CHAPTER

12

Measurement and Geometry

Further trigonometry

Trigonometry begins with the study of relationships between sides and angles in a right-angled triangle.

In this chapter, we will review the basics of the trigonometry of right-angled triangles, look at applications to three-dimensional problems, and extend our study of trigonometry to triangles that are not right-angled.

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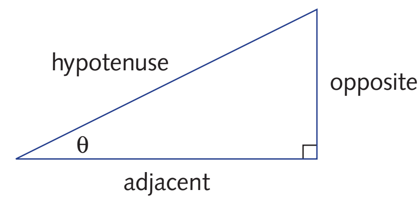
12A Review of the basic trigonometric ratios

By similarity, the ratio of any two sides in a right-angled triangle is always the same, once we have fixed the angles.



We choose one of the two acute angles and call it the **reference angle**.

The side opposite the reference angle is called the **opposite**, the side opposite the right angle is called the **hypotenuse** and the remaining side, which is between the reference angle and the right-angle, is called the **adjacent**.

The three basic trigonometric ratios are the **sine**, **cosine** and **tangent** ratios.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

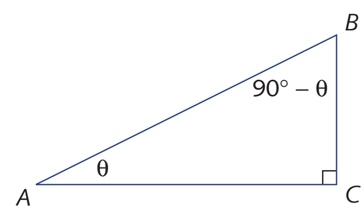
You should learn the three ratios for sine, cosine and tangent by heart and remember them. A simple mnemonic is:

SOHCAHTOA

for **S**ine: **O**pposite/**H**ypotenuse, **C**osine: **A**djacent/**H**ypotenuse, **T**angent: **O**pposite/**A**djacent

Complementary angles

In the diagram, the angles at A and B are complementary; that is, they add to 90°.



The side opposite A is the side adjacent to B and vice versa. Hence, the sine of θ is the cosine of (90° – θ) and vice versa.

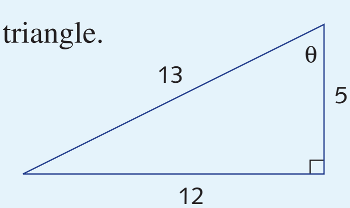
$$\sin\theta = \cos(90^\circ - \theta)$$

$$\cos\theta = \sin(90^\circ - \theta)$$

For example, $\sin 60^\circ = \cos 30^\circ$ and $\cos 10^\circ = \sin 80^\circ$.

Example 1

Write down the sine, cosine and tangent ratios for the angle θ in this triangle.



Solution

$$\sin\theta = \frac{12}{13} \qquad \cos\theta = \frac{5}{13} \qquad \tan\theta = \frac{12}{5}$$

Once a reference angle is given, the approximate numerical value of each of the three ratios can be obtained from a calculator. We can use this idea to find unknown sides in a right-angled triangle.

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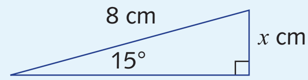
Example 2

Find, correct to two decimal places, the value of the pronumeral in each triangle.

a



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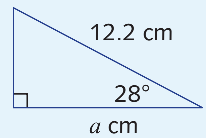

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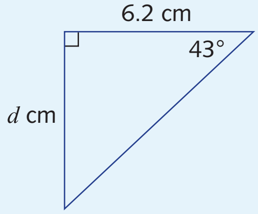
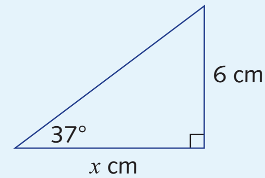


d



Solution

a

$$\sin 15° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 15° = \frac{x}{8}$$

$$x = 8 \times \sin 15°$$

$$\approx 2.07$$

b

$$\cos 28° = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 28° = \frac{a}{12.2}$$

$$a = 12.2 \times \cos 28°$$

$$\approx 10.77$$

c

$$\tan 43° = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 43° = \frac{d}{6.2}$$

$$d = 6.2 \tan 43°$$

$$\approx 5.78$$

d

$$\tan 37° = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 37° = \frac{6}{x}$$

$$x \tan 37° = 6$$

$$x = \frac{6}{\tan 37°}$$

$$\approx 7.96$$

Finding angles

In order to apply trigonometry to finding angles rather than side lengths in right-angled triangles, we need to be able to go from the value of sine, cosine or tangent back to the angle.

What is the acute angle whose sine is 0.5?

The calculator gives $\sin 30° = 0.5$, so we write $\sin^{-1} 0.5 = 30°$.

The opposite process of finding the sine of an angle is to find the **inverse sine** of a number.

When θ is an acute angle, the statement $\sin^{-1} x = \theta$ means $\sin \theta = x$.

This notation is standard, but is rather misleading. The index -1 does NOT mean *one over,* as it normally does in algebra. To help you avoid confusion, you should always read $\sin^{-1} x$ as *inverse sine of x* and $\tan^{-1} x$ as *inverse tan of x,* and so on.

For example, the calculator gives $\cos^{-1} 0.8192 \approx 35°$ (read this as *inverse cosine of 0.8192 is approximately 35°*).

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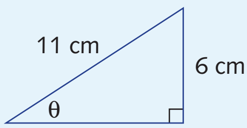
Example 3

Calculate the value of θ, correct to one decimal place.

a



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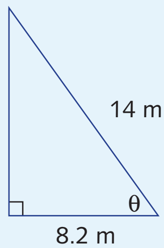

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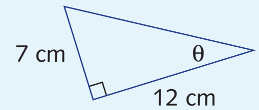


Solution

a $\sin\theta = \dfrac{6}{11}$

$\theta = \sin^{-1}\left(\dfrac{6}{11}\right)$

$\approx 33.1°$

b $\cos\theta = \dfrac{8.2}{14}$

$\theta = \cos^{-1}\left(\dfrac{8.2}{14}\right)$

$\approx 54.1°$

c $\tan\theta = \dfrac{7}{12}$

$\theta = \tan^{-1}\left(\dfrac{7}{12}\right)$

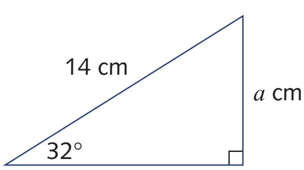
$\approx 30.3°$

Exercise 12A

Example 2

1 Calculate the value of each pronumeral, correct to two decimal places.

a



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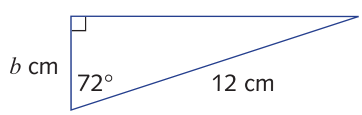

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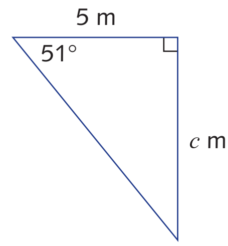

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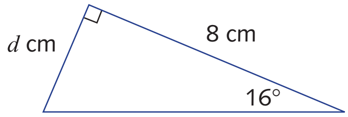

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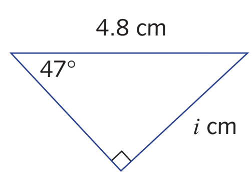

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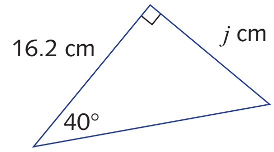
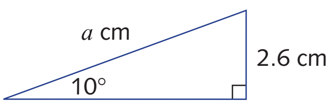


2 Calculate the value of the pronumeral, correct to two decimal places.

a



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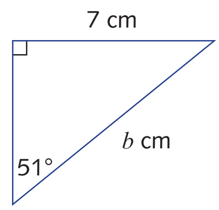

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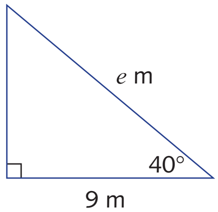

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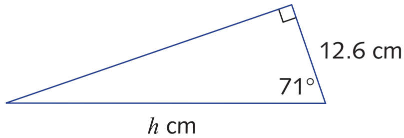

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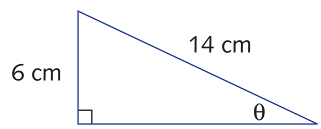
Example 3

3 Calculate the value of θ, correct to one decimal place.

a



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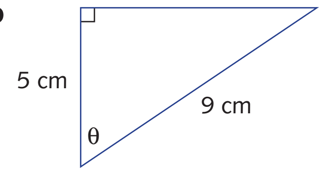

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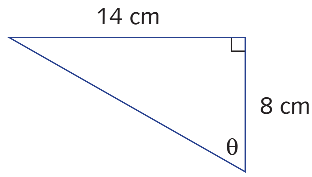

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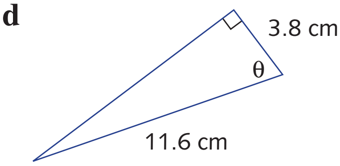

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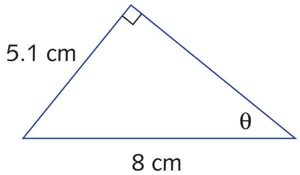

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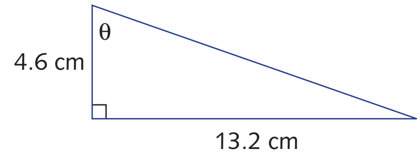
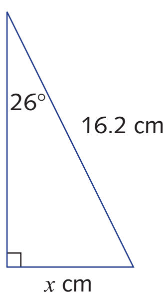


4 Calculate the value of each pronumeral. Give side lengths correct to two decimal places and angles correct to one decimal place.

a



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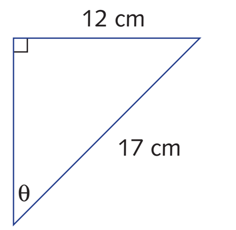

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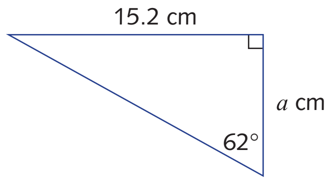

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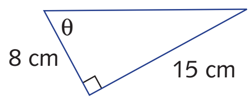

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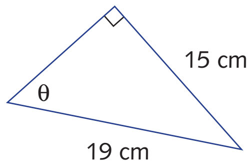

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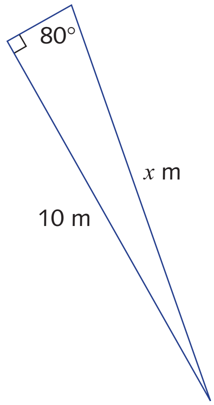

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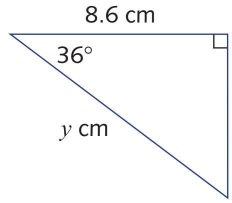

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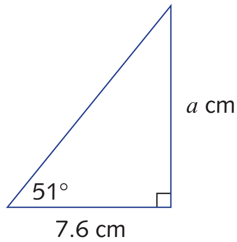

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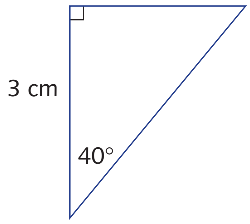
5 Find all sides, correct to two decimal places, and all angles, correct to one decimal place.

a



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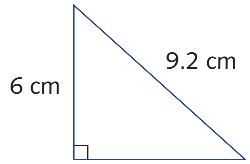

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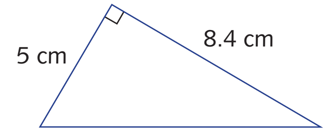
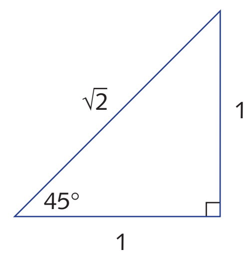


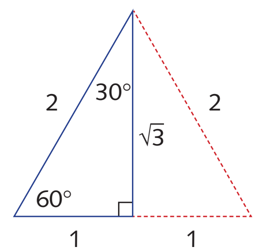
12B Exact values

The trigonometric ratios for the angles 30°, 45° and 60° occur very frequently and can be expressed using surds.

The value of the trigonometric ratios for 45° can be found from the diagram opposite. It is an isosceles triangle with shorter sides 1.



The values of the trigonometric ratios for 30° and 60° can be found by drawing an altitude in an equilateral triangle.



The values are given in the table.

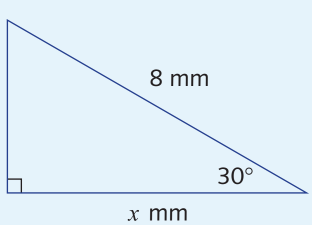
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Check the details in the triangles and the entries in the table. You can either learn the table or remember the diagrams to construct the table.

Example 4

Find the exact value of x.

a



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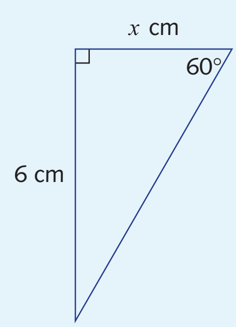

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Solution

a We have $\cos 30° = \dfrac{x}{8}$

$$x = 8\cos 30°$$
$$= 8 \times \frac{\sqrt{3}}{2}$$
$$= 4\sqrt{3}$$

b We have $\tan 60° = \dfrac{6}{x}$

so $\dfrac{6}{x} = \sqrt{3}$

Hence $x = \dfrac{6}{\sqrt{3}}$

$$= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \text{ (Rationalise the denominator.)}$$
$$= 2\sqrt{3}$$

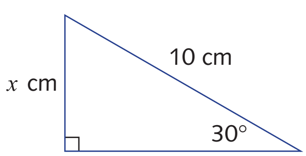
Alternatively, we could have worked with the complementary angle, $\tan 30° = \dfrac{x}{6}$.

Exercise 12B

Example 4

1 Find the exact value of x.

a



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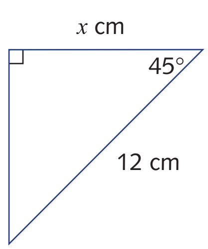

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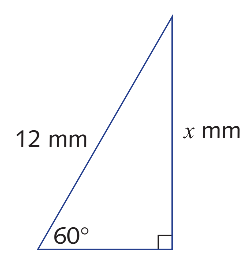

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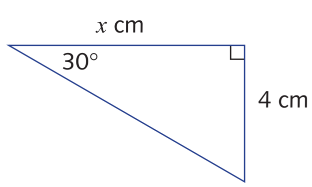

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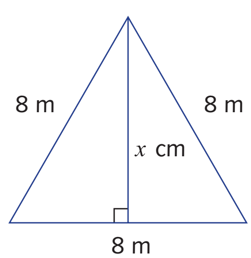

2 Find the exact values, rationalising the denominator where appropriate.

a $(\sin 60°)^2 + (\cos 60°)^2$

b $(\tan 30°)^2 - \dfrac{1}{(\cos 30°)^2}$

c $\dfrac{\tan 60° - \tan 45°}{1 + \tan 60° \times \tan 45°}$

d $\sin 45° \times \cos 60° + \cos 45° \times \sin 60°$

e $2 \sin 30° \times \cos 30°$

f $2(\cos 45°)^2 - 1$

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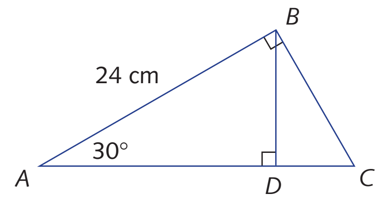
3 $ABCD$ is a rhombus with $\angle ABD = 30°$. Find the exact length of each diagonal if the side lengths are 10 cm.

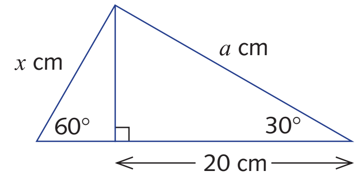
4 Find exact values of:

a AC **b** AD **c** BC

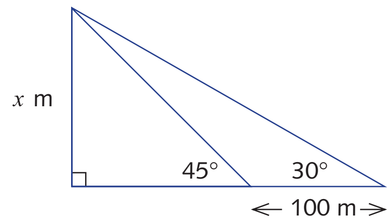
d DC **e** BD



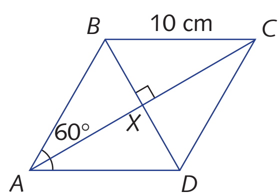
5 Find the exact values of a and x.



6 Find the exact value of x.



7 $ABCD$ is a rhombus with sides 10 cm. Find AX.



12C Three-dimensional trigonometry

We can apply our knowledge of trigonometry to solve problems in three dimensions. To do this you will need to draw careful diagrams and look for right-angled triangles. Sometimes it is helpful to draw a separate diagram showing the right-angled triangle.

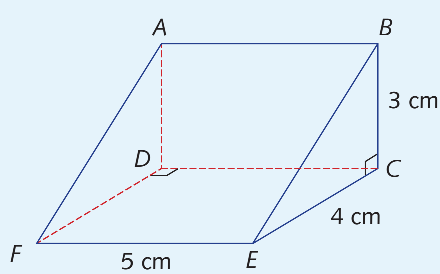
Example 5

In the triangular prism shown, find:

a the length CF

b the length BF

c the angle BFC, correct to one decimal place.



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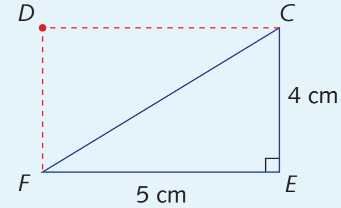
Solution

a Applying Pythagoras' theorem to ΔCEF:

$$\begin{aligned} CF^2 &= 4^2 + 5^2 \\ &= 41 \end{aligned}$$

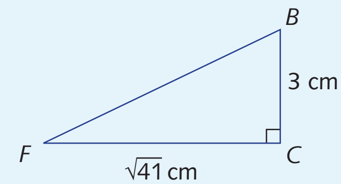
Hence, $CF = \sqrt{41}$ cm



b Applying Pythagoras' theorem to ΔBCF:

$$\begin{aligned} BF^2 &= 3^2 + \left(\sqrt{41}\right)^2 \\ &= 50 \end{aligned}$$

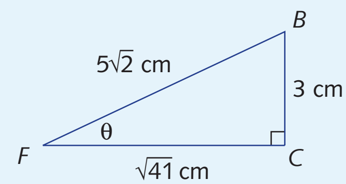
Hence, $BF = 5\sqrt{2}$ cm



c To find the angle BFC, draw ΔBCF and let $\angle BFC = \theta$.

Now $\tan\theta = \dfrac{3}{\sqrt{41}}$

so $\theta \approx 25.1°$ (Correct to one decimal place.)

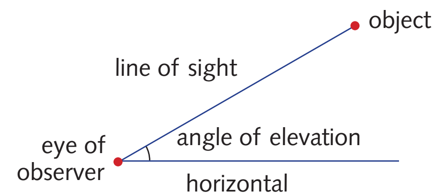


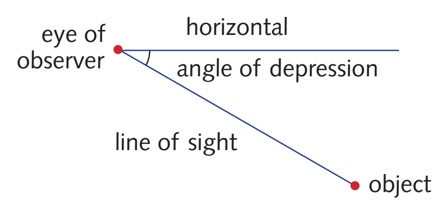
Angles of elevation and depression

When a person looks at an object that is higher than the person's eye, the angle between the line of sight and the horizontal is called the **angle of elevation**.

On the other hand, when the object is lower than the person's eye, the angle between the horizontal and the line of sight is called the **angle of depression**.

In practice, 'eye of observer' is replaced by a point on the ground.

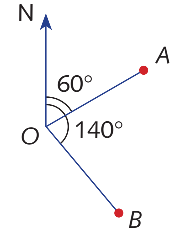




Bearings

Bearings are used to indicate the direction of an object from a fixed reference point, O. **True bearings** give the angle $\theta°$ from north, measured clockwise. We write a true bearing of $\theta°$ as $\theta°$T, where $\theta°$ is an angle between 0° and 360°. It is customary to write the angle using three digits, so 0°T is written 000°T, 15°T is written 015°T, and so on.

For example, in the diagram opposite, the true bearing of A from O is 060°T, and the true bearing of B from O is 140°T.



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Example 6

A tower is situated due north of a point A and due west of a point B. From A, the angle of elevation of the top of the tower is 18°. In addition, B (which is on the same level as A) is 52 metres from A and has a bearing of 064°T from A. Find, correct to one decimal place:

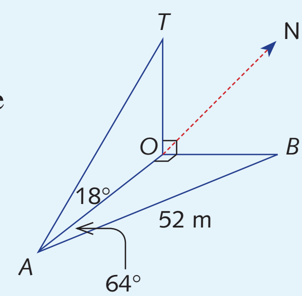
a the distance from A to the base of the tower

b the height of the tower

c the angle of elevation of the top of the tower from B.

Solution

Draw the tower OT and mark the point A level with the base of the tower. The line AO then points north. We can then mark all the given information on the diagram. The triangle AOT is vertical and triangle AOB is horizontal.



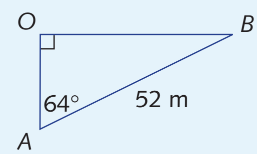
a In ΔAOB, $\cos 64° = \dfrac{OA}{52}$

so $OA = 52 \cos 64°$

$= 22.795\ldots$ (Keep this in your calculator for part **b**.)

≈ 22.8 (Correct to one decimal place.)

A is approximately 22.8 metres from the base of the tower.



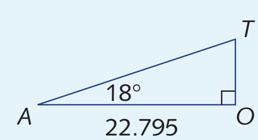
b In ΔAOT, $\tan 18° = \dfrac{OT}{OA}$

that is, $OT = OA \times \tan 18°$

$= 7.406\ldots$ (Keep this in your calculator for part **c**.)

≈ 7.4 (Correct to one decimal place.)

The tower is approximately 7.4 metres high.



c In ΔTOB, $\tan\theta = \dfrac{OT}{OB}$

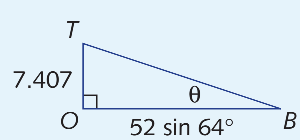
Now from ΔAOB, $OB = 52 \sin 64°$

Hence, $\tan\theta = \dfrac{OT}{52 \sin 64°}$

≈ 0.1585

so $\theta \approx 9.0°$ (Correct to one decimal place.)

The angle of elevation of the top of the tower from B is approximately 9.0°.



Do not re-enter a rounded result into your calculator; it is much more accurate to store the un-rounded number and use it in subsequent steps.

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Exercise 12C

Example 5

1 In the rectangular prism shown opposite, find:

a BN

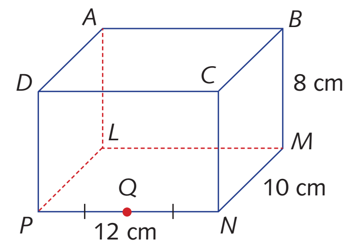
b $\angle BNM$ (correct to one decimal place)

c BP

d the angle BPM (correct to one decimal place)

e MQ, where Q is the midpoint of PN

f the angle BQM (correct to one decimal place)



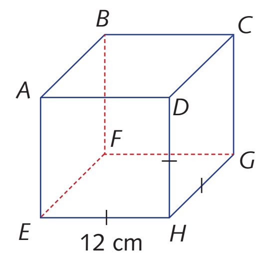
2 In the cube shown opposite, find:

a CE

b $\angle CEG$ (correct to one decimal place)

c $\angle CBE$

d $\angle CEB$ (correct to one decimal place)



3 In the square pyramid shown opposite, find:

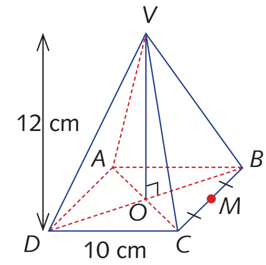
a AC **b** OC **c** VC

d $\angle VCO$ (correct to one decimal place)

e OM, where M is the midpoint of BC

f $\angle VMO$ (correct to one decimal place)

g $\angle VBM$ (correct to one decimal place)

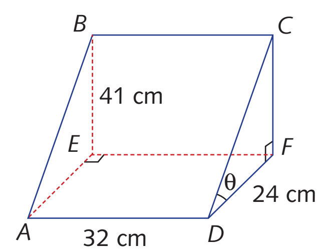


4 $AEFD$ is a horizontal rectangle. $ABCD$ is a rectangle inclined at an angle θ to the horizontal. $AD = 32$ cm, $AE = 24$ cm and $BE = 41$ cm. Find, correct to one decimal place where necessary:

a DC

b AF

c $\angle CAF$



Example 6

5 The base of a tree is situated 50 metres due north of a point P. The angle of elevation of the top of the tree from P is 32°.

a Find the height of the tree, correct to one decimal place.

b Q is a point 100 metres due east of P. Find:

i the distance of Q from the base of the tree

ii the angle of elevation of the top of the tree from Q, correct to one decimal place

iii the bearing of the tree from Q, correct to one decimal place

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6 Dillon and Eugene are both looking at a tower of height 35 metres. Dillon is standing due south of the tower and he measures the angle of elevation from the ground to the top of the tower to be 15°. Eugene is standing due east of the tower and he measures the angle of elevation from the ground to the top of the tower to be 20°. Find, correct to one decimal place:

a the distance Dillon is from the foot of the tower

b the distance Eugene is from the foot of the tower

c the distance between Dillon and Eugene

d the bearing of Dillon from Eugene

7 From a point A, a lighthouse is on a bearing of 026°T and the top of the lighthouse is at angle of elevation of 20.25°. From a point B, the lighthouse is on a bearing of 296°T and the top of the lighthouse is at angle of elevation of 10.20°. If A and B are 500 metres apart, find the height of the lighthouse, correct to the nearest metre.

8 From the top of a cliff that runs north–south, the angle of depression of a yacht, 200 metres out to sea and due east of the observer, is 20°. When the observer next looks at the yacht, he notices that it has sailed 150 metres parallel to the cliff.

a Find the height of the cliff, correct to the nearest metre.

b Find the distance the yacht is from the observer after it has sailed 150 metres parallel to the cliff, correct to the nearest metre.

c Find the angle of depression of the yacht from the top of the cliff when it is in its new position, correct to the nearest degree.

9 A mast is held in position by means of two taut ropes running from the ground to the top of the mast. One rope is of length 40 metres and makes an angle of 58° with the ground. Its anchor point with the ground is due south of the mast. The other rope is 50 metres long and its anchor point is due east of the mast. Find the distance, correct to the nearest metre, between the two anchor points.

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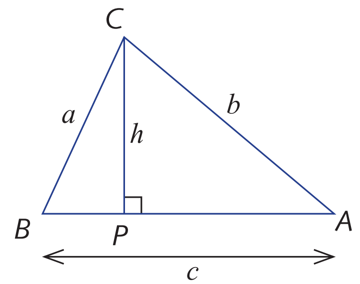
12D The sine rule

In many situations we encounter triangles that are not right-angled. We can use trigonometry to deal with these triangles as well. One of the two key formulas for doing this is known as the sine rule.

We begin with an acute-angled triangle, ABC, with side lengths a, b and c, as shown below. (It is standard to write a lower case letter on a side and the corresponding upper case letter on the angle opposite that side.) Drop a perpendicular, CP, of length h, from C to AB.



In ΔAPC we have $\sin A = \frac{h}{b}$, so $h = b\sin A$.

Similarly, in ΔCPB we have $\sin B = \frac{h}{a}$, so $h = a\sin B$.

Equating these expressions for h, we have:

$$b\sin A = a\sin B$$

which we can write as:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The same result holds for the side c and angle C, so we can write:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This is known as the **sine rule**.

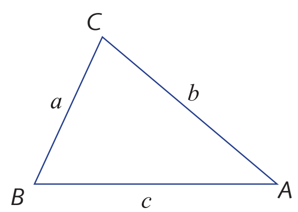
In words, this says 'any side of a triangle over the sine of the opposite angle equals any other side of the triangle over the sine of its opposite angle'.

This result also holds in an obtuse-angled triangle. We will look at that case later.

The sine rule

In any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



For example, the sine rule can be used to find an unknown length of a side of a triangle when a side length and the angles are known. This is closely related to the AAS congruence test.

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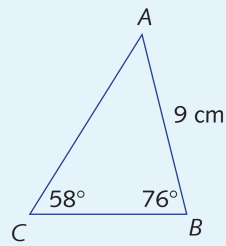
Example 7

In $\Delta ABC = AB = 9$ cm, $\angle ABC = 76°$ and $\angle ACB = 58°$.

Find, correct to two decimal places:

a AC **b** BC



Solution

a Apply the sine rule:

$$\frac{AC}{\sin 76°} = \frac{9}{\sin 58°}$$

$$AC = \frac{9\sin 76°}{\sin 58°}$$

$$\approx 10.30 \text{ cm}$$

b To find BC, we need the angle $\angle CAB$ opposite it.

$$\angle CAB = 180° - 58° - 76°$$

$$= 46°$$

Then by the sine rule:

$$\frac{BC}{\sin 46°} = \frac{9}{\sin 58°}$$

$$BC = \frac{9\sin 46°}{\sin 58°}$$

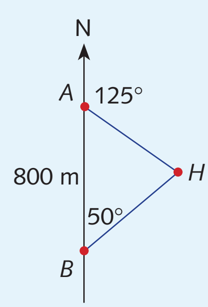
$$\approx 7.63 \text{ cm}$$

Example 8

From two points A and B, which are 800 metres apart on a straight north–south road, the bearings of a house are 125°T and 050°T, respectively. Find how far each point is from the house, correct to the nearest metre.

Solution

We draw a diagram to represent the information.



We can find the angles in ΔAHB.

$$\angle HAB = 180° - 125°$$

$$= 55°$$

and $\angle AHB = 180° - 50° - 55°$

$$= 75°$$

Apply the sine rule to ΔABH:

$$\frac{BH}{\sin 55°} = \frac{800}{\sin 75°}$$

$$BH = \frac{800\sin 55°}{\sin 75°}$$

$$\approx 678.44 \text{ m}$$

Similarly, $\dfrac{AH}{\sin 50°} = \dfrac{800}{\sin 75°}$

and so $AH = \dfrac{800\sin 50°}{\sin 75°}$

$$\approx 634.45 \text{ m}$$

Thus, A and B are approximately 634 metres and 678 metres from the house, respectively.

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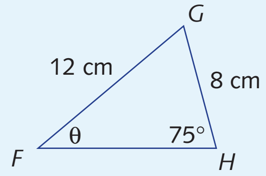
Finding angles

The sine rule can also be used to find angles in a triangle, provided that one of the known sides is opposite a known angle.

At this stage we can only deal with acute angled triangles.

Example 9

Find the angle θ in the triangle FGH, correct to the nearest degree.



Solution

Apply the sine rule to ΔFGH:

$$\frac{8}{\sin\theta} = \frac{12}{\sin 75^\circ}$$

To make the algebra easier, take the reciprocal of both sides:

$$\frac{\sin\theta}{8} = \frac{\sin 75^\circ}{12}$$

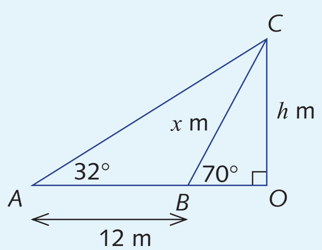
Hence, $\sin\theta = \dfrac{8\sin 75^\circ}{12}$

$= 0.6440\ \ldots$

Hence, $\theta \approx 40^\circ$ (Correct to the nearest degree.)

Example 10

Find the length of OC in the diagram, correct to one decimal place.



Solution

$OC = h$ m and $BC = x$ m.

$\angle ACB + 32° = 70°$ (exterior angle of ΔABC)

The angle $\angle ACB = 38°$

Applying the sine rule:

$$\frac{x}{\sin 32°} = \frac{12}{\sin 38°}$$

$$x = \frac{12 \sin 32°}{\sin 38°}$$

$$= 10.3287\ldots \quad \text{(Keep this in your calculator.)}$$

In triangle BCO:

$$\sin 70° = \frac{h}{x}$$

so

$$h = x \times \sin 70°$$

$$\approx 9.7 \quad \text{(Correct to one decimal place.)}$$

The length OC is approximately 9.7 m.

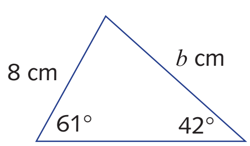
Note: Alternatively, h can be calculated directly as $\dfrac{12 \sin 32°}{\sin 38°} \times \sin 70°$.

Exercise 12D

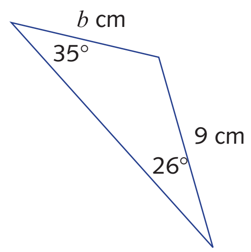
Example 7

1 Find the value of b, correct to two decimal places.

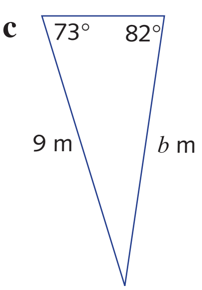
a



b

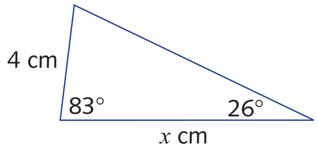


c

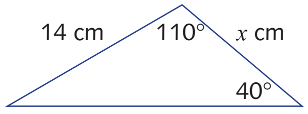


2 Find the value of x, correct to two decimal places.

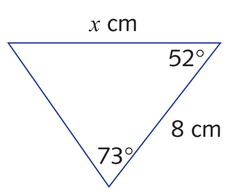
a



b



c



3 **a** In ΔABC, $A = 62°$, $B = 54°$ and $a = 8$. Find b, correct to two decimal places.

b In ΔABC, $B = 47°$, $C = 82°$ and $b = 10$. Find a, correct to two decimal places.

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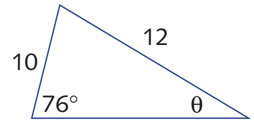
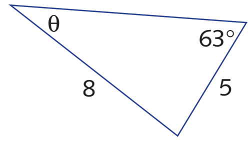
Example 9

4 Find the value of θ, correct to the nearest degree.

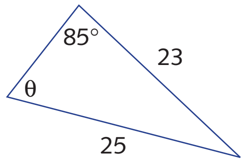
a



b



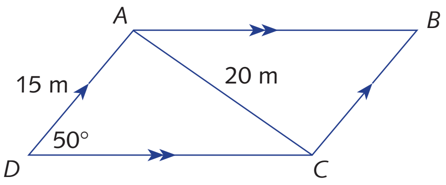
c



5 In ΔABC, $A = 71°$, $a = 18$ cm and $b = 14$ cm. Find, correct to two decimal places:

a B **b** C **c** c

6 $ABCD$ is a parallelogram with $\angle ADC = 50°$. The shorter diagonal, AC, is 20 m, and $AD = 15$ m. Find $\angle ACD$ and hence the length of the side DC, correct to two decimal places.



Example 8

7 Two hikers, Paul and Sayo, are both looking at a distant landmark. From Paul, the bearing of the landmark is 222°T and, from Sayo, the bearing of the landmark is 300°T. If Sayo is standing 800 m due south of Paul, find, correct to the nearest metre:

a the distance from Paul to the landmark

b the distance from Sayo to the landmark

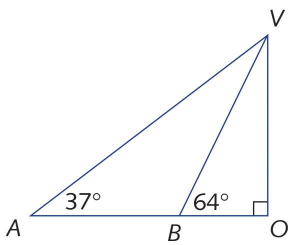
8 An archaeologist wishes to determine the height of an ancient temple. From a point A at ground level, she measures the angle of elevation of V, the top of the temple, to be 37°.

She then walks 100 m towards the temple to a point B. From here, the angle of elevation of V from ground level is 64°. Find:

a $\angle AVB$

b VB, correct to two decimal places

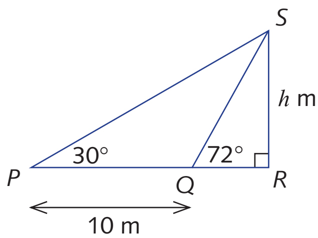
c OV, the height of the temple, to the nearest metre



9 A hillside is inclined at 26° to the horizontal. From the bottom of the hill, Alex observes a vertical tree whose base is 40 m up the hill from the point where Alex is standing. If the angle of elevation of the top of the tree is 43° from the point where Alex is standing, find the height of the tree, correct to the nearest metre.

Example 10

10 Find h, correct to the nearest centimetre.



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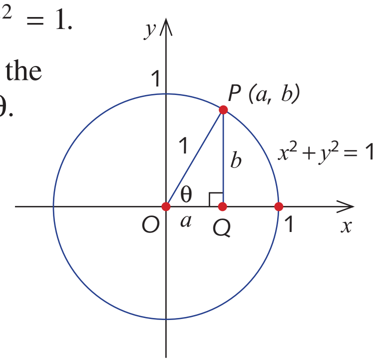
12E Trigonometric ratios of obtuse angles

We have seen that we can use the sine rule to find sides and angles in acute-angled triangles. What happens when one of the angles is obtuse? We can extend our definition of the basic trigonometric functions to obtuse angles by using coordinate geometry.

We begin by drawing a circle of radius 1 in the Cartesian plane, with its centre at the origin. The equation of the circle is $x^2 + y^2 = 1$.

Take a point $P(a, b)$ on the circle in the first quadrant and form the right-angled triangle POQ with O at the origin. Let $\angle POQ$ be θ.



$$\cos\theta = \frac{OQ}{OP} = \frac{a}{1} = a, \text{ and}$$

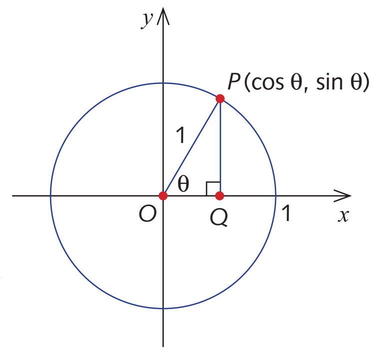
$$\sin\theta = \frac{QP}{OP} = \frac{b}{1} = b$$

But a is the x-coordinate of P and b is the y-coordinate of P.

Hence, the coordinates of the point P are $(\cos\theta, \sin\theta)$.

We can now turn this idea around and say that if θ is the angle between OP and the positive x-axis, then:

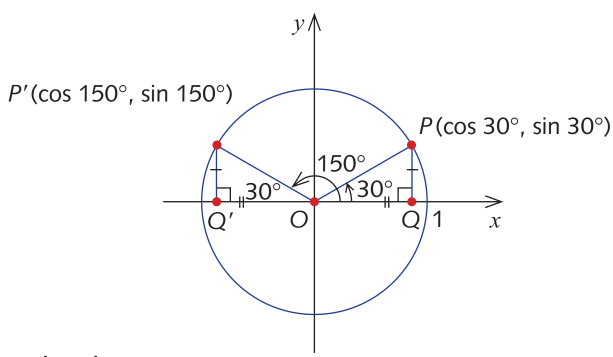
- the cosine of θ is defined to be the x-coordinate of the point P on the unit circle
- the sine of θ is defined to be the y-coordinate of the point P on the unit circle.



This definition can be applied to all angles θ, but in this chapter we will restrict the angle θ to $0° \le \theta \le 180°$.

Now take θ to be 30°, so P has coordinates $(\cos 30°, \sin 30°)$.

Suppose that we move the point P around the circle to P' so that P' makes an angle of 150° with the positive x-axis. (Recall that 30° and 150° are supplementary angles.)



The coordinates of P' are $(\cos 150°, \sin 150°)$. But we can see that triangles OPQ and $OP'Q'$ are congruent, so the y-coordinates of P and P' are the same. That is:

$$\sin 150° = \sin 30°$$

The x-coordinates have the same magnitude but opposite sign, so:

$$\cos 150° = -\cos 30°$$

From this example, we can see the following rules.

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Supplementary angles

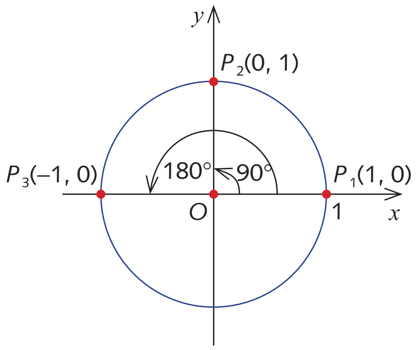
- The sines of two supplementary angles are the same.
- The cosines of two supplementary angles are opposite in sign.
- In symbols:

  $\sin\theta = \sin(180° - \theta)$ and $\cos\theta = -\cos(180° - \theta)$

We can extend the definition of sine and cosine to angles beyond 180°. This will be done later in this book.

The angles 0°, 90° and 180°

We have defined $\cos\theta$ and $\sin\theta$ as the x- and y-coordinates of the point P on the unit circle. Taking the axis intercepts P_1, P_2 and P_3 from the unit circle diagram to the right, we obtain the following table of values. These values should be memorised.



θ	0°	90°	180°
$\sin\theta$	0	1	0
$\cos\theta$	1	0	–1

Example 11

Find the exact value of:

a sin 150° **b** cos 150° **c** sin 120° **d** cos 120°

Solution

a $\sin 150° = \sin(180 - 150)° = \sin 30° = \dfrac{1}{2}$

b $\cos 150° = -\cos(180 - 150)° = -\cos 30° = -\dfrac{\sqrt{3}}{2}$

c $\sin 120° = \sin(180 - 120)° = \sin 60° = \dfrac{\sqrt{3}}{2}$

d $\cos 120° = -\cos(180 - 120)° = -\cos 60° = -\dfrac{1}{2}$

Note: You can verify these results using your calculator.

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Example 12

Find, correct to the nearest degree, the acute and obtuse angle whose sine is:

a approximately 0.7431 **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{\sqrt{3}}{2}$

Solution

a If $\sin\theta = 0.7431$ and θ is acute, then the calculator gives $\theta = \sin^{-1} 0.7431 \approx 48°$.
Hence, the solutions are 48° and 132°, correct to the nearest degree, because 132° is the supplement of 48°.

b If $\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = 45°$ or $\theta = 180° - 45°$

That is, $\theta = 45°$ or $\theta = 135°$

c If $\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = 60°$ or $\theta = 180° - 60°$

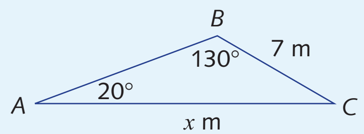
That is, $\theta = 60°$ or $\theta = 120°$

More on the sine rule

The sine rule also holds in obtuse-angled triangles. A proof is given in question **7** of Exercise 12E. We now see how to apply the sine rule in obtuse-angled triangles.

Example 13

Find the value of x, correct to one decimal place.



Solution

Apply the sine rule to ΔABC:

$$\frac{x}{\sin 130°} = \frac{7}{\sin 20°}$$

$$x = \frac{7\sin 130°}{\sin 20°} \quad (\sin 130° = \sin 50°)$$

$$\approx 15.7 \quad \text{(Correct to one decimal place.)}$$

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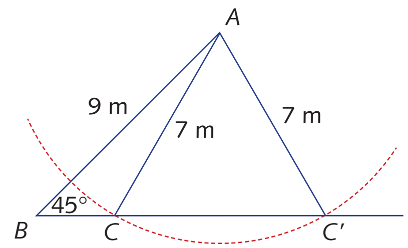
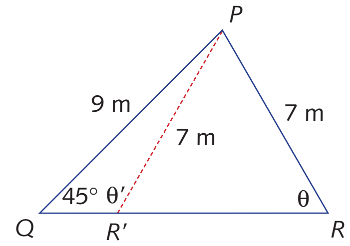
The ambiguous case

You are given the following information about a triangle.
A triangle has side lengths 9 m and 7 m and an angle of 45° between the 9 m side and the unknown side.
How many triangles satisfy these properties?



In the diagram, the triangles ABC and ABC' both have sides of length 9 m and 7 m, and both contain an angle of 45° opposite the side of length 7 m. Despite this, the triangles are different. (Recall that the included angle was required in the SAS congruence test.)



Hence, given the data that a triangle $\angle PQR$ has $PQ = 9$ m, $\angle PQR = 45°$ and $PR = 7$ m, the angle opposite PQ is not determined. There are two non-congruent triangles that satisfy the given data.

Let $PR' = 7$ m so that $\theta = \angle PRQ$ is acute and $\theta' = \angle PR'Q$ is obtuse.

Applying the sine rule to the ΔPRQ, we have:

$$\frac{9}{\sin\theta} = \frac{7}{\sin 45°}$$

$$\sin\theta = \frac{9\sin 45°}{7}$$

$$= 0.9091\ \ldots$$

The calculator tells us that $\sin^{-1}(0.9091\ \ldots)$ is approximately 65°. Hence $\theta \approx 65°$.

The triangle $PR'R$ is isosceles, so $\angle PR'R$ is 65° and $\theta' = 180° - 65° = 115°$. Since $\sin 65° = \sin 115°$, the triangle $PR'Q$ also satisfies the given data.

Exercise 12E

1 Copy and complete:

a $\sin 115° = \sin$ ___

b $\cos 123° = -\cos$ ___

c $\sin 138° = \sin$ ___

d $\cos 95° = -\cos$ ___

Example 11

2 Find the exact value of:

a $\sin 135°$

b $\cos 135°$

Example 12

3 **a** Find the acute and the obtuse angle whose sine is $\frac{1}{2}$.

b Find, correct to the nearest degree, two angles whose sine is approximately 0.5738.

c Find, correct to the nearest degree, an angle whose cosine is approximately -0.8746.

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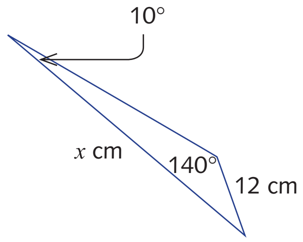
4 Copy and complete:

θ	30°	120°	150°	90°	135°
sin θ		$\frac{\sqrt{3}}{2}$			$\frac{1}{\sqrt{2}}$
cos θ			$-\frac{\sqrt{3}}{2}$	0	

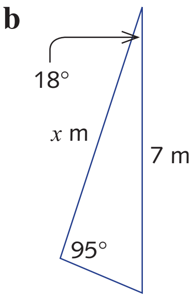
Example 13

5 Use the sine rule to find the value of x, correct to two decimal places.

a

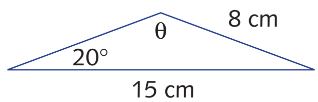


b



6 Given that θ is an obtuse angle, find its value, correct to the nearest degree.

a



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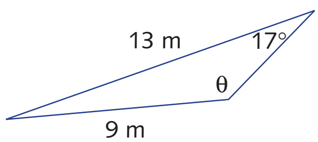


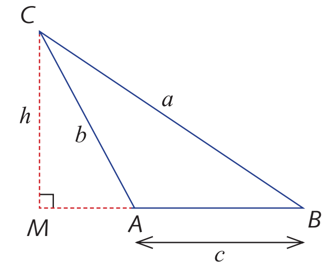
7 Suppose that $\angle A$ in triangle ABC is obtuse.

a Explain why $\sin \angle A = \sin \angle CAM$.

b Use triangle ACM to find a formula for h in terms of A and b.

c Use triangle BCM to find a formula for h in terms of B and a.

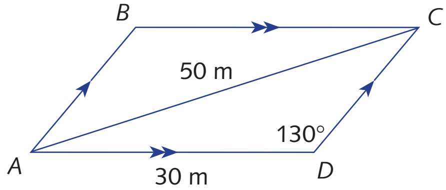
d Deduce that $\frac{a}{\sin A} = \frac{b}{\sin B}$.



That is, we have proved the sine rule holds in obtuse-angled triangles.

8 The angle between the two sides of a parallelogram is 93°. If the longer side has length 12 cm and the longer diagonal has length 14 cm, find the angle between the long diagonal and the short side of the parallelogram, correct to the nearest degree.

9 $ABCD$ is a parallelogram. $\angle CDA = 130°$, the long diagonal AC is 50 m and $AD = 30$ m. Find the length of the side DC, correct to one decimal place.



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10 Sonia starts at O and walks 600 metres due east to point A. She then walks on a bearing of 250°T to point B, 750 metres from O. Find:

a the bearing of B from O, correct to the nearest degree

b the distance from A to B, correct to the nearest metre

11 A point M is one kilometre due east of a point C. A hill is on a bearing of 028°T from C and is 1.2 km from M. Find:

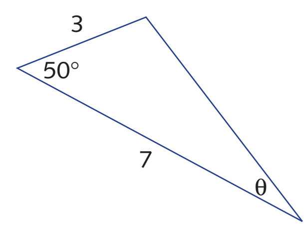
a the bearing of the hill from M, correct to the nearest degree

b the distance, correct to the nearest metre, between C and the hill

12F The cosine rule

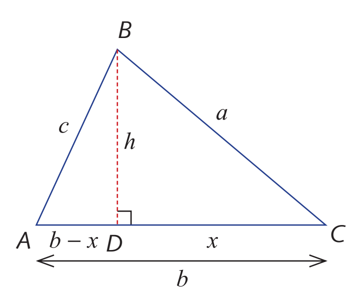
We know, from the SAS congruence test, that a triangle is completely determined if we are given two of its sides and the included angle. If we want to know the third side and the two other angles, the sine rule does not help us.

You can see from the diagram that there is not enough information to apply the sine rule. This is because the known angle is not opposite one of the known sides.



Fortunately there is another rule called the **cosine rule** which we can use in this situation.

Suppose that ABC is a triangle and that the angles A and C are acute. Drop a perpendicular from B to AC and mark the side lengths as shown in the diagram.



In ΔBDA, Pythagoras' theorem tells us that:

$$c^2 = h^2 + (b - x)^2$$

Also in ΔCBD, by Pythagoras' theorem we have:

$$h^2 = a^2 - x^2$$

Substituting this expression for h^2 into the first equation and expanding:

$$\begin{aligned} c^2 &= a^2 - x^2(b - x)^2 \\ &= a^2 - x^2 + b^2 - 2bx + x^2 \\ &= a^2 + b^2 - 2bx \end{aligned}$$

Finally, from ΔCBD, we have $\dfrac{x}{a} = \cos C$. That is, $x = a \cos C$ and so:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

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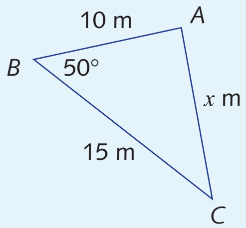
Notes:

- By relabelling the sides and angle, we could also write $a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = a^2 + c^2 - 2ac \cos B$.
- If $C = 90°$, then, since $\cos 90° = 0$, we obtain Pythagoras' theorem. Thus the cosine rule can be thought of as 'Pythagoras' theorem with a correction term'.
- The cosine rule is also true if C is obtuse. This is proven in the exercises.

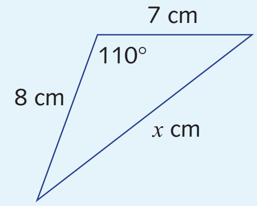
Example 14

Find the value of x, correct to one decimal place.

a



b



Solution

a Applying the cosine rule to ΔABC:

$$x^2 = 10^2 + 15^2 - 2 \times 10 \times 15 \times \cos 50°$$
$$= 132.16 \ldots$$

so $x \approx 11.5$

b Applying the cosine rule:

$$x^2 = 7^2 + 8^2 - 2 \times 7 \times 8 \times \cos 110°$$
$$= 151.30 \ldots$$

so $x \approx 12.3$ (Correct to one decimal place.)

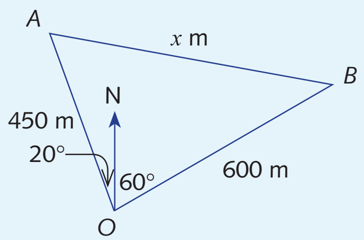
Note that in Example 14a, $x^2 < 10^2 + 15^2$ since $\cos 50°$ is positive. In Example 14b, $x^2 > 7^2 + 8^2$ since $\cos 110°$ is negative.

Example 15

A tower at A is 450 metres from O on a bearing of 340°T and a tower at B is 600 metres from O on a bearing of 060°T. Find, correct to the nearest metre, the distance between the two towers.

Solution

We draw a diagram to represent the information.



Now $\angle AOB = 80°$. Let $AB = x$ m.

Applying the cosine rule:

$$x^2 = 450^2 + 600^2 - 2 \times 450 \times 600 \times \cos 80°$$
$$= 468\,729.98 \ldots$$

that is, $x \approx 684.63 \ldots$

Hence, the towers are 685 metres apart, correct to the nearest metre.

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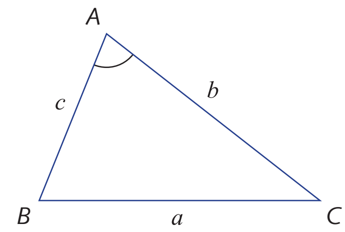
The cosine rule

In any triangle ABC:

$a^2 = b^2 + c^2 - 2bc\cos A$,

where A is the angle opposite a.

The cosine rule can be used to find the length of the third side of a triangle when the lengths of two sides and the size of the included angle are known.

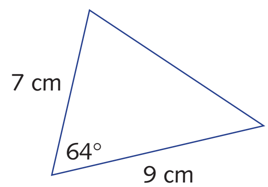


Exercise 12F

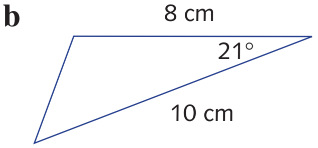
Example 14

1 In each triangle, calculate the unknown side length, giving your answer correct to two decimal places.

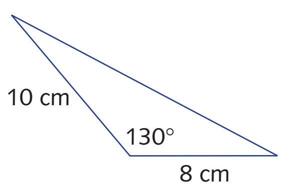
a



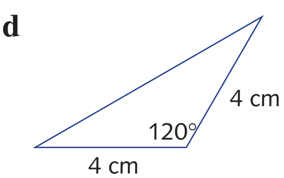
b



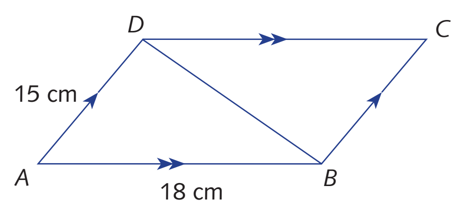
c



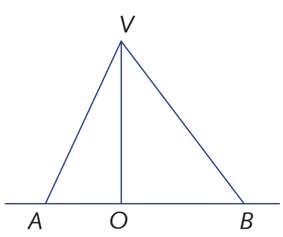
d



2 $ABCD$ is a parallelogram with sides 15 cm and 18 cm. The angle at A is 65°. Find the length of the shorter diagonal, correct to two decimal places.



3 A vertical pole OV is being held in position by two ropes, VA and VB. If $VA = 6$ m, $VB = 6.5$ m, $\angle OVB = 32°$ and $\angle OVA = 27°$ find, correct to one decimal place, the distance AB.



Example 15

4 A ship is 300 km from port on a bearing of 070°T. A second ship is 400 km from the same port and on a bearing of 140°T. How far apart, correct to the nearest kilometre, are the two ships?

5 A pilot flies a plane on course for an airport 600 km away. Unfortunately, due to an error, his bearing is out by 2°. After travelling 700 km he realises he is off course. How far from the airport is he, correct to the nearest kilometre?

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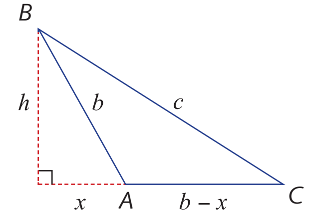
6 A rhombus $PQRS$ has side lengths 8 m, and contains an angle of 128°.

a Find the length of the longer diagonal, correct to two decimal places.

b Find the length of the shorter diagonal, correct to two decimal places.

c Find the area of the rhombus, correct to two decimal places.

7 Prove the cosine rule when the included angle, A is obtuse.

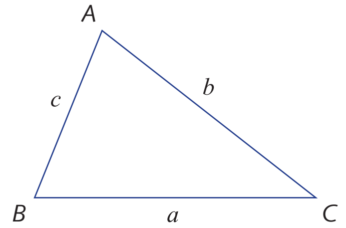


12G Finding angles using the cosine rule

The SSS congruence test tells us that once three sides of a triangle are known, the angles are uniquely determined. The question is, how do we find them?

Given three sides of a triangle, we can substitute the information into the cosine rule and rearrange to find the cosine of one of the angles and hence the angle.

If you prefer, you can learn or derive another form of the cosine rule, with $\cos C$ as the subject.



Rearranging $c^2 = a^2 + b^2 - 2ab\cos C$ we have:

$$2ab\cos C = a^2 + b^2 - c^2$$

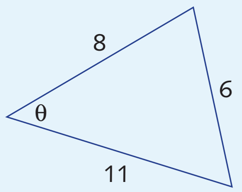
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 16

A triangle has side lengths 6 cm, 8 cm and 11 cm. Find the smallest angle in the triangle.

Solution

The smallest angle in the triangle is opposite the smallest side.



Applying the cosine rule:

$$6^2 = 8^2 + 11^2 - 2 \times 8 \times 11 \times \cos\theta$$

$$\cos\theta = \frac{8^2 + 11^2 - 6^2}{2 \times 8 \times 11}$$

$$= \frac{149}{172}$$

and so $\theta \approx 32.2°$ (Correct to one decimal place.)

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There is no ambiguous case when we use the cosine rule to find an angle. In the following example, the unknown angle is obtuse.

Example 17

In ABC, $a = 6$, $b = 20$ and $c = 17$. Find the size of $\angle ABC$, correct to one decimal place.

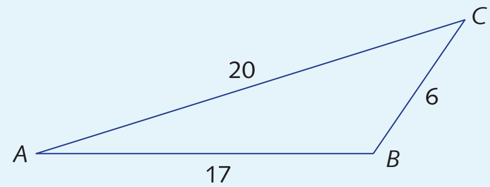
Solution

Applying the cosine rule:

$$20^2 = 6^2 + 17^2 - 2 \times 6 \times 17 \cos B$$

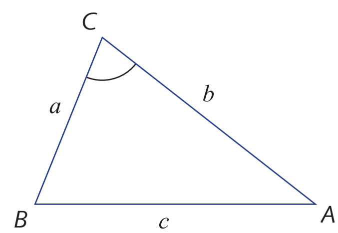
$$\cos B = \frac{6^2 + 17^2 - 20^2}{2 \times 6 \times 17} = \frac{-75}{204}$$

and so $B = 111.6°$ (Correct to one decimal place.)



Using the cosine rule to find an angle

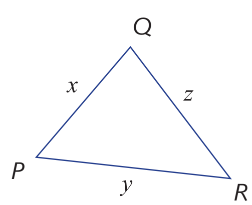
- The cosine rule can be used to determine the size of any angle in a triangle where the three side lengths are known.
- In any triangle $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, where C is the opposite angle C.



Exercise 12G

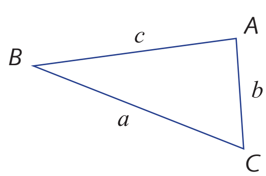
1 Copy and complete the statement of the cosine rule.

a



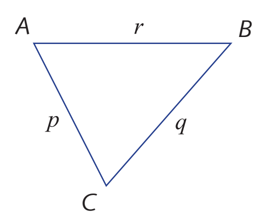
$x^2 =$

b



$b^2 =$

c

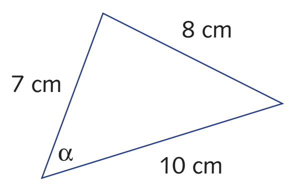


$p^2 =$

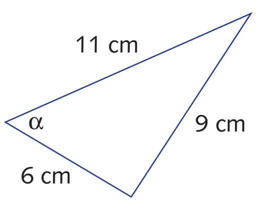
Example 16

2 Calculate α, giving the answer correct to one decimal place.

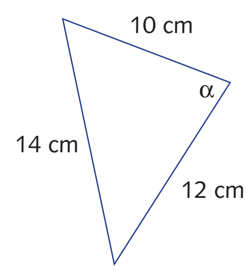
a



b



c



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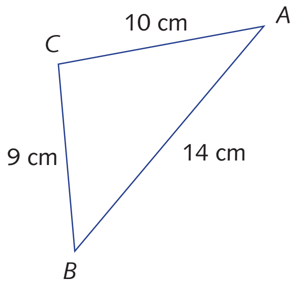
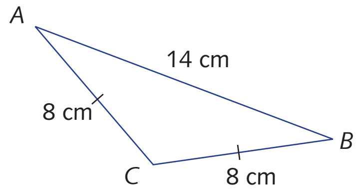
Example 17

3 Find all angles. Give answers correct to one decimal place.

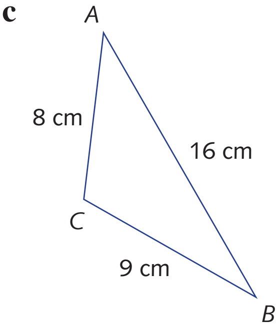
a



b



c



4 Calculate the size of the smallest angle of the triangle whose side lengths are 30 mm, 70 mm and 85 mm. Give your answer correct to one decimal place.

5 A triangle has sides of length 9 cm, 13 cm and 18 cm. Calculate the size of the largest angle, correct to one decimal place.

6 Find all the angles of a triangle whose sides are in the ratio 4 : 8 : 11, to the nearest degree.

7 A parallelogram has sides of length 12 cm and 18 cm. The longer diagonal has length 22 cm. Find, correct to one decimal place, the size of the obtuse angle between the two sides.

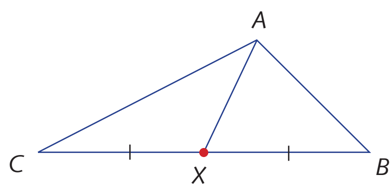
8 In ΔABC, $AB = 6$ cm, $AC = 10$ cm, $BC = 14$ cm and X is the midpoint of side BC.

a Find, correct to one decimal place:

i $\angle ACB$ **ii** the length AX

AX is called a **median** of the triangle. A median is the line segment from a vertex to the midpoint of the opposite side.

b Find the length of the other two medians, correct to one decimal place.



12H Area of a triangle

If we know two sides and an included angle of a triangle, then by the SAS congruence test the area is determined. We will now find a formula for the area.

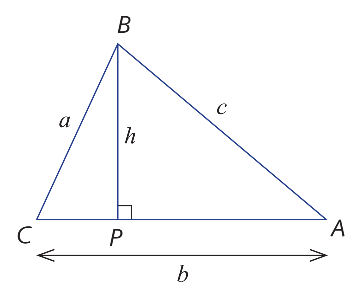
In ΔABC on the right, drop a perpendicular from A to BC. Then in ΔAPC:

$$\frac{h}{a} = \sin C$$

That is:

$$h = a\sin C$$

Hence, the area of $\Delta ABC = \frac{1}{2}bh = \frac{1}{2}ab\sin C$



Thus, the area of a triangle is half the product of any two sides times the sine of the included angle.

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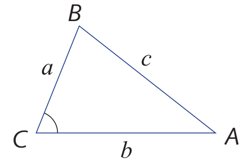
Area of a triangle

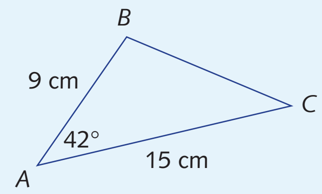
Area $= \frac{1}{2}ab\sin C$, where C is included angle.



Note that if $C = 90°$, then since $\sin 90° = 1$, the area formula becomes $\frac{1}{2}ab$. The formula also applies when the angle is obtuse. This is proved in the exercises.

Example 18

Calculate the area of the triangle ABC, correct to one decimal place.



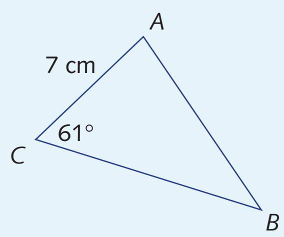
Solution

$$\text{Area of } \Delta ABC = \frac{1}{2} \times 9 \times 15 \times \sin 42°$$
$$\approx 45.2 \quad \text{(Correct to one decimal place.)}$$

So the area of the triangle is 45.2 cm^2.

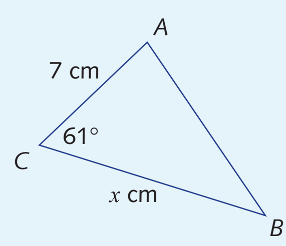
Example 19

The triangle shown has area 34 cm^2. Find the length of BC, correct to two decimal places.



Solution

Let $BC = x$ cm



$$34 = \frac{1}{2} \times 7 \times x \times \sin 61°$$
$$x = \frac{68}{7\sin 61°}$$
$$\approx 11.11 \quad \text{(Correct to two decimal places.)}$$
$$BC \approx 11.11 \text{ cm}$$

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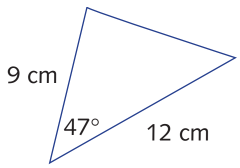
Exercise 12H

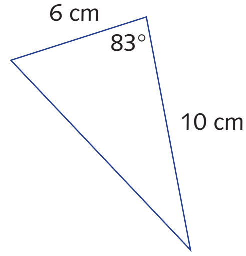
Example 18

1 Calculate each area, correct to one decimal place.

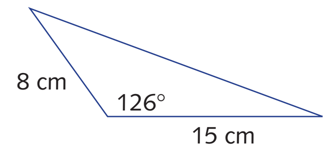
a



b

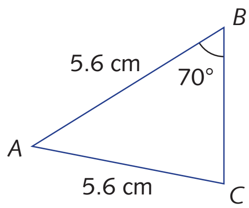


c



2 Calculate each area, correct to two decimal places.

a



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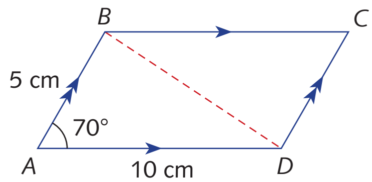
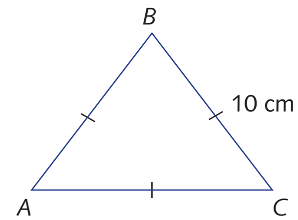


c

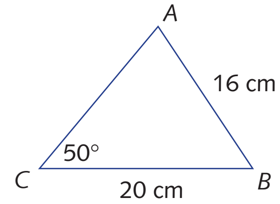


3 In ΔABC shown opposite, $\angle CAB$ is an acute angle.

a Use the sine rule to find $\angle CAB$, correct to one decimal place.

b Find $\angle ABC$, correct to one decimal place.

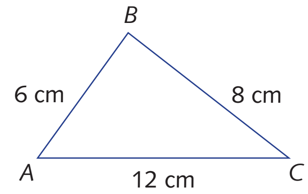
c Find the area of the triangle, correct to the nearest square centimetre.



4 In ΔABC shown opposite:

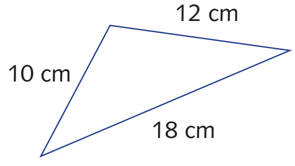
a use the cosine rule to find $\angle BAC$

b find the area of the triangle, correct to the nearest square centimetre



5 Calculate the area of each triangle, correct to the nearest square centimetre.

a



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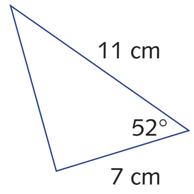
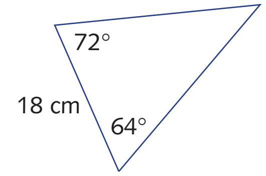


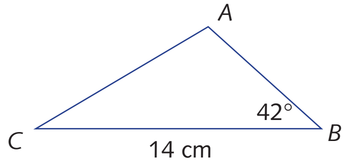
c



Example 19

6 In ΔABC shown opposite, the area of the triangle is 40 cm^2. Find, correct to one decimal place:

a AB **b** AC **c** $\angle ACB$



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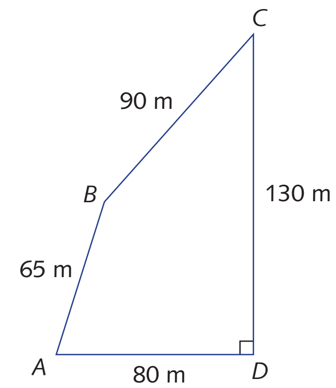
7 An acute-angled triangle of area 60 cm^2 has side lengths of 16 cm and 20 cm. What is the magnitude of the included angle, correct to the nearest degree?

8 An irregular block of land, *ABCD*, has dimensions shown opposite. Calculate, correct to one decimal place:

a the length *AC*

b $\angle ABC$

c the area of the block



9 *ABCDE* is a regular pentagon with side lengths 10 cm. Diagonals *AD* and *AC* are drawn. Find:

a $\angle AED$

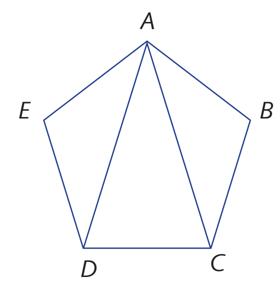
b the area of ΔADE, correct to two decimal places

c *AD*, correct to two decimal places

d $\angle ADE$ **e** $\angle ADC$ **f** $\angle DAC$

g the area of ΔADC, correct to two decimal places

h the area of the pentagon, correct to two decimal places



10 A quadrilateral has diagonals of length 12 cm and 18 cm. If the angle between the diagonals is 65°, find the area of the quadrilateral, correct to the nearest square centimetre.

11 The sides of a triangle *ABC* are enlarged by a factor, *k*. Use the area formula to show that the area is enlarged by the factor, k^2.

12 Prove that the formula Area $= \frac{1}{2}ab\sin C$ gives the area of a triangle when *C* is obtuse.

13 A triangle has sides of length 8 cm, 11 cm and 15 cm.

a Find the size of the smallest angle in the triangle, correct to two decimal places.

b Calculate, correct to two decimal places, the area of the triangle.

c Calculate the perimeter of the triangle.

d Let s = half the perimeter of the triangle. The area of the triangle can be found using Heron's formula: Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where *a*, *b* and *c* are the lengths of the three sides. Use this formula to calculate the area of the triangle, correct to two decimal places.

e Check that your answers to parts **b** and **d** are the same.

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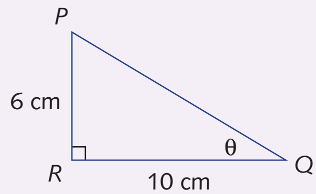
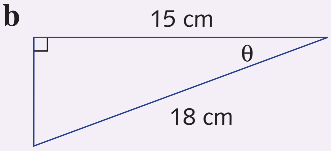
Review exercise

1 Calculate the value of the pronumeral in each triangle. Give all side lengths correct to two decimal places and all angles correct to one decimal place.

a



b



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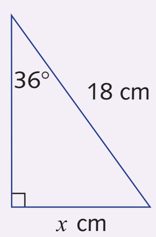
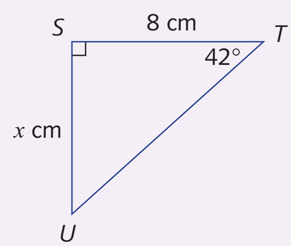
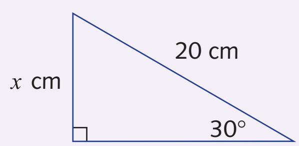


d



2 Find the exact value of the pronumeral in each triangle.

a



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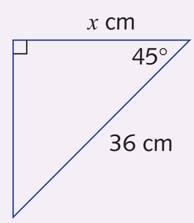

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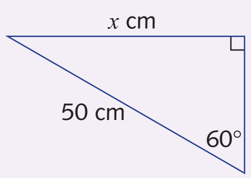
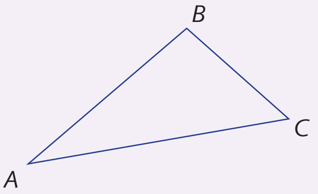


3 $AB = 8$ cm, $BC = 6$ cm and $AC = 12$ cm. Find the magnitude of each of the angles of triangle ABC correct to one decimal place.



4 A triangular region is enclosed by straight fences of lengths 42.8 metres, 56.6 metres and 72.1 metres.

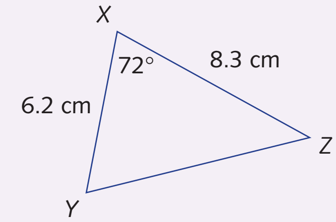
a Find the angle between the 42.8 m and the 56.6 m fences, correct to the nearest degree.

b Find the area of the region, correct to the nearest square metre.

5 In a triangle ABC, $\sin A = \frac{1}{8}$, $\sin B = \frac{3}{4}$ and $a = 8$. Find, using the sine rule, the value of b.

6 In a triangle, ABC, $a = 5$, $b = 6$ and $\cos C = \frac{1}{5}$. Find c.

7 Find the area of triangle XYZ, correct to two decimal places.



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8 For a triangle ABC, $AC = 16.2$ cm, $AB = 18.6$ cm and $\angle ACB = 60°$. Find, correct to one decimal place:

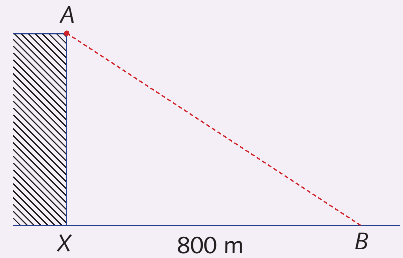
a $\angle ABC$

b $\angle BAC$

c the length of CB

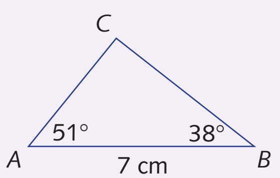
d the area of the triangle

9 The angle of depression from a point A to a ship at point B is 10°. If the distance BX from B to the foot of the cliff at X is 800 m, find the height of the cliff, correct to the nearest metre.



10 Calculate the lengths of the unknown sides and the sizes of the unknown angles, correct to two decimal places.

a



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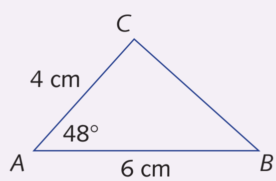

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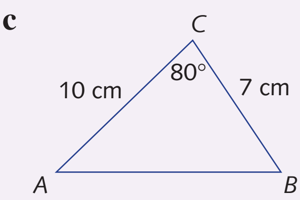

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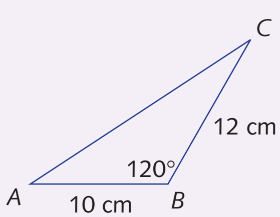
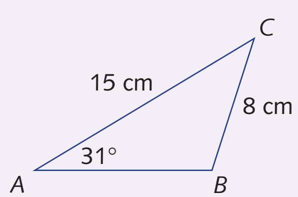


e



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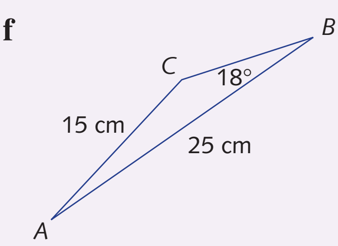
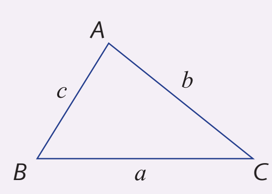


Challenge exercise

1 Write down two formulas for the area of triangle ABC and deduce the sine rule from those two formulas.



2 Simi is standing 200 metres due east of Ricardo. From Ricardo, the angle of elevation from the ground to the top of a building due north of Ricardo is 12°. From Simi, the angle of elevation from the ground to the top of the building is 9°.

a Let the height of the building be h metres. Express the following in terms of h:

i the distance from Ricardo to the foot of the building

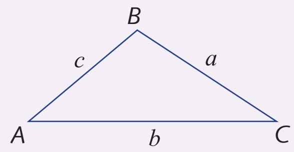
ii the distance from Simi to the foot of the building

b Use your answers to part **a** and Pythagoras' theorem to find the height of the building correct to one decimal place.

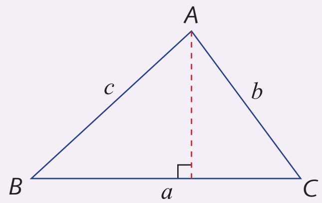
c On what bearing is the building from Simi?

3 For triangle ABC, show that

$$\text{Area} = \frac{a^2 \sin B \sin C}{2 \sin A}$$



4 Here is an alternative proof of the cosine rule. Assume ΔABC is acute-angled.



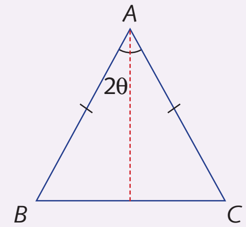
a Prove that $a = b\cos C + c\cos B$.

b Write down corresponding results for b and c.

c Show that $a^2 = a(b\cos C + c\cos B)$ and, using corresponding results for b^2 and c^2, prove the cosine rule.

d Check that a similar proof works for an obtuse-angled triangle.

5 ABC is an isosceles triangle, with $AB = AC = 1$. Suppose $\angle BAC = 2\theta$.



a Show that $BC^2 = 2(1 - \cos 2\theta)$.

b Show that $BC = 2\sin\theta$.

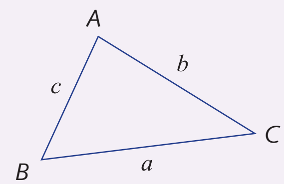
c Deduce that $1 - \cos 2\theta = 2(\sin\theta)^2$.

d Deduce that $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

6 **a** Use the cosine rule to show that

$$1 + \cos A = \frac{(b+c)^2 - a^2}{2bc} \quad \text{and}$$

$$1 - \cos A = \frac{a^2 - (b-c)^2}{2bc}$$



b Let $s = \dfrac{a+b+c}{2}$.

Show that $1 + \cos A = \dfrac{2s(s-a)}{bc}$

and $1 - \cos A = \dfrac{2(s-b)(s-c)}{bc}$

c Use the fact that $(\sin A)^2 = 1 - (\cos A)^2$ to show that the square of the area of ΔABC is $s(s-a)(s-b)(s-c)$ and deduce Heron's formula for the area of a triangle:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

7 Given two sides and a non-included angle, describe the conditions for 0, 1, or 2 triangles to exist.

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