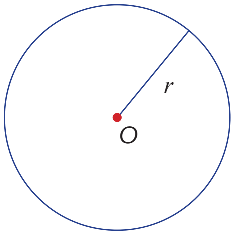
CHAPTER

11

Number and Algebra

Circles, hyperbolas and simultaneous equations

A circle with centre O and radius r is the set of all points whose distance from the centre O is equal to r.



In this chapter, we study circles using the techniques of coordinate geometry.

We also introduce rectangular hyperbolas, and describe methods for finding the coordinates of the points of intersection of hyperbolas, parabolas and circles with straight lines.

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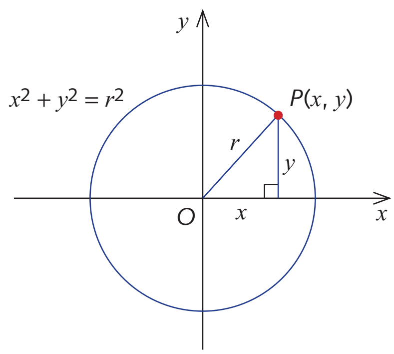
11A Cartesian equation of a circle

Circles with centre the origin

Consider a circle in the coordinate plane with centre the origin and radius r. Throughout this chapter, we will always assume that $r > 0$.



If $P(x, y)$ is a point on the circle, then its distance from the origin is r. By Pythagoras' theorem, this gives $x^2 + y^2 = r^2$.

Conversely, if a point $P(x, y)$ satisfies the equation $x^2 + y^2 = r^2$, then its distance from $O(0, 0)$ is $\sqrt{x^2 + y^2} = r$, so it lies on the circle with centre the origin and radius r.

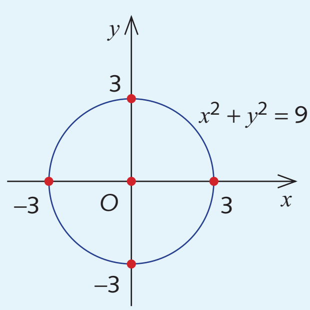
Example 1

Sketch the graphs of the circles with the following equations.

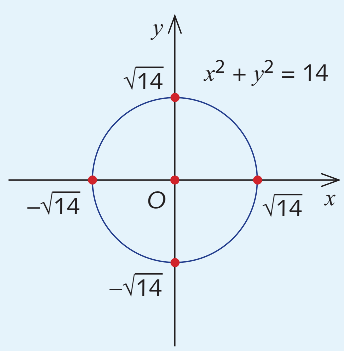
a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 14$

Solution

a Centre is (0,0) and radius is 3.



b Centre is (0,0) and radius is $\sqrt{14}$.



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Example 2

Sketch the graph of the circle $x^2 + y^2 = 25$ and verify that the points (3, 4), (−3, 4), (−3, −4) and (4, −3) lie on the circle.

Solution

The circle has centre the origin and radius 5.

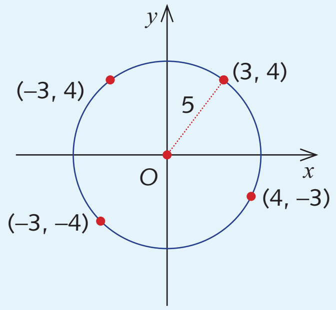
To verify that a point lies on the circle, we substitute the coordinates into $x^2 + y^2 = 25$.

The point (3, 4) lies on the circle, since $3^2 + 4^2 = 25$.

The point (−3, 4) lies on the circle, since $(-3)^2 + 4^2 = 25$.

The point (−3, −4) lies on the circle, since $(-3)^2 + (-4)^2 = 25$.

The point (4, −3) lies on the circle, since $(4)^2 + (-3)^2 = 25$.

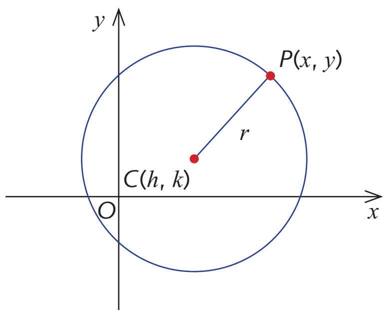


Circles with centre not the origin

Now take a circle in the coordinate plane with centre at the point $C(h, k)$ and radius r.

If $P(x, y)$ is a point on the circle, then by the distance formula:

$$(x - h)^2 + (y - k)^2 = r^2$$



Conversely, if a point $P(x, y)$ satisfies the equation $(x - h)^2 + (y - k)^2 = r^2$, then its distance from (h, k) is r, so it lies on a circle with centre $C(h, k)$ and radius r.

We call $(x - h)^2 + (y - k)^2 = r^2$ the **standard form for the equation of a circle**.

Circles

- The circle with centre $O(0,0)$ and radius r has equation:

  $x^2 + y^2 = r^2$

- The standard form for the equation of the circle with centre (h, k) and radius r is:

  $(x - h)^2 + (y - k)^2 = r^2$

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Example 3

Sketch the graph of each circle, showing any intercepts.

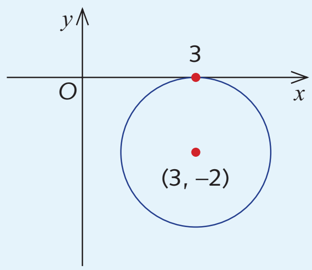
a $(x-3)^2 + (y+2)^2 = 4$

b $(x+1)^2 + (y-3)^2 = 25$

Solution

a The circle has centre $(3, -2)$ and radius 2, and hence the circle touches the x-axis. That is, it meets the x-axis but does not cross it.

The circle does not meet the y-axis.



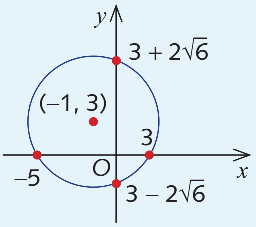
b The circle has centre $(-1,3)$ and radius 5.

Put $y = 0$ into the equation to find where the circle cuts the x-axis.

$$(x+1)^2 + (0-3)^2 = 25$$
$$(x+1)^2 + 9 = 25$$
$$(x+1)^2 = 16$$
$$x+1 = 4 \quad \text{or } x+1 = -4$$
$$x = 3 \quad \text{or} \quad x = -5$$

Put $x = 0$ into the equation to find where the circle cuts the y-axis.

$$(0+1)^2 + (y-3)^2 = 25$$
$$1 + (y-3)^2 = 25$$
$$(y-3)^2 = 24$$
$$y-3 = 2\sqrt{6} \quad \text{or } y-3 = -2\sqrt{6}$$
$$y = 3 + 2\sqrt{6} \quad \text{or} \quad y = 3 - 2\sqrt{6}$$



Note: The circle $(x-3)^2 + (y+2)^2 = 4$ is a translation of the circle $x^2 + y^2 = 4$, three units to the right and two units down.

The circle $(x+1)^2 + (y-3)^2 = 25$ is the image of the circle $x^2 + y^2 = 25$ under a translation of 1 unit to the left and 3 units up.

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Finding the centre and radius of a circle by completing the square

In Example 3a, we sketched the graph of $(x-3)^2+(y+2)^2=4$.
Expanding the brackets, we obtain:

$$x^2-6x+9+y^2+4y+4=4$$

which simplifies to:

$$x^2+y^2-6x+4y+9=0$$

This is still the equation of the circle with centre $(3, -2)$ and radius 2, but in this form it is not clear what the centre and radius are.

Completing the square enables us to reverse the process and to express the equation in standard form. We can then read off the centre and radius.

Converting to standard form

To find the centre and radius of a circle, complete the square in both x and y to write the equation in standard form. Then read off the centre and the radius.

Example 4

Express each equation in the standard form $(x-h)^2+(y-k)^2=r^2$ and hence write down the centre and the radius of the circle.

a $x^2+4x+y^2+6y+4=0$

b $x^2-4x+y^2+8y-5=0$

Solution

a

$$(x^2+4x)+(y^2+6y)=-4$$

(Group together the x-terms and y-terms on one side of the equation.)

$$(x^2+4x+4)+(y^2+6y+9)=-4+4+9$$

(Complete the square for the quadratic in x and the quadratic in y.)

$$\begin{aligned}(x+2)^2+(y+3)^2&=9\\&=3^2\end{aligned}$$

Hence, the centre of the circle is $(-2, -3)$ and the radius is 3.

b

$$(x^2-4x)+(y^2+8y)=5$$

$$(x^2-4x+4)+(y^2+8y+16)=5+4+16$$

(Complete the square.)

$$\begin{aligned}(x-2)^2+(y+4)^2&=25\\&=5^2\end{aligned}$$

Hence, the centre of the circle is $(2, -4)$ and the radius is 5.

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Exercise 11A

Example 1

1 Sketch the graph of each circle, marking any intercepts.

a $x^2 + y^2 = 25$ b $x^2 + y^2 = 1$ c $x^2 + y^2 = 2$ d $x^2 + y^2 = 3$

2 Sketch the graphs, marking any intercepts.

a $y^2 = 4 - x^2$ b $y^2 = -x^2 + 10$ c $x^2 = 5 - y^2$ d $x^2 = -y^2 + 8$

Example 2

3 Check whether or not each point lies on the circle $x^2 + y^2 = 100$.

a $(6,8)$ b $(10,10)$ c $(20,80)$

d $(-6,8)$ e $(5\sqrt{2},5\sqrt{2})$ f $(10,0)$

4 Check whether or not each point lies on the circle $x^2 + y^2 = 169$.

a $(5,12)$ b $(100,69)$ c $(-5,-12)$

d $(-5,12)$ e $(-13\sqrt{2},13\sqrt{2})$ f $(0,13)$

Example 3

5 Sketch the graphs, showing the x- and y-intercepts.

a $(x-1)^2 + (y-2)^2 = 4$ b $(x-3)^2 + (y-4)^2 = 25$ c $(x-2)^2 + (y-3)^2 = 9$

d $(x-3)^2 + (y-1)^2 = 16$ e $(x-1)^2 + y^2 = 4$ f $x^2 + (y-4)^2 = 16$

Example 4

6 Complete the square in x and y to find the coordinates of the centre and the radius of each circle.

a $x^2 + 4x + y^2 + 6y + 9 = 0$ b $x^2 - 2x + y^2 + 8y + 4 = 0$

c $x^2 - 6x + y^2 - 8y = 39$ d $x^2 - 14x + y^2 - 8y + 40 = 0$

e $x^2 - 8x + y^2 - 6y + 15 = 0$ f $x^2 - 8x + y^2 - 4y + 10 = 0$

7 Write down the equation of the circle with:

a centre $(1,3)$ and radius 3 b centre $(-2,1)$ and radius 4

c centre $(4,-1)$ and radius 1 d centre $(2,0)$ and radius 2

8 Show that the point $(17,17)$ lies on the circle with centre $(5,12)$ and radius 13. Find the equation of the circle.

9 Find the equation of the circle with centre $(3,-4)$ passing through the origin.

10 a Find the equation of the circle with centre $(6,7)$ that touches the y-axis.

b Find the equation of the circle with centre $(6,7)$ that touches the x-axis.

11 The interval AB joins the points $A(2,6)$ and $B(8,6)$. Find:

a the distance AB b the midpoint of AB

c the equation of the circle with diameter AB

12 The interval AB joins the points $A(1,6)$ and $B(3,-8)$. Find:

a the distance AB b the midpoint of AB

c the equation of the circle with diameter AB

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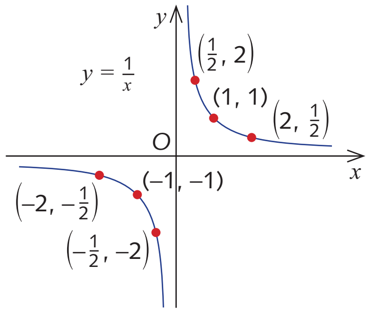
The basic rectangular hyperbola

In Chapter 7, we called $y = x^2$ the basic parabola and then we showed how to obtain other parabolas from the basic parabola by using transformations.

Similarly, we shall call the hyperbola $y = \frac{1}{x}$ the **basic rectangular hyperbola**.

To see what the graph of $y = \frac{1}{x}$ looks like, begin by considering the table of values below.



x	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$-\frac{1}{2}$	–1	–2	–	2	1	$\frac{1}{2}$

Since division by zero is not allowed, there is no y-value when $x = 0$.

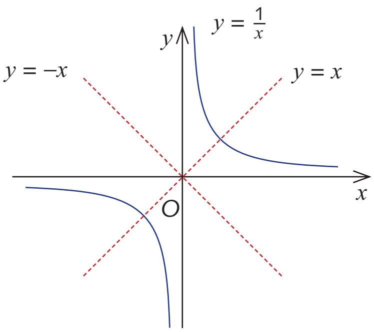
To see more clearly what is happening to the curve close to zero, we produce the table of values for $y = \frac{1}{x}$ below.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
y	–10	–100	–1000	–	1000	100	10

Can you see how these values are reflected in the graph close to the y-axis?

And as x gets larger and larger in the positive and negative directions (moves further away from zero), y gets smaller and smaller. For example, when $x = 100$, $y = 0.01$ and when $x = 10\,000$, $y = 0.0001$. Similarly, when $x = -100$, $y = -0.01$ and when $x = -10\,000$, $y = -0.0001$.

Features of $y = \frac{1}{x}$



- There are no x-intercepts and no y-intercepts.
- When x is a large positive number, y is a small positive number.
- When x is a small positive number, y is a large positive number.
- Similar results hold for large and small negative values of x.
- The x-axis and the y-axis are called **asymptotes** to the graph. The graph gets very close to each of these lines, but never meets them.
- The lines $y = x$ and $y = -x$ are axes of symmetry for the graph of $y = \frac{1}{x}$.

The types of transformations applied to parabolas in Chapter 7 will now be applied to a **rectangular hyperbola**. The word 'rectangular' means that the asymptotes are perpendicular.

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Reflection in the x-axis

In Chapter 7, we saw that $y = -x^2$ is the reflection of $y = x^2$ in the x-axis. Similarly, $y = -\frac{1}{x}$ is the reflection of $y = \frac{1}{x}$ in the x-axis.

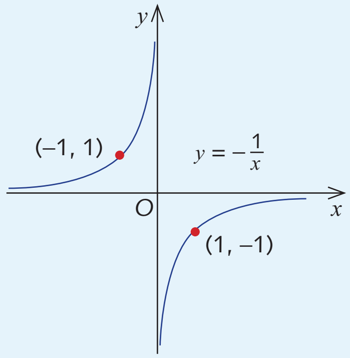
Example 5

Sketch the graph of $y = -\frac{1}{x}$.

Solution

The graph of $y = -\frac{1}{x}$ is the reflection of $y = \frac{1}{x}$ in the x-axis.

The graph of $y = -\frac{1}{x}$ has been drawn.



Horizontal translations

In Chapter 7, we saw that the graph of $y = x^2$ becomes:

- the graph of $y = (x - 5)^2$ when translated 5 units to the right
- the graph of $y = (x + 4)^2$ when translated 4 units to the left.

In a similar way, the graph of $y = \frac{1}{x}$ becomes:

- the graph of $y = \frac{1}{x - 5}$ when translated 5 units to the right
- the graph of $y = \frac{1}{x + 4}$ when translated 4 units to the left.

Example 6

Sketch the graph of $y = \frac{1}{x - 3}$.

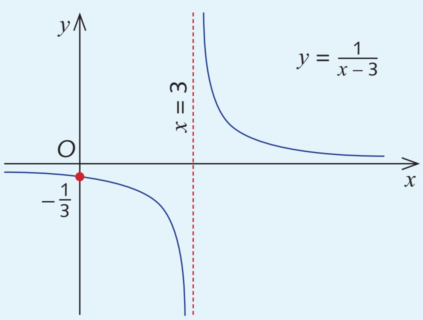
Solution

The graph is obtained by translating the graph of $y = \frac{1}{x}$ three units to the right.

The vertical asymptote has equation $x = 3$.

The horizontal asymptote remains $y = 0$.

The y-intercept is found by putting $x = 0$ into the equation, and so it is $-\frac{1}{3}$. There are no x-intercepts.



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Vertical translations

In Chapter 7, we saw that the graph of $y = x^2$ becomes:

- the graph of $y = x^2 + 5$ when translated 5 units up
- the graph of $y = x^2 - 4$ when translated 4 units down.

In a similar way, the graph of $y = \frac{1}{x}$ becomes:

- the graph of $y = \frac{1}{x} + 5$ when translated 5 units up
- the graph of $y = \frac{1}{x} - 4$ when translated 4 units down.

Example 7

Sketch the graph of $y = \frac{1}{x} + 2$.

Solution

The graph of $y = \frac{1}{x} + 2$ is obtained by translating the graph of $y = \frac{1}{x}$ two units up.

The horizontal asymptote has equation $y = 2$. The vertical asymptote remains $x = 0$.

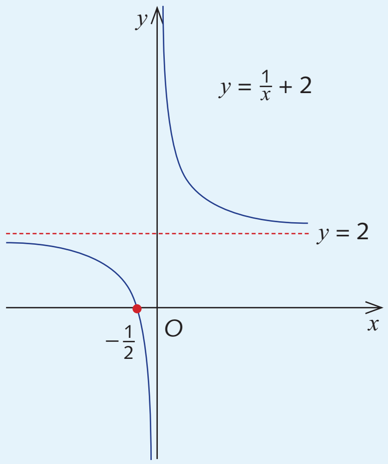
To find the x-intercept, put $y = 0$ into the equation.

$$0 = \frac{1}{x} + 2$$

$$\frac{1}{x} = -2$$

$$x = -\frac{1}{2}$$

There is no y-intercept.



Translations of the basic rectangular hyperbola

- The graph of $y = \frac{1}{x - h}$, where h is a positive number, can be obtained by translating the graph of $y = \frac{1}{x}$ by h units to the right. The equation of the vertical asymptote is $x = h$.
- The graph of $y = \frac{1}{x} + k$, where k is a positive number, can be obtained by translating the graph of $y = \frac{1}{x}$ by k units up. The equation of the vertical asymptote is $y = k$.
- Similar statements apply for translations to the left and translations down.

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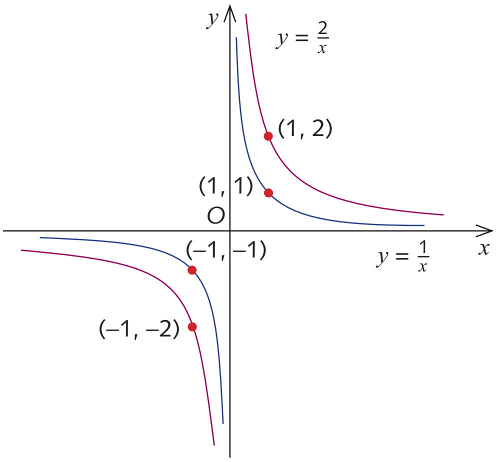
The rectangular hyperbola $y = \dfrac{a}{x}$

The graph of $y = \dfrac{2}{x}$ is obtained from the graph of $y = \dfrac{1}{x}$ by transforming each point $\left(p, \dfrac{1}{p}\right)$, where $p \neq 0$, to $\left(p, \dfrac{2}{p}\right)$. The y-coordinate is multiplied by 2.

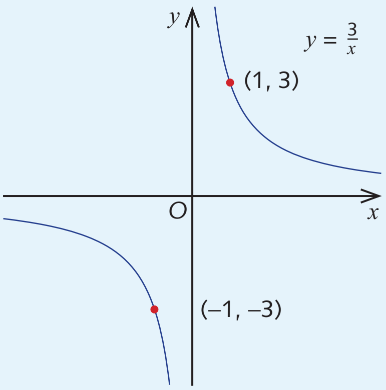
The rectangular hyperbola $y = \dfrac{2}{x}$ is obtained by stretching the basic hyperbola $y = \dfrac{1}{x}$ by a factor of 2 from the x-axis.

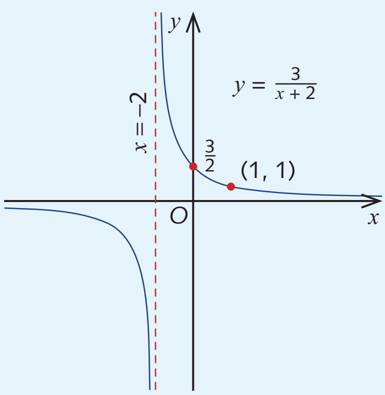


Example 8

Sketch the graph of $y = \dfrac{3}{x+2}$ by first sketching the graph of $y = \dfrac{3}{x}$.

Solution





The graph of $y = \dfrac{3}{x+2}$ is obtained by translating the graph of $y = \dfrac{3}{x}$ two units to the left.

Exercise 11B

1 Given that $y = \dfrac{1}{x}$, find y when:

a $x = 2$ **b** $x = -2$ **c** $x = \dfrac{1}{2}$ **d** $x = \dfrac{3}{2}$ **e** $x = -\dfrac{2}{3}$

2 Given that $y = \dfrac{12}{x}$, find y when:

a $x = 3$ **b** $x = 4$ **c** $x = -\dfrac{1}{2}$ **d** $x = -\dfrac{3}{2}$ **e** $x = \dfrac{3}{4}$

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3 Given that $y = -\frac{1}{x}$, find y when:

a $x = -1$ **b** $x = -2$ **c** $x = -\frac{1}{2}$ **d** $x = \frac{3}{2}$ **e** $x = -\frac{3}{2}$

4 Given that $y = \frac{1}{x-3}$, find y when:

a $x = 4$ **b** $x = 2$ **c** $x = 5$ **d** $x = 3\frac{1}{2}$ **e** $x = 2\frac{3}{4}$

5 **a** Sketch the graph of $y = \frac{4}{x}$.

b Find the values of y when $x = -4, -2, -1, 1, 2$ and 4, and plot the corresponding points on the graph.

6 **a** Sketch the graph of $y = \frac{1}{x+2}$.

b Find the values of y when $x = -4, -3, 2\frac{1}{2}, 1\frac{1}{2}, -1$ and 0, and plot the corresponding points on the graph.

7 On the hyperbola $y = \frac{12}{x}$, find the value of y when x equals:

a –0.001 **b** –0.2 **c** 3 **d** 24 **e** 144

8 On the hyperbola $y = \frac{6}{x-3}$, find the value of y when x equals:

a 0 **b** 1 **c** 2.99 **d** 3.01 **e** 1000

9 On the hyperbola $y = \frac{12}{x+3}$, find the value of y when x equals:

a 0 **b** –2.9 **c** –2.99 **d** –3.01 **e** –3.001

Example 5 **10** Sketch each graph, and indicate two points on each graph.

a $y = \frac{3}{x}$ **b** $y = \frac{3}{2x}$ **c** $y = -\frac{1}{x}$ **d** $y = -\frac{3}{x}$

Example 6 **11** Sketch each graph, showing the asymptotes and any intercepts.

a $y = \frac{1}{x-4}$ **b** $y = \frac{1}{x-2}$ **c** $y = \frac{1}{x+3}$ **d** $y = \frac{-1}{x+1}$

Example 7 **12** Sketch each graph, showing the asymptotes and any intercepts.

a $y = \frac{1}{x} + 1$ **b** $y = \frac{1}{x} - 3$ **c** $y = -\frac{1}{x} + 4$ **d** $y = \frac{1}{x} - 1$

Example 8 **13** Sketch each graph, showing the asymptotes and any intercepts.

a **i** $y = \frac{6}{x}$ **ii** $y = \frac{6}{x-3}$

b **i** $y = \frac{10}{x}$ **ii** $y = \frac{10}{x-5}$

c **i** $y = \frac{4}{x}$ **ii** $y = \frac{4}{x+2}$

d **i** $y = \frac{-3}{x}$ **ii** $y = \frac{-3}{x+1}$

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14 Sketch each graph, showing the asymptotes and any intercepts.

a $y = \dfrac{2}{x} + 1$ **b** $y = \dfrac{4}{x} - 3$ **c** $y = -\dfrac{12}{x} + 4$ **d** $y = \dfrac{2}{x} - 1$

11C Intersections of graphs

In this section we will look at the intersections of:

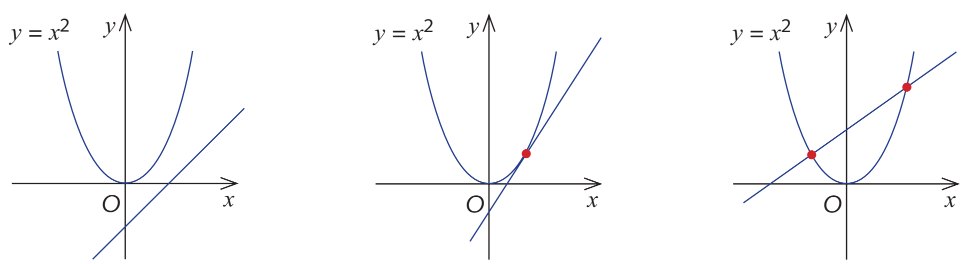
- lines and parabolas
- lines and circles
- lines and rectangular hyperbolas.

In Chapter 4, we looked at the intersections of lines.

Two distinct lines meet at zero points or 1 point. In the situations listed above, there are always 0, 1 or 2 points of intersection. We find these points of intersection by solving simultaneous equations.

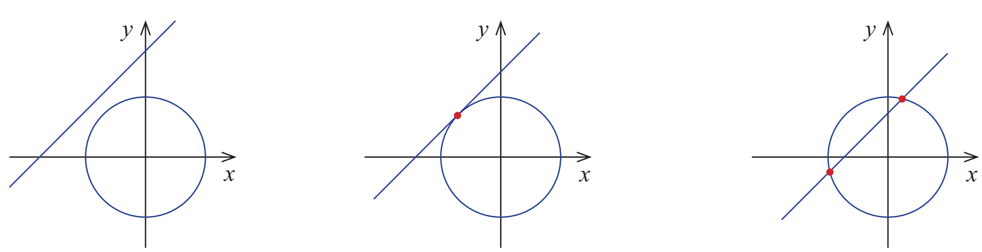
That is, we shall be using algebra to solve problems in geometry.

A straight line and a parabola



0 points of intersection 1 point of intersection 2 points of intersection

A straight line and a circle



0 points of intersection 1 point of intersection 2 points of intersection

When there is just one point of intersection between a circle and a line, the line is called a **tangent** to the circle (see Chapter 13).

A straight line and a rectangular hyperbola

The following diagrams show that when a straight line and a rectangular hyperbola are drawn there may be 0, 1 or 2 points of intersection.

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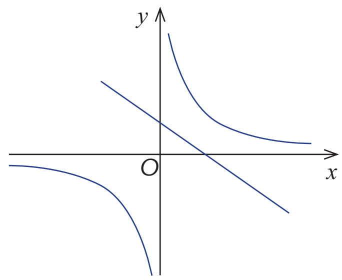
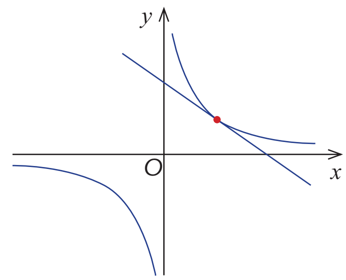
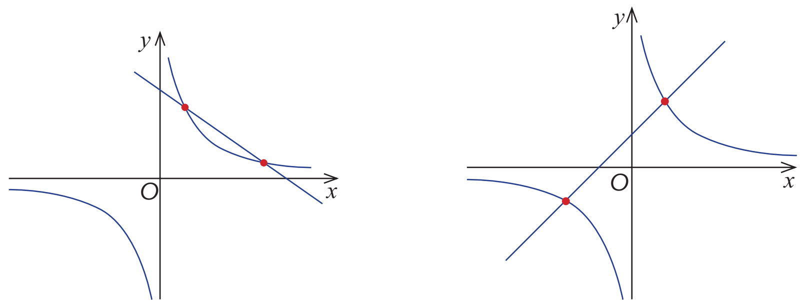


0 points of intersection



1 point of intersection



2 points of intersection

Example 9

Find the coordinates of the points of intersection of the graphs of $y = 4 - x^2$ and $y = 4 - x$, and illustrate your answer graphically.

Solution

To find the points of intersection, solve the equations simultaneously.

$y = 4 - x^2 \qquad (1)$

$y = 4 - x \qquad (2)$

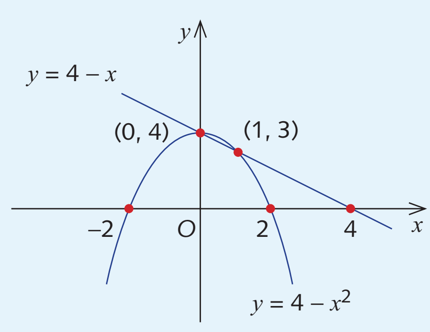
At the points of intersection, the y-values are the same, so:

$$4 - x^2 = 4 - x$$
$$x^2 - x = 0$$
$$x(x - 1) = 0$$
$$x = 0 \text{ or } x = 1$$

When $x = 0, y = 4$.

When $x = 1, y = 3$.

So the two points of intersection are $(0, 4)$ and $(1, 3)$.



Equating the two expressions for y is called **eliminating *y***. We have used this previously for simultaneous linear equations. For all three curves, parabolas, circles and rectangular hyperbolas, a quadratic equation results from eliminating y. This equation will have 0, 1 or 2 solutions, each situation graphically related to 0, 1 or 2 points of intersection, respectively.

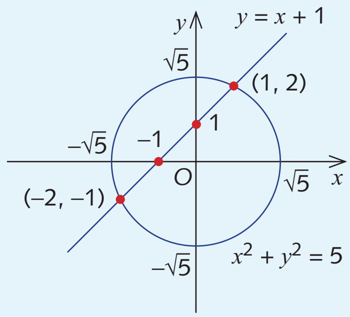
Example 10

Find the points of intersection of the circle $x^2 + y^2 = 5$ and the line $y = x + 1$. Illustrate this graphically.

Solution

We have $x^2 + y^2 = 5$ (1)

$y = x + 1$ (2)



Substituting the right-hand side of (2) into (1):

$$x^2 + (x + 1)^2 = 5$$
$$x^2 + x^2 + 2x + 1 = 5$$
$$2x^2 + 2x - 4 = 0$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } x = 1$$

To find the y-values, substitute the values of x into equation (2).

When $x = -2$, $y = -1$.

When $x = 1$, $y = 2$.

So the line cuts the circle at the points $(-2, -1)$ and $(1, 2)$.

Substituting the x-values into equation (1) does not determine the y-values. Check what happens yourself.

Example 11

Find the point of intersection of the line $y = 2x + 5$ and the circle $x^2 + y^2 = 5$. Illustrate this graphically.

Solution

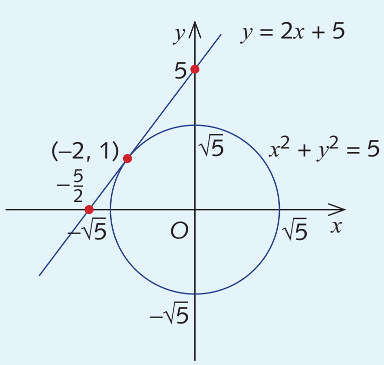
$y = 2x + 5$ (1)

$x^2 + y^2 = 5$ (2)

Substituting from (1) into (2):

$$x^2 + (2x + 5)^2 = 5$$
$$x^2 + 4x^2 + 20x + 25 = 5$$
$$5x^2 + 20x + 20 = 0$$
$$x^2 + 4x + 4 = 0$$
$$(x + 2)^2 = 0$$
$$x = -2$$

and $y = 2 \times (-2) + 5 = 1$



Thus the line meets the circle at one point $(-2, 1)$.

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Example 12

Show that the line $y = x + 4$ does not meet the circle $x^2 + y^2 = 1$. Illustrate this graphically.

Solution

$$y = x + 4 \quad (1)$$
$$x^2 + y^2 = 1 \quad (2)$$

Substituting from (1) into (2):

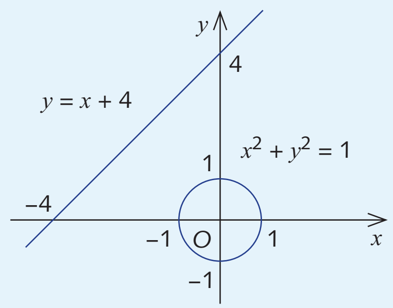
$$x^2 + (x + 4)^2 = 1$$
$$x^2 + x^2 + 8x + 16 = 1$$
$$2x^2 + 8x + 15 = 0$$

For this quadratic equation:

$$\Delta = b^2 - 4ac = 64 - 4 \times 2 \times 15$$
$$= 64 - 120$$
$$= -56 < 0$$

so $b^2 - 4ac < 0$ and there are no solutions to the quadratic equation.

Hence, the line does not meet the circle.



Example 13

Find where the hyperbola $y = \frac{2}{x}$ meets the line $y = x + 1$ and illustrate this graphically.

Solution

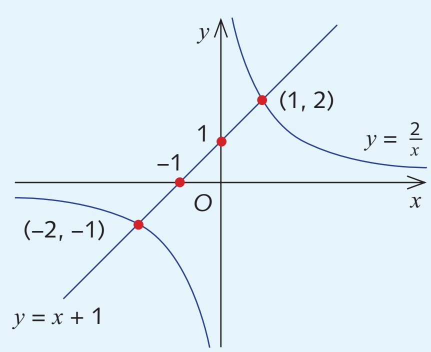
$$y = \frac{2}{x} \quad (1)$$
$$y = x + 1 \quad (2)$$

Eliminating y from equations (1) and (2):

$$x + 1 = \frac{2}{x}$$
$$x^2 + x = 2$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } x = 1$$

Thus, $y = -1$ or $y = 2$ (either from equation (1) or (2))

Hence, the hyperbola meets the line at $(-2, -1)$ and $(1, 2)$.



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Intersection of graphs

- To find the points of intersection of graphs, solve the equations simultaneously.
- A line meets a parabola, a rectangular hyperbola or a circle at 0, 1 or 2 points.

Exercise 11C

Example 9

1 Find the coordinates of the points of intersection of:

a $y = x^2$ and $y = 4$

b $y = x^2$ and $y = 1$

c $y = (x - 1)^2$ and $y = 2x - 3$

d $y = x^2$ and $y = 7x - 12$

2 Find the coordinates of the points of intersection of:

a $y = x^2 + 3x + 3$ and $y = x + 2$

b $y = x^2 + 5x + 2$ and $y = x + 7$

c $y = x^2 + 2x + 4$ and $y = x + 6$

d $y = 2x^2 + 3x + 1$ and $y = 2x + 1$

e $y = 3x^2 + x + 2$ and $y = 3x + 3$

f $y = 6x^2 + 9x + 5$ and $y = 2x + 3$

Example 10

3 Find the coordinates of the points of intersection of:

a $x^2 + y^2 = 4$ and $x = 2$

b $x^2 + y^2 = 9$ and $y = 0$

c $x^2 + y^2 = 32$ and $y = x$

d $x^2 + y^2 = 81$ and $y = 2\sqrt{2}x$

Example 10, 11, 12

4 For each pair of curves, find the points of intersection and illustrate with a graph.

a $x^2 + y^2 = 10$ and $y = x + 2$

b $x^2 + y^2 = 17$ and $y = 3 - x$

c $x^2 + y^2 = 26$ and $x + y = 4$

d $x^2 + y^2 = 20$ and $y = 2x$

e $x^2 + y^2 = 5$ and $y = 2x - 3$

f $x^2 + y^2 = 8$ and $y = x + 4$

g $x^2 + y^2 = 18$ and $x + y = 6$

h $x^2 + y^2 = 25$ and $3x + 4y = 25$

i $x^2 + y^2 = 4$ and $x + y = 6$

j $x^2 + y^2 = 9$ and $y = 2x + 8$

Example 13

5 For each pair of curves, find the coordinates of the points of intersection and illustrate with a graph.

a $y = x - 2$ and $y = \frac{3}{x}$

b $y = 2x - 1$ and $y = \frac{1}{x}$

c $y = 3x - 1$ and $y = \frac{2}{x}$

d $y = -\frac{1}{x}$ and $y = -x$

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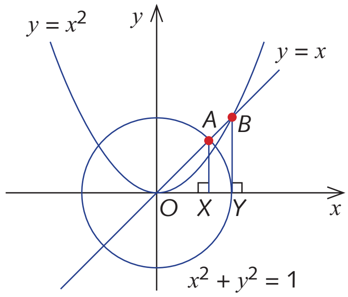
6 The circle $x^2 + y^2 = 1$, the parabola $y = x^2$ and the line $y = x$ are drawn on the same axes. Let A and B be the points of intersection of $y = x$ with the circle and parabola, respectively. XA and YB are drawn perpendicular to the x-axis.



a Find the coordinates of A and B.

b Find the area of triangles OAX and OBY.

7 Where does the line $3y - x = 7$ meet the circle $(x - 3)^2 + y^2 = 10$?

8 Show that the line $y = 2x$ does not meet the circle $(x - 5)^2 + y^2 = 4$.

9 Find the values of a for which the graphs of $y = x + a$ and $x^2 + y^2 = 9$ intersect at:

a one point **b** two points **c** no points.

10 Find the points of intersection of the circles $x^2 + y^2 = 9$ and $(x - 2)^2 + y^2 = 9$.

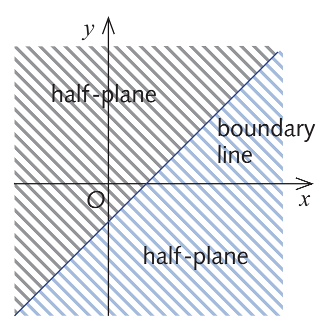
11D Regions of the plane

When we plot a set of points satisfying an inequality, we generally obtain a region of the plane, not a curve or a line.

Half-planes

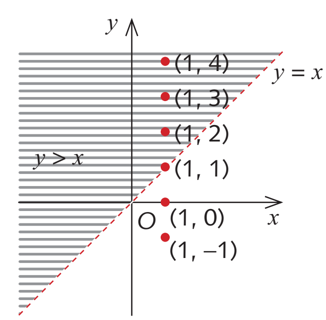
A straight line divides the plane into three non-overlapping regions:

- the points that lie on the line
- the points that lie on one side of the line
- the points that lie on the other side of the line.



Regions consisting of all the points on one side of a line are called **half-planes**.

The region may or may not include the points on the line. The line is often called the **boundary line** of the half-plane.



The region of the plane defined by the inequality $y > x$ consists of all the points (x, y) whose y-coordinate is greater than the x-coordinate. The points $(1, 2)$, $(1, 3)$ and $(1, 4)$ are all in this region, whereas $(1, 0)$ and $(1, -1)$ are not in the region.

The region $y > x$ contains all the points above the line $y = x$. This is because if you choose any point on the line $y = x$ (for example, $(1, 1)$), then all the points (x, y) above the point $(1, 1)$ have $y > x$, and those below have $y < x$.

The region $y > x$ is shown above. The line $y = x$ is dashed to show that it is not included in the region $y > x$.

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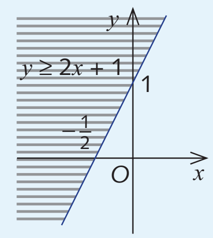
Example 14

Sketch the region defined by the inequality $y \geq 2x + 1$.

Solution

We first sketch the boundary line $y = 2x + 1$.



The boundary line has been drawn as a solid line since it is included in the required region.

Method 1 – Using the inequality with *y* the subject

If you choose any point on the line $y = 2x + 1$, for example, $(0, 1)$, then all the points above $(0,\ 1)$ have $y \geq 2x + 1$ and those below have $y < 2x + 1$. Hence, we shade the region above the line $y = 2x + 1$.

Method 2 – Using a test point *not* on the boundary line

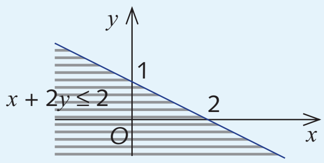
Test the point $(0, 0)$. Since $0 \leq 2 \times 0 + 1$, the point $(0, 0)$ does not belong to the region. Hence, the required region is above the line.

Example 15

Sketch the region defined by the inequality $x + 2y \leq 2$.

Solution

First sketch the boundary line $x + 2y = 2$.



Method 1

The inequality can be rearranged to make y the subject:

$$y \leq -\frac{1}{2}x + 1$$

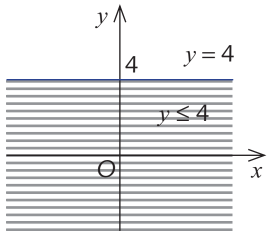
Hence, we shade the region below the line.
We include the line, since points on the line satisfy $y = -\frac{1}{2}x + 1$.

Method 2

Test the point $(0, 0)$. Since $0 + 2 \times 0 \leq 2$, the point $(0, 0)$ does belong to the region. Hence, the required region is below the line.

Boundaries parallel to the x-axis

The inequality $y \leq 4$ describes the half-plane with boundary line $y = 4$. All of the points below $y = 4$ and the points on $y = 4$ are included in the region.



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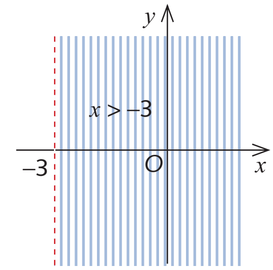
Boundaries parallel to the y-axis



The inequality $x > -3$ describes the half-plane with the boundary line $x = -3$. All of the points to the right of $x = -3$ are in the half-plane. The points on $x = -3$ are not included, so the line is dashed.

Intersection of regions

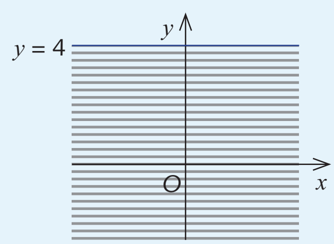
To sketch the intersection of two regions, sketch the regions and see which points they have in common. **Corner points** are those points where the boundary lines of the half-planes meet. They should always be labelled in the sketch.

Example 16

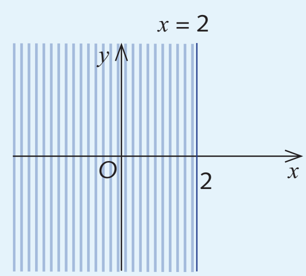
Sketch the region of the plane defined by $y \leq 4$ and $x \leq 2$.

Solution

Sketch the region $y \leq 4$ and the region $x \leq 2$.

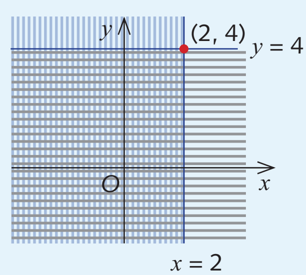


$y \leq 4$



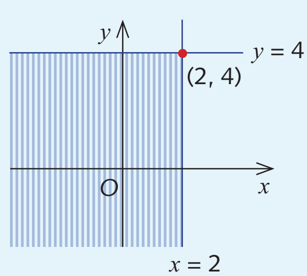
$x \leq 2$

The boundary lines $x = 2$ and $y = 4$ intersect at the corner point $(2, 4)$.



The region is $y \leq 4$ and $x \leq 2$.

Alternatively, it can be shown as:



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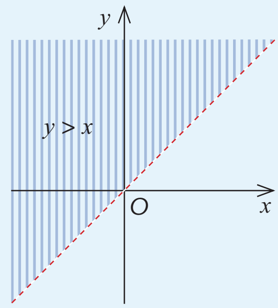
Example 17

Sketch the region defined by the inequalities $y > x$ and $x + y \leq 4$.

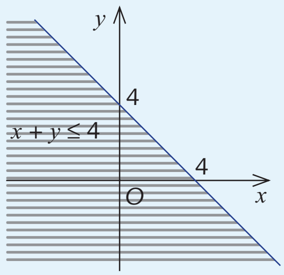
Solution

First sketch the region $y > x$.

Draw the line $y = x$ and shade the region above the line.



Draw $x + y = 4$ and test the origin, $0 + 0 \leq 4$.



To find the corner point, solve the simultaneous equations.

$$y = x \qquad (1)$$

$$x + y = 4 \qquad (2)$$

Substitute (1) into (2):

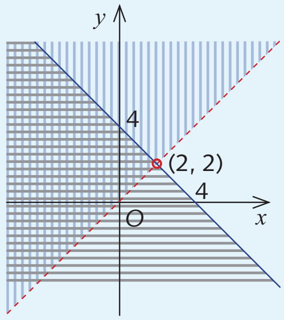
$$x + x = 4$$

$$x = 2$$

From equation (1):

$$y = 2$$

The corner point is $(2, 2)$ but it does not lie in the required region, since it is not a member of $y > x$. It is therefore indicated by an open circle. The region $y > x$ and $x + y \leq 4$ is ▦.



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Discs

A circle divides the plane into three regions. The points in the plane are either on the circle, inside the circle or outside the circle. The set of points inside and on a circle is called a **disc**.

Example 18

Sketch the regions.

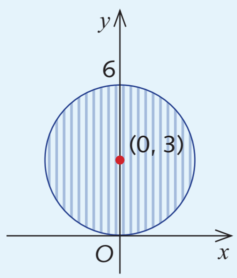
a $x^2 + (y - 3)^2 \leq 9$

b $x^2 + (y - 3)^2 > 9$

Solution

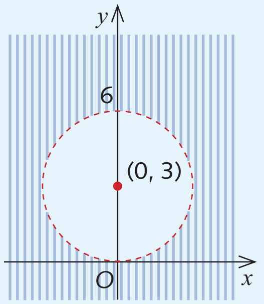
a First draw the circle $x^2 + (y - 3)^2 = 9$. This circle has centre $(0, 3)$ and radius 3. The region is the set of points whose distance from $(0, 3)$ is less than or equal to 3 units.

The region is the shaded disc.



b The region $x^2 + (y - 3)^2 > 9$ is the set of points whose distance from $(0, 3)$ is greater than to 3.

The region is the shaded area outside the disc.



Exercise 11D

Example 14, 15

1 Sketch each region.

a $y > x + 1$

b $y < 2x + 3$

c $y \leq 2x - 1$

d $y > 1 - x$

e $2x + y \leq 4$

f $3x - 2y > 6$

g $3x + y > 1$

h $x - 2y < 1$

i $y \leq 2x$

j $y \geq 3x$

k $x \geq 3$

l $x < 1$

m $y < 2$

n $y \leq -2$

o $4x + 3y \leq 12$

p $2x - y \leq 8$

q $2x - y \geq 4$

r $y < 3 - 2x$

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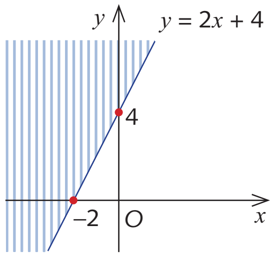
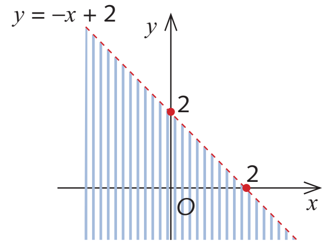
2 Write down the inequalities that describe each given region.

a



b



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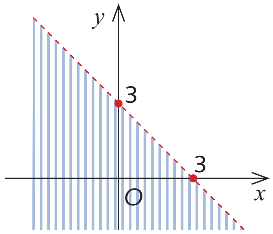

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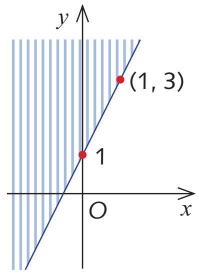


Example 16, 17

3 Sketch the regions satisfying the given inequalities. Find the coordinates of the corner points.

a $y > x$ and $x + y \le 6$

b $y \le 2x$ and $2x + y > 4$

c $x + y \le 4$ and $2x + y \le 6$

d $x + 2y \le 8$ and $3x + y \le 9$

e $y \ge x + 1$ and $y > 3x - 5$

f $y \le 1 - 2x$ and $y \ge \frac{1}{2}x$

g $x \le 2$ and $y \le 1$

h $x \le -2$ and $y \ge 2$

i $x \ge 1,\ x \le 3,\ y \ge 0$ and $y \le 4$

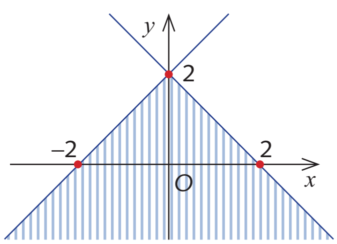
j $x \ge 0,\ y \ge 0$ and $x + y \le 4$

k $x \ge 0,\ y \ge 0,\ x + y \le 6$ and $x + 2y \le 8$

l $y \ge x, y \ge 0$ and $y \le \frac{1}{2}x + 2$

4 Write down the inequalities whose intersections are the shaded region.

a



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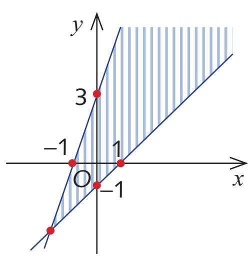

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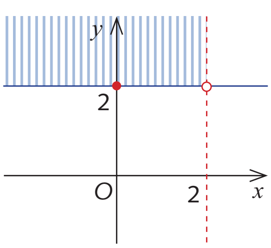

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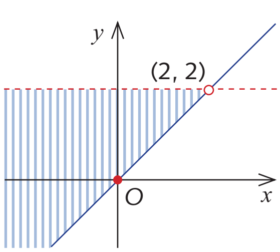
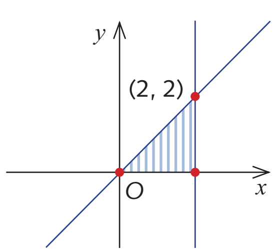
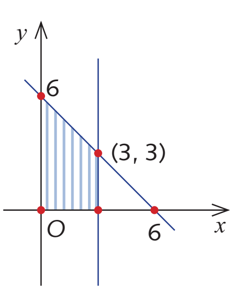


e



f



Example 18

5 Sketch each region.

a $x^2 + y^2 < 4$

b $(x - 2)^2 + y^2 \ge 9$

c $(x + 3)^2 + (y - 1)^2 > 16$

d $(x + 1)^2 + (y + 2)^2 \le 1$

6 Sketch $y > \frac{1}{x}$.

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Review exercise

1 Sketch each graph.

a $x^2 + y^2 = 9$ **b** $2x^2 + 2y^2 = 8$ **c** $x^2 + y^2 = 5$ **d** $x^2 + y^2 = \frac{9}{4}$

2 Sketch each graph.

a $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 1$ **b** $(x + 1)^2 + (y + 1)^2 = 4$

c $(x + 3)^2 + (y + 4)^2 = 25$ **d** $x^2 + (y + 4)^2 = 16$

e $(x - 3)^2 + (y + 5)^2 = 4$ **f** $(x - 1)^2 + (y - 1)^2 = 25$

3 Complete the square to find the centre and the radius of each circle.

a $x^2 + 4x + y^2 + 8y = 0$ **b** $x^2 + y^2 + 4x + 2y - 5 = 0$

c $2x^2 + 2y^2 - 8x + 5y + 3 = 0$ **d** $x^2 + y^2 - 4x + 6y - 37 = 0$

4 Write down the equation of the circle with:

a centre $(0, 0)$ and radius $= 3$ **b** centre $(-1, 4)$ and radius $= 6$

c centre $(2, 5)$ and radius $= 1$ **d** centre $(-2, -6)$ and radius $= 4$

5 Sketch the graph of each rectangular hyperbola.

a $y = \frac{4}{x}$ **b** $y = \frac{5}{2x}$ **c** $y = -\frac{4}{x}$

6 Sketch the graph of each rectangular hyperbola. Include the asymptotes.

a $y = \frac{1}{x + 2}$ **b** $y = \frac{1}{x - 4}$ **c** $y = \frac{1}{x + 5}$ **d** $y = \frac{-1}{x - 1}$

7 Find the points where each parabola meets the line.

a $y = 2x^2 - 3x + 4$ and $y = 12 - 3x$ **b** $y = 2 - x - 3x^2$ and $y = -7x + 2$

8 Find the points of intersection of:

a $x^2 + y^2 = 9$ and $x = 3$ **b** $x^2 + y^2 = 16$ and $y = 0$

c $x^2 + y^2 = 16$ and $y = \sqrt{3}x$

9 Find the points of intersection of:

a $3y + 4x = 25$ and $x^2 + y^2 = 25$ **b** $x^2 + y^2 = 29$ and $y = 3x - 1$

10 Sketch each region.

a $y > x + 2$ **b** $y \geq 2x - 4$ **c** $y > 2 - x$ **d** $2x + y \leq 6$

e $3x + 2y > 6$ **f** $y \leq -1$ **g** $x < -1$ **h** $y \leq 3x$

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11 Sketch each region and find the coordinates of the corner points.

a $x > 4$ and $y \leq -3$

b $y \leq 2x$ and $x \leq 6$

c $x + y \leq 4$ and $y \leq 2x$

d $y \leq 1 - 2x$ and $y > x + 2$

e $2x + y \leq 6$ and $x + y \geq 4$

f $x + y \leq 6$ and $y \geq -2x + 3$

12 Sketch the regions.

a $(x - 1)^2 + y^2 \leq 1$

b $(x - 3)^2 + (y - 4)^2 \leq 25$

c $x^2 + y^2 > 36$

d $(x - 2)^2 + y^2 > 9$

13 Find the points where the hyperbola meets the line.

a $y = x - 1,\ y = \dfrac{12}{x}$

b $y = 2x - 7,\ y = -\dfrac{3}{x}$

c $y = \dfrac{6}{x},\ x = 3$

d $y = \dfrac{9}{x},\ y = 4 - x$

Challenge exercise

1 By considering suitable translations, sketch the graph of:

a $y = 1 + \dfrac{1}{x + 4}$

b $y = 2 + \dfrac{1}{x - 3}$

2 By considering suitable transformations, sketch the graph of:

a $y = 2 + \dfrac{3}{x - 4}$

b $y = 4 + \dfrac{2}{x - 5}$

3 By considering suitable transformations, sketch the graph of:

a $y = 1 - \dfrac{1}{x + 4}$

b $y = 3 - \dfrac{1}{x + 2}$

c $y = 2 + \dfrac{3}{x - 2}$

d $y = 4 - \dfrac{5}{x + 2}$

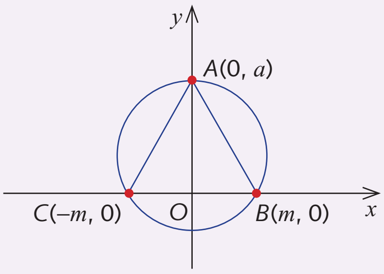
4 Triangle ABC is equilateral with vertices $A(0, a)$, $B(m, 0)$ and $C(-m, 0)$. First show $a = \sqrt{3}m$.

a Find, in terms of a, the equation of the perpendicular bisector of:

i AC

ii AB

b Show that the two perpendicular bisectors meet at $X\left(0, \dfrac{a}{3}\right)$.



c Find the distance AX in terms of a.

d Find the equation of the circle with centre $X\left(0, \dfrac{a}{3}\right)$ and radius AX.

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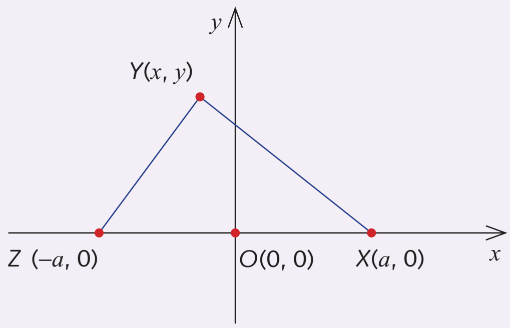
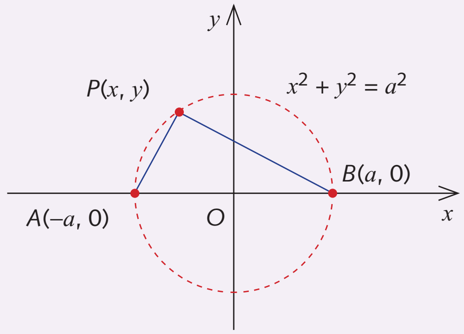
5 **a** XYZ is a right-angled triangle with the right angle at Y. $O(0,0)$ is the midpoint of XZ. The coordinates of X and Z are $(a,0)$ and $(-a,0)$, respectively.



i Use the fact that XY is perpendicular to ZY to show that $x^2 + y^2 = a^2$.

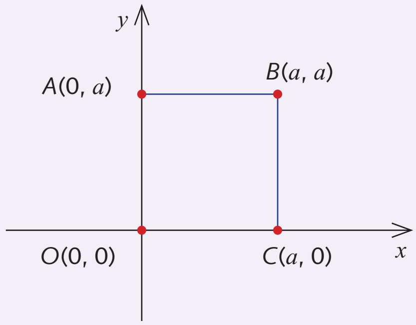
ii Hence, show that $OX = OY = OZ$.

b $P(x,y)$ is a point on the circle $x^2 + y^2 = a^2$. Show that PA is perpendicular to PB.



Note: This proves the important result that the diameter of a circle subtends a right-angle at the circumference. You will encounter this result ag ain in Chapter 13.

6 $ABCO$ is a square of side length a. Show that the equation of the circle passing through all four vertices is $x^2 + y^2 - ax - ay = 0$.



7 The points $O(0,0)$, $A(a,0)$ and $B(0,b)$ lie on a circle.

a Find the equation of the perpendicular bisector of:

i OA **ii** OB

b Find the coordinates of the point of intersection of the perpendicular bisectors of OA and OB.

c Show that the perpendicular bisector of AB also passes through this point.

d Find the equation of the circle passing through O, A and B.

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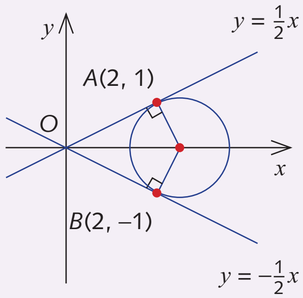
8 Find the equation of the circle that passes through the points (a, b), $(a, -b)$ and $(a + b, a - b)$.

9 The lines $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$ meet the circle at $(2,1)$ and $(2, -1)$, as shown in the diagram. Find the equation of the circle.



10 Sketch each graph.

a $(x - 4)(y - 3) = 2$

b $(x - 2)(y - 3) = 2$

11 Sketch each graph.

a $(x - y)(x + y) = 0$

b $(y - x^2)(y + x^2) = 0$

c $(x^2 - y^2)(x + y^2) = 0$

d $(y^2 - x)(y^2 + x) = 0$

12 Show that the circles $x^2 + y^2 - 2x - 3y = 0$ and $x^2 + y^2 + x - y = 6$ intersect on the x-axis and y-axis.

13 Find the points of intersection of the circles $x^2 + y^2 + x - 3y = 0$ and $2x^2 + 2y^2 - x - 2y - 15 = 0$.

14 The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Find the equation of the circle passing through the points $(-1,3)$, $(2,2)$ and $(1,4)$.

15 Show that $y = ax + b$, where $a > 0$, always meets $y = \frac{1}{x}$.

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