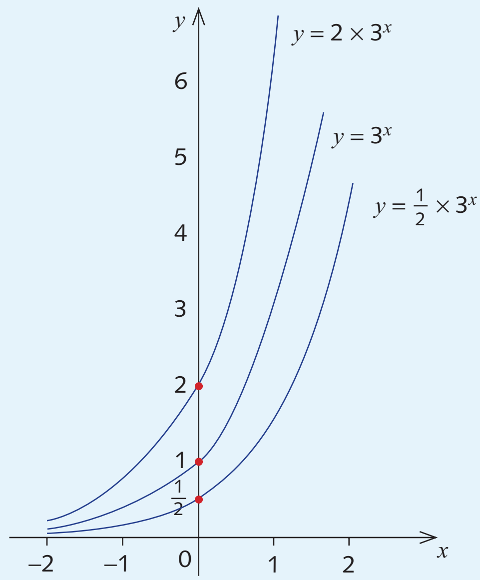
In applications, exponential functions often occur multiplied by a constant.

Example 18

Draw the graphs of $y = 3^x$, $y = \frac{1}{2} \times 3^x$ and $y = 2 \times 3^x$ on the same set of axes. (Produce a table of values first.)

Solution

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\frac{1}{2} \times 3^x$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{9}{2}$
2×3^x	$\frac{2}{9}$	$\frac{2}{3}$	2	6	18



The different graphs in Example 18 are roughly the same shape and the y-intercept of the curve is the constant that multiplies the exponential function.

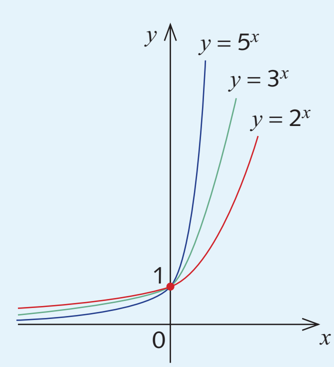
Next, we will investigate how exponential functions change for different values of the base.

Example 19

Draw the graphs of $y = 2^x$, $y = 3^x$ and $y = 5^x$ on the same set of axes.

Solution

x	-3	-2	-1	0	1	2	3
2^x	0.125	0.25	0.5	1	2	4	8
3^x	0.037	0.111	0.333	1	3	9	27
5^x	0.008	0.04	0.2	1	5	25	125



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All three graphs in Example 19 pass through the point $(0, 1)$ but they have different 'gradients'. That is, 5^x increases more quickly than 3^x, which increases more quickly than 2^x. So, for example, $2^x < 5^x$ if $x > 0$, but $2^x > 5^x$ if $x < 0$.

Graphs of exponential functions

- To graph an exponential function, first create a table of values, then plot the points on a set of axes.
- If the exponential is multiplied by a constant, the y-intercept is that constant.
- The graph of $y = a^{-x}$, where $a > 0$, is the reflection of the graph of $y = a^x$ in the y-axis.
- The x-axis is an asymptote of the graph of $y = a^x$ and of $y = a^{-x}$, where $a > 0$ and $a \neq 1$.

Exercise 9D

Example 17

1 For each function, produce a table of values for $x = -2, -1, 0, 1, 2$, and use it to draw a graph.

a $y = 2^x$ **b** $y = 2^{-x}$ **c** $y = 4^x$ **d** $y = 5^{-x}$

Example 18

2 Sketch the graphs of $y = 4^x$, $y = 2 \times 4^x$ and $y = 3 \times 4^x$ on a single set of axes.

3 Sketch the graphs of $y = 2^x$, $y = 2 \times 2^x$ and $y = \frac{1}{2} \times 2^x$ on a single set of axes.

Example 19

4 Sketch the graph of $y = 2^{-x}$, $y = 3^{-x}$ and $y = 5^{-x}$ on the one set of axes.

9E Exponential equations

From the previous section, we have seen that the graph of $y = 2^x$ is increasing and the graph of

$y = 2^{-x} = \left(\frac{1}{2}\right)^x$ is decreasing.

In general, suppose that a is a positive number different from 1. Since the graphs of $y = a^x$ are either increasing or decreasing (unless $a = 1$), there is only one value of x for each value of y. Hence, we know that if $a^c = a^d$, then $c = d$.

In the following examples, this fact is used to solve **exponential equations**. From the above discussion it can be seen that there is only one solution for x to the equation $a^x = y$, provided that y is positive.

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Example 20

Solve each equation for x.

a $2^x = 32$ **b** $10^x = 10\,000$ **c** $5^x = 625$

Solution

a $2^x = 32$

Since $32 = 2^5$

$2^x = 2^5$

$x = 5$

b $10^x = 10\,000$

Since $10\,000 = 10^4$

$10^x = 10^4$

$x = 4$

c $5^x = 625$

Since $625 = 5^4$

$5^x = 5^4$

$x = 4$

Example 21

Solve each equation for x.

a $2^x = \frac{1}{8}$ **b** $7^x = \frac{1}{343}$ **c** $7^x = 1$

Solution

a $2^x = \frac{1}{8}$

Since $\frac{1}{8} = 2^{-3}$

$2^x = 2^{-3}$

$x = -3$

b $7^x = \frac{1}{343}$

Since $\frac{1}{343} = 7^{-3}$

$7^x = 7^{-3}$

$x = -3$

c $7^x = 1$

Since $7^0 = 1$

$x = 0$

In Example 22, we first write each side of the equation as a power with the same base.

Example 22

Solve each equation for x.

a $16^x = 32$ **b** $81^x = 243$ **c** $256^x = 32$

Solution

a $16^x = 32$

$(2^4)^x = 32$

$2^{4x} = 2^5$

$4x = 5$

$x = \frac{5}{4}$

b $81^x = 243$

$(3^4)^x = 3^5$

$4x = 5$

$x = \frac{5}{4}$

c $256^x = 32$

$(2^8)^x = 2^5$

$8x = 5$

$x = \frac{5}{8}$

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Example 23

Solve:

a $3^{2x-1} = 81$ **b** $6^{x-1} = 36\sqrt{6}$

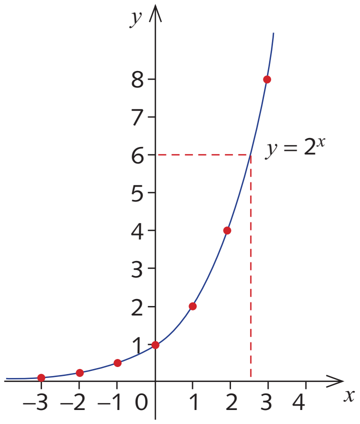
Solution

a
$$\begin{aligned} 3^{2x-1} &= 81 \\ 3^{2x-1} &= 3^4 \\ 2x - 1 &= 4 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

b
$$\begin{aligned} 6^{x-1} &= 36\sqrt{6} \\ 6^{x-1} &= 6^2 \times 6^{\frac{1}{2}} \\ 6^{x-1} &= 6^{\frac{5}{2}} \\ x - 1 &= \frac{5}{2} \\ x &= \frac{7}{2} \end{aligned}$$

Consider the exponential equation $2^x = 6$.
Since $2^2 = 4$ and $2^3 = 8$, x must be between 2 and 3. From the graph opposite, we can estimate x to be about 2.5. From a calculator, one obtains 2.58 as a better approximation. The value of x is called $\log_2 6$, which is $\approx 2.584\,962$.

We will discuss logarithms in a later section of this chapter.



Example 24

Between which two integers does x lie if:

a $2^x = 70$? **b** $2^x = 200$?

Solution

The graph of $y = 2^x$ is increasing.

a $2^6 = 64$ and $2^7 = 128$

Therefore, x lies between 6 and 7.

b $2^7 = 128$ and $2^8 = 256$

Therefore, x lies between 7 and 8.

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Solving exponential equations

For any positive value a, if $a^c = a^d$, then $c = d$.

For $a > 0$ and $a \neq 1$, the equation $a^x = y$, where $y > 0$, can be solved, and there is only one solution for x.

Exercise 9E

Example 20

1 Solve:

a $2^x = 8$ **b** $2^x = 512$ **c** $3^x = 243$ **d** $10^x = 100$

e $11^x = 1331$ **f** $20^x = 400$ **g** $6^x = 216$ **h** $10^x = 100\,000$

i $5^x = 125$ **j** $3^x = 729$ **k** $4^x = 256$ **l** $4^x = 1024$

Example 21

2 Solve:

a $2^x = \frac{1}{16}$ **b** $4^x = \frac{1}{256}$ **c** $5^x = 1$ **d** $10^x = 0.001$

e $10^x = \frac{1}{100\,000}$ **f** $7^x = \frac{1}{343}$ **g** $3^x = \frac{1}{243}$ **h** $2^x = \frac{1}{1024}$

Example 22

3 Solve:

a $121^x = 11$ **b** $121^x = 1331$ **c** $9^x = 27$ **d** $64^x = 16$

e $25^x = 125$ **f** $125^x = 25$ **g** $1000^x = 100$ **h** $10\,000^x = 1000$

4 Solve:

a $27^a = 243$ **b** $4^b = 128$ **c** $128^c = 32$ **d** $625^d = 125$

e $1000^e = 10$ **f** $\left(\frac{1}{8}\right)^f = 4$ **g** $27^x = \frac{1}{3}$ **h** $(0.01)^x = 1000$

Example 23

5 Solve:

a $3^{x-2} = 27$ **b** $5^{1-x} = 125$ **c** $4^{3x-1} = 64$ **d** $32^{3x+1} = 128$

e $2^{3-x} = \sqrt{8}$ **f** $(\sqrt{7})^x = 343$ **g** $4^{x-1} = \frac{1}{16\sqrt{2}}$ **h** $3^{3-x} = 27^{x-1}$

Example 24

6 Identify which two integers x lies between if:

a $2^x = 19$ **b** $5^x = 30$ **c** $2^x = 40$ **d** $10^x = 500$

e $3^x = 90$ **f** $7^x = 50$ **g** $11^x = 100$ **h** $13^x = 200$

i $2^{-x} = 0.1$ **j** $5^{-x} = 2$ **k** $5^{-x} = 0.3$ **l** $10^{-x} = 0.045$

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9F Exponential growth and decay

We begin by looking at two examples.

Exponential growth

The first example is a mathematical model of the number of bacteria in a culture.

Initially, there are 1000 bacteria in a culture. The number of bacteria is doubling every hour.

Therefore:

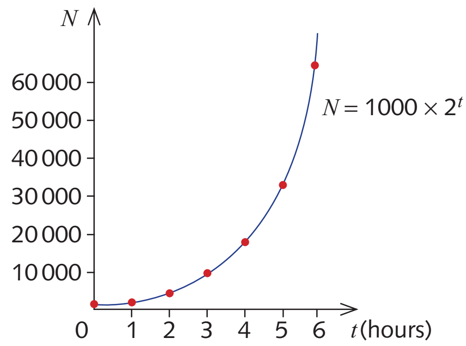
- after 1 hour there are 1000×2 bacteria
- after 2 hours there are $1000 \times 2 \times 2 = 1000 \times 2^2$ bacteria
- after 3 hours there are $1000 \times 2^2 \times 2 = 1000 \times 2^3$ bacteria.

Following this pattern, there are 1000×2^t bacteria after t hours. This can be written as a formula. Let N be the number of bacteria after t hours. Then:

$$N = 1000 \times 2^t$$

A graph can be plotted by first producing a table of values.

t	0	1	2	3	4	5	6
N	1000	2000	4000	8000	16 000	32 000	64 000



This is an example of **exponential growth**.

Exponential decay

Radioactivity is a natural phenomenon in which atoms of one element 'decay' to form atoms of another element by emitting a particle such as an alpha particle.

A sample of a radioactive substance that is widely used in medical radiology initially has a mass of 100 g. The substance decays over time, its quantity halving every hour. Let M grams be the mass present after t hours.

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Therefore:

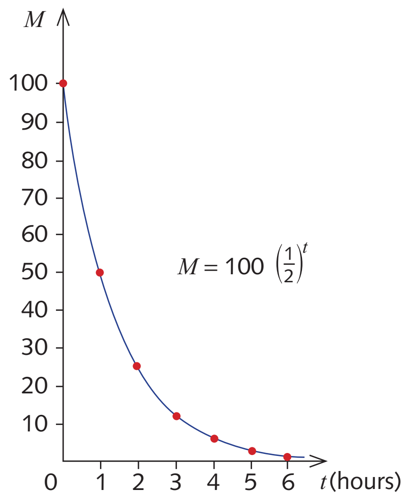
- after 1 hour the mass is $100 \times \frac{1}{2}$ g
- after 2 hours the mass is $100 \times \frac{1}{2} \times \frac{1}{2} = 100 \times \left(\frac{1}{2}\right)^2$ g
- after 3 hours the mass is $100 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} = 100 \times \left(\frac{1}{2}\right)^3$ g.

Following this pattern, there are $100 \times \left(\frac{1}{2}\right)^t$ grams of the radioactive substance after t hours. So:

$$M = 100\left(\frac{1}{2}\right)^t$$

A table is constructed and the graph is plotted.

t	0	1	2	3	4	5	6
M	100	50	25	12.5	6.25	3.13	1.56



This is an example of **exponential decay**.

Formulas for exponential growth and decay

The two previous examples concern populations or quantities that can be described by a formula of the form:

$$P = A \times B^t$$

In this formula, A and B are positive constants and t is a variable that is usually time measured in seconds, hours or years, depending on the application.

If $t = 0$, then $P = A$, so A is the initial amount.

If $B = 1$, then $P = A$ for all values of t.

If $B > 1$, we say that P grows exponentially.

If $B < 1$, we say that P decays exponentially.

It is possible to estimate both future and past sizes of the population by substituting positive and negative values for t.

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Updated June 2022

Example 25

For the rule $y = 20 \times 3^t$:

a Complete the table of values.

t	0	1	2	3
y				

b Plot the graph of y against t.

c Find the value y, correct to two decimal places, when:

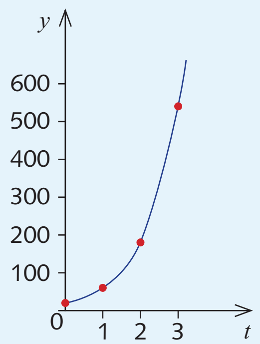
i $t = 0.5$ **ii** $t = 2.5$ **iii** $t = 2.8$

Solution

a Complete the table of values.

t	0	1	2	3
y	20	60	180	540

b



c Using a calculator:

i When $t = 0.5$, $y = 34.64$ **ii** When $t = 2.5$, $y = 311.77$

iii When $t = 2.8$, $y = 433.48$

Exercise 9F

Example 25

1 For the formula $y = 200 \times 2^t$:

a Complete the table of values.

t	0	1	2	3	4	5
y						

b Plot the graph of y against t.

c Using your calculator, find the value of y, correct to two decimal places, when:

i $t = 0.6$ **ii** $t = 2.2$ **iii** $t = 3.5$

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2 For the formula $y = 200 \times \left(\frac{1}{2}\right)^t$:

a Complete the table of values.

t	0	1	2	3	4	5
y						

b Plot the graph of y against t.

c Using your calculator, find the value of y, correct to two decimal places, when:

i $t = 0.6$ **ii** $t = 3.2$ **iii** $t = 4.6$

3 **a** For $y = 60 \times 8^t$, find the value of y when:

i $t = 0$ **ii** $t = 2$ **iii** $t = 2.5$

b For $y = 1000 \times (0.1)^t$, find the value of y when:

i $t = 0$ **ii** $t = 1$ **iii** $t = 3$ **iv** $t = 4$

4 On 1 January 2011, the population of the world was estimated to be $7\,074\,000\,000 = 7.074 \times 10^9 = A$. Assume that the population of the world is increasing at the rate of 3% per year, so that $N = A(1.03)^t$ after t years.

a Estimate what the population of the world will be on 1 January 2016.

b Estimate the population on 1 January 2111.

5 A liquid cools from its original temperature of $95°C$ to a temperature $T°C$ in t minutes. Given that $T = 95(0.96)^t$, find:

a the value of T when $t = 10$

b the value of T when $t = 20$

6 The number of finches on an island, N, at time t years after 1 January 2010 is approximately described by the rule $N = 80\,000 \times (1.008)^t$.

a Identify (from the rule) the annual percentage increase in finches on the island after 1 January 2010.

b How many finches were there on the island on 1 January 2010?

c How many finches will there be on the island on 1 January 2020?

7 The number of bacteria, N, in a certain culture is halving every hour, so $N = A \times \left(\frac{1}{2}\right)^t$, where t is the time in hours after 2 p.m. on a particular day. Assume that there are initially 1000 bacteria.

a State the value of A.

b Estimate the number of bacteria in the culture when:

i $t = 2$ **ii** $t = 3$ **iii** $t = 5$

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9G Logarithms

In Section 9D, we saw how to sketch the graph of $y = 2^x$. When we wish to determine the value of y for particular values of x, for example, $6 = 2^x$, the concept of a logarithm is used.

Consider the number fact $2^3 = 8$. When making the exponent the subject of this relationship, we express it as $\log_2 8 = 3$.

This is read as either 'log to the base 2 of 8 is (equal to) 3' or 'the log of 8 to the base 2 is (equal to) 3'.

For example:

- $3^5 = 243$ is equivalent to $\log_3 243 = 5$
- $10^2 = 100$ is equivalent to $\log_{10} 100 = 2$
- $5^{-2} = \frac{1}{25}$ is equivalent to $\log_5 \frac{1}{25} = -2$
- $8^{\frac{2}{3}} = 4$ is equivalent to $\log_8 4 = \frac{2}{3}$

The logarithm of a number to base a is the index to which a is raised to give that number.

In general, the logarithm can be defined as follows.

If $a > 0$ and $a \neq 1$ and $a^x = y$, then $\log_a y = x$.

Logarithms were invented in the seventeenth century to assist in astronomical calculations. They have a number of important properties, which will be discussed in detail in Chapter 14.

Example 26

Evaluate these logarithms.

a $\log_2 32$ **b** $\log_3 81$ **c** $\log_{10} 1000$ **d** $\log_2 1024$

Solution

a $2^5 = 32$, so $\log_2 32 = 5$

b $3^4 = 81$, so $\log_3 81 = 4$

c $10^3 = 1000$, so $\log_{10} 1000 = 3$

d $2^{10} = 1024$, so $\log_2 1024 = 10$

Example 27

Evaluate these logarithms.

a $\log_4 \frac{1}{16}$ **b** $\log_{10} 0.001$ **c** $\log_3 \frac{1}{27}$ **d** $\log_2 \frac{1}{1024}$

Solution

a $4^{-2} = \frac{1}{16}$, so $\log_4 \frac{1}{16} = -2$

b $10^{-3} = 0.001$, so $\log_{10} 0.001 = -3$

c $3^{-3} = \frac{1}{27}$, so $\log_3 \frac{1}{27} = -3$

d $2^{-10} = \frac{1}{1024}$, so $\log_2 \frac{1}{1024} = -10$

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On most calculators, the button labelled 'log' calculates $\log_{10} x$, for any positive number x.

Example 28

Calculate these logarithms correct to four decimal places.

a $\log_{10} 3$

b $\log_{10} 842$

c $\log_{10} 2$

d $\log_{10} 0.0005$

Solution

a $\log_{10} 3 \approx 0.4771$

b $\log_{10} 842 \approx 2.9253$

c $\log_{10} 2 \approx 0.3010$

d $\log_{10} 0.0005 \approx -3.3010$

In general, simple logarithmic equations are best solved by first converting them into their equivalent exponential form.

Example 29

Find the value of x.

a $\log_2 32 = x$

b $\log_8 \frac{1}{64} = x$

c $\log_2 x = 5$

d $\log_x 16 = 2$

e $\log_{36} x = -\frac{1}{2}$

f $\log_7 x = 2$

Solution

a $\log_2 32 = x$, is equivalent to $2^x = 32$, so $x = 5$

b $\log_8 \frac{1}{64} = x$, is equivalent to $8^x = \frac{1}{64}$, so $x = -2$.

c $\log_2 x = 5$, is equivalent to $2^5 = x$, so $x = 32$.

d $\log_x 16 = 2$, is equivalent to $x^2 = 16$, so $x = 4$ (since $x > 0$).

e $\log_{36} x = -\frac{1}{2}$, is equivalent to $36^{-\frac{1}{2}} = x$, so $x = \frac{1}{6}$.

f $\log_7 x = 2$, is equivalent to $7^2 = x$, so $x = 49$.

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Logarithms

The logarithm of a number to base a is the index to which a is raised to give this number.

If $a > 0$ and $a \neq 1$ and $a^x = y$, then $\log_a y = x$.

Exercise 9G

1 Copy and complete:

a $2^3 = 8$ is equivalent to $\log_2 8 = \ldots$
b $10^2 = 100$ is equivalent to $\log_{10} 100 = \ldots$
c $7^2 = 49$ is equivalent to $\log_7 \ldots = \ldots$
d $3^4 = \ldots$ is equivalent to $\log_3 \ldots = \ldots$
e $5^3 = \ldots$ is equivalent to $\log_5 \ldots = \ldots$
f $7^3 = \ldots$ is equivalent to $\log_7 \ldots = \ldots$
g $2^5 = \ldots$ is equivalent to $\log_2 \ldots = \ldots$
h $10^4 = \ldots$ is equivalent to $\log_{10} \ldots = \ldots$
i $10^{-3} = \ldots$ is equivalent to $\log_{10} \ldots = \ldots$
j $2^{-1} = \ldots$ is equivalent to $\log_2 \ldots = \ldots$

Example 26

2 Evaluate each logarithm.

a $\log_2 4$ **b** $\log_2 64$ **c** $\log_2 128$ **d** $\log_2 4096$
e $\log_2 1$ **f** $\log_2 256$ **g** $\log_{10} 1000$ **h** $\log_5 25$

3 Evaluate:

a $\log_3 27$ **b** $\log_5 625$ **c** $\log_4 64$ **d** $\log_8 64$
e $\log_6 216$ **f** $\log_7 1$ **g** $\log_6 1296$ **h** $\log_9 729$

Example 27

4 Evaluate:

a $\log_2 \frac{1}{4}$ **b** $\log_5 \frac{1}{5}$ **c** $\log_3 \frac{1}{9}$ **d** $\log_{11} \frac{1}{121}$
e $\log_5 \frac{1}{125}$ **f** $\log_4 \frac{1}{1024}$ **g** $\log_3 \frac{1}{81}$ **h** $\log_7 \frac{1}{343}$

5 Evaluate:

a $\log_{10} 10$ **b** $\log_{10} 1$ **c** $\log_{10} 1000$ **d** $\log_{10} 100\,000$
e $\log_{10} 10^{100}$ **f** $\log_{10} \frac{1}{100}$ **g** $\log_{10} 0.000\,0001$ **h** $\log_{10} 10^{-13}$

6 If $a > 0$, what is $\log_a a$?

7 If $a > 0$, what is $\log_a 1$?

Example 28

8 Use your calculator to evaluate each logarithm correct to four decimal places.

a $\log_{10} 789$ **b** $\log_{10} 0.0003$ **c** $\log_{10} 72\,000\,000$
d $\log_{10}(5.3950 \times 10^{-3})$ **e** $\log_{10}(635 \times 10^{54})$ **f** $\log_{10} 0.000\,123\,45$

Example 29

9 Find the value of x.

a $\log_2 64 = x$ **b** $\log_3 243 = x$ **c** $\log_4 \frac{1}{256} = x$ **d** $\log_{10} \frac{1}{1000} = x$
e $\log_3 x = 3$ **f** $\log_5 x = 2$ **g** $\log_2 x = -3$ **h** $\log_{25} x = -\frac{1}{2}$
i $\log_x 16 = 4$ **j** $\log_x 16 = 2$ **k** $\log_x 125 = 3$ **l** $\log_x \frac{1}{8} = -3$

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Review exercise

1 Simplify:

a $(a^3)^4 \times a^5$ **b** $(2m^3)^4 \times (3m)^4$ **c** $\dfrac{(a^3b^2)^4}{(a^2b^2)^3}$

2 Evaluate:

a 4^{-2} **b** $6a^0$ **c** 10^{-4} **d** $\left(\dfrac{2}{3}\right)^{-4}$

3 Simplify each expression, writing each pronumeral with a positive index.

a $a^{-3} \times a^{-5}$ **b** $2a^3 \times 7a^{-6}$ **c** $\dfrac{12a^4}{3a^6}$ **d** $(2a^{-1})^2 \times (4a^2)^{-2}$

4 Write each term with positive indices only.

a b^{-3} **b** $2x^{-4}$ **c** $5x^{-3}$

d $\dfrac{x^{-3}}{2}$ **e** $\dfrac{a^{-4}}{5}$ **f** $\dfrac{2}{x^{-2}}$

g $\dfrac{4}{x^{-3}}$ **h** $\dfrac{4a^{-2}}{b^{-3}}$ **i** $\dfrac{5m^{-1}}{6m^{-4}}$

5 Express each power as a fraction.

a 6^{-2} **b** 4^{-3} **c** 2^{-4} **d** 5^{-1} **e** 10^{-2}

6 Simplify each expression.

a 5^0 **b** $5a^0$ **c** $(5a)^0$

d $6 + a^0$ **e** $(4 + a)^0$ **f** $2 + 3b^0$

g $\left(\dfrac{2}{3}\right)^0$ **h** $\dfrac{2^0}{3}$ **i** $\dfrac{4a^0}{(7b)^0}$

7 Simplify each expression, giving your answers with positive indices.

a $\dfrac{2a^2(2b)^3}{2ab^2}$ **b** $\dfrac{a^2b^3}{ab} \times \dfrac{a^2b^5}{a^2b^2}$ **c** $\dfrac{(2a)^2 \times 8b^3}{16a^2b^2}$

d $\dfrac{2a^2b^3}{8a^2b^2} \div \dfrac{16(ab)^2}{2ab}$ **e** $\dfrac{8a^6}{6a^3} \div \dfrac{4(a^2)^4}{(3a)^3}$ **f** $\dfrac{3a^3}{6a^{-1}}$

8 Write $\dfrac{2^n \times 8^n}{2^{2n} \times 16}$ in the form 2^{an+b}.

9 Write $2^{-x} \times 3^{-x} \times 6^{2x} \times 3^{2x} \times 2^{2x}$ as a power of 6.

10 Simplify each product.

a $2^{\frac{1}{3}} \times 2^{\frac{1}{6}} \times 2^{-\frac{2}{3}}$ **b** $a^{\frac{1}{4}} \times a^{\frac{2}{5}} \times a^{-\frac{1}{10}}$

c $2^{\frac{1}{3}} \times (2^{\frac{2}{5}})^5$ **d** $(2^{\frac{1}{3}})^2 \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}}$

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11 Write each number in scientific notation.

a 4200 **b** 0.0062 **c** 740 000 000 **d** 0.000 0002

12 Write each number in decimal notation.

a 5.4×10^3 **b** 11.2×10^4 **c** 6.8×10^{-2} **d** 9.7×10^{-3}

e 1.8×10^{-1} **f** 6.4×10^{-5} **g** 7.41×10^6 **h** 4.02×10^2

13 Write each number correct to the number of significant figures specified in the brackets.

a 18 (1) **b** 495 (1) **c** 416 (2)

d 34 200 (2) **e** 0.006 81 (2) **f** 0.049 21 (3)

g 475.2 (2) **h** 598.7 (2) **i** 0.006 842 (1)

14 Evaluate:

a $\log_2 8$ **b** $\log_2 16$ **c** $\log_2 \frac{1}{4}$ **d** $\log_3 1$

e $\log_5 \frac{1}{25}$ **f** $\log_4 \frac{1}{64}$ **g** $\log_3 \frac{1}{81}$ **h** $\log_7 \frac{1}{343}$

15 Evaluate:

a $\log_{10} 10$ **b** $\log_{10} 100\,000$ **c** $\log_{10} 10^{15}$

d $\log_{10} \frac{1}{10}$ **e** $\log_{10} \frac{1}{100}$ **f** $\log_{10} 10^{-6}$

16 Solve each equation for x.

a $4^x = 32^{x+1}$ **b** $(3^{x+2})^3 = \frac{1}{3}$ **c** $3^{x+1} = \frac{1}{81^x}$

d $5^{3x} \div 5^{2(x-1)} = 1$ **e** $2^7 \times 4^x = \frac{1}{8}$ **f** $9x = 27^4$

17 Evaluate:

a $2^{\frac{1}{3}} \times 12^{\frac{2}{3}} \times 6^{\frac{1}{3}}$ **b** $2^{-3} \times 4^{\frac{1}{2}} \times 8^{\frac{1}{3}}$ **c** $8^{-\frac{4}{3}} \times 32^{\frac{6}{5}}$

d $8^{-\frac{2}{3}} \times 16^{-\frac{5}{4}}$ **e** $16^{-\frac{3}{4}} \times 4$ **f** $7^{\frac{3}{2}} \times 7^{-1}$

18 Simplify, expressing your answers with positive indices.

a $\left(\frac{3^{-2}a}{b^2}\right)^{-2} \times \frac{27}{a^2b^2}$ **b** $\left(\frac{x^2}{y^{-2}}\right)^{-3} \times \left(\frac{x^2}{y^3}\right)$

c $\frac{(3a^2)^3}{(2ab^2)^2} \times \frac{(2b)^{-5}}{(3a)^{-4}}$ **d** $\frac{(a^2b)^3 \times (ab^3)^{-1}}{(a^{-1}b)^{-4}}$

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19 Solve for x.

a $\log_2 x = 5$
b $\log_3 x = 7$
c $\log_5 x = 0$
d $\log_7 x = 2$
e $\log_{10} x = -1$
f $\log_5 x = -\frac{1}{2}$
g $\log_x 25 = 2$
h $\log_x 81 = 4$
i $\log_x 10\,000 = 4$

20 The population of a town is initially 8000. Every year the population increases by 5%. What is the population of the town after:

a 1 year?
b 3 years?
c n years?

Challenge exercise

1 a If $2^y = x$, what is $15 \times 2^{y+3}$, in terms of x?

b If $3^x = 2$, find 3^{7x}.

c If $4^y = x$, what is 4^{y-2}, in terms of x?

2 Find the value of x if $a^x = \dfrac{\sqrt{a^8}}{\left(\sqrt{a^6}\right)^4}$.

3 Find the value of x if $t^x = \sqrt[3]{\dfrac{t}{\sqrt{t}}}$.

4 Find the value of x if $\dfrac{4^{\frac{1}{2}}}{\sqrt[3]{8^2}} = 2^x$.

5 a Evaluate $2^8 + 2^{11} + 2^n$ for n between 1 and 8.

b Find the value of $n > 8$ such that $2^8 + 2^{11} + 2^n$ is a perfect square.

6 a Prove that the index laws hold for negative integer exponents. (Use the laws for positive integer exponents.)

For example, the product-of-powers result can be proved in the following way for negative integer exponents.

Consider $a^{-p}a^{-q}$ where p and q are positive integers.

$$\begin{aligned} a^{-p}a^{-q} &= \frac{1}{a^p} \times \frac{1}{a^q} \\ &= \frac{1}{a^p a^q} \\ &= \frac{1}{a^{p+q}} && \text{(Index law 1 for positive integers)} \\ &= a^{-(p+q)} \\ &= a^{-p+(-q)} \end{aligned}$$

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b Prove that the index laws hold for fractional exponents.

For example, $a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{1}{n}+\frac{1}{m}}$ can be proved in the following way.

$$\begin{aligned} a^{\frac{1}{n}} \times a^{\frac{1}{m}} &= a^{\frac{m}{nm}} \times a^{\frac{n}{nm}} \\ &= \sqrt[nm]{a^m} \times \sqrt[nm]{a^n} \\ &= \sqrt[nm]{a^m \times a^n} \qquad \text{(Index law 1)} \\ &= \sqrt[nm]{a^{m+n}} \\ &= a^{\frac{m+n}{nm}} \\ &= a^{\frac{1}{n}+\frac{1}{m}} \end{aligned}$$

The result can easily be extended to $a^{\frac{p}{n}} \times a^{\frac{q}{m}} = a^{\frac{p}{n}+\frac{q}{m}}$.

7 Solve each pair of equations for x and y.

a $25^x = 125^y$, $16^x \div 8 = 2 \times 4^{2y}$

b $3^y - 1 = 9^x$, $4^y \times 64^x = 128$

c $10^{5y} = 10^5 \times 100^x$, $49^y = 7 \times 7^x$

d $a^{2x} = a^{y-1}$, $b^{2+y} = b^{3x}$

8 Simplify:

a $\dfrac{a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} - c^{\frac{2}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}} - c^{\frac{1}{3}}}$

b $\dfrac{x^2 + x^{-2} - 1}{x + x^{-1} - 3^{\frac{1}{2}}}$

9 Expand:

a $\left(3a^{\frac{2}{3}} - 2a^{-\frac{2}{3}}b^{\frac{1}{2}} - b^{-\frac{1}{2}}\right)\left(a^{\frac{1}{3}} - 2b\right)$

b $\left(a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}}\right)\left(a^{\frac{1}{4}} - b^{\frac{1}{2}}\right)$

10 Without using a calculator, list the numbers $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$ and $5^{\frac{1}{5}}$ in order from greatest to least.

11 The areas of the side, front and bottom faces of a rectangular prism are $2x$, $\dfrac{y}{2}$ and xy. Find the volume of the prism in terms of x and y.

12 Simplify $\dfrac{5^{3x+1} - 5^{3x-1} + 24}{24 \times 5^{3x} + 120}$.

13 Find the sum of the digits of $10^{2008} - 2008$.

ICE-EM Mathematics 10 3ed ISBN 978-1-108-40434-1