CHAPTER

13

Measurement and Geometry

Circle geometry

You have already seen how powerful Euclidean geometry is when working with triangles. For example, Pythagoras' theorem and all of trigonometry arise from Euclidean geometry.

When applied to circles, geometry also produces beautiful and surprising results. In this chapter, you will see how useful congruence and similarity are in the context of circle geometry.

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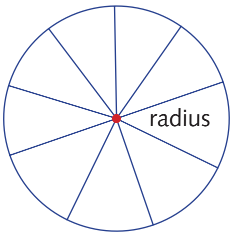
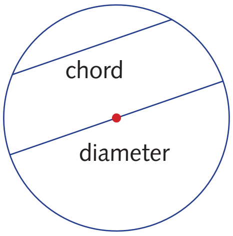
13A Angles at the centre and the circumference

A **circle** is the set of all points that lie a fixed distance (called the **radius**) from a fixed point (called the **centre**).



While we use the word 'radius' to mean this fixed distance, we also use 'radius' to mean any interval joining a point on the circle to the centre. The radii (plural of radius) of a circle radiate out from the centre, like the spokes of a bicycle wheel. (The word *radius* is a Latin word meaning 'spoke' or 'ray'.)

A **chord** of a circle is the interval joining any two points on the circle. The word *chord* is a Greek word meaning 'cord' or 'string'. A tightly stretched string that is plucked gives out a musical note, which is the origin of the word *chord* in music.

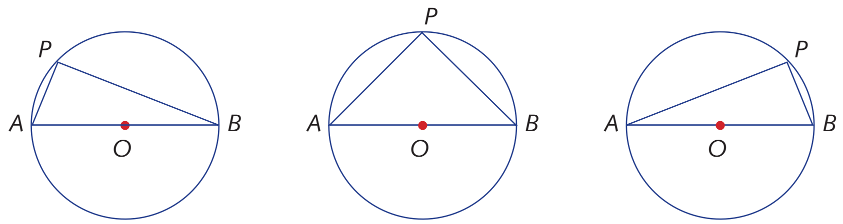


A chord that passes through the centre of the circle is called a **diameter**.

Angles in a semicircle

We will start this chapter with an important result about circles. The discovery of this result was attributed to Thales (~ 600 BC) by later Greek mathematicians, who claimed that it was the first theorem ever consciously stated and proved in mathematics.

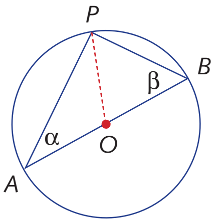
In each diagram below, the angle $\angle P$ is called an **angle in a semicircle**. It is formed by taking a diameter AOB, choosing any other point P on the circle, and joining the chords PA and PB.



These diagrams lead us to ask the question, 'What happens to $\angle P$ as P takes different positions around the semicircle?' Thales discovered a marvellous fact: $\angle P$ is always a right angle.

Theorem: An angle in a semicircle is a right angle.

Proof: Draw the radius OP.



Because the radii are equal, we have two isosceles triangles ΔAOP and ΔBOP.

Let $\angle BAP = \alpha$ and $\angle ABP = \beta$.

Then $\angle OPA = \alpha$ (base angles of isosceles ΔOPA)

and $\angle OPB = \beta$ (base angles of isosceles ΔOPB).

Adding up the interior angles of the triangle ΔABP,

$$\alpha + \alpha + \beta + \beta = 180^\circ$$

$$\alpha + \beta = 90^\circ$$

So $\angle APB = 90^\circ$, which is the required result.

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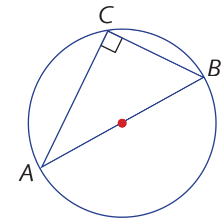
Angles in a semicircle (Thales' theorem)

An angle in a semicircle is a right angle.



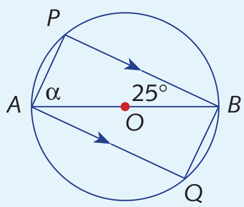
Two slightly different proofs of this result are given in Exercise 13A.

Example 1

In the diagram shown, O is the centre of the circle.

a Find α.

b Prove that $APBQ$ is a rectangle.



Solution

a First, $\angle P = 90°$ (angle in a semicircle)
so $\alpha = 65°$ (angle sum of ΔAPB)

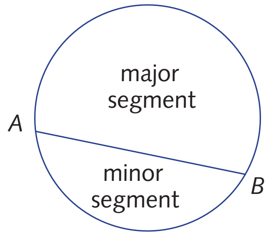
b Also, $\angle Q = 90°$ (angle in a semicircle)
so $\angle PAQ = 90°$ and $\angle PBQ = 90°$ (co-interior angles, $AQ \parallel BP$)
so $APBQ$ is a rectangle, being a quadrilateral with interior angles that are all 90°.

Arcs and segments

Our next result needs some additional words.

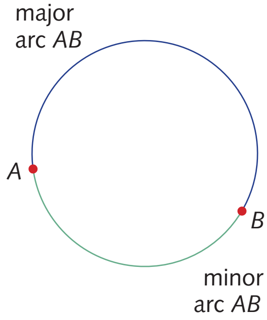
A chord AB divides the circle into two regions called **segments**.

If AB is not a diameter, the regions are unequal. The larger region is called the **major segment** and the smaller region is called the **minor segment**.



Similarly, the points A and B divide the circumference into two pieces called **arcs**.

If AB is not a diameter, the arcs are unequal. The larger piece is called the **major arc** and the smaller piece is called the **minor arc**. Notice that the phrase 'the arc AB' could refer to either arc, and we often need to clarify which arc we mean.



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Angles at the centre and the circumference

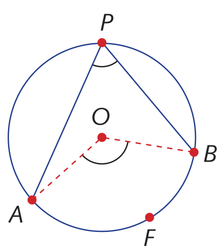
Thales' theorem about an angle in a semicircle is a special case of a more general result.

Consider a fixed arc AFB. Consider a point P on the other arc. Join the chords AP and BP to form the angle $\angle APB$.

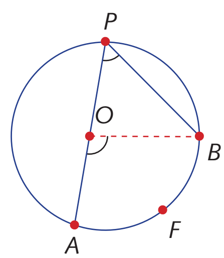
We call this angle $\angle APB$ **an angle at the circumference subtended by the arc** AFB. The word *subtends* comes from Latin and literally means 'stretches under' or 'holds under'. We also say $\angle APB$ **stands on the arc** AFB.

As with angles in a semicircle, we ask, 'What happens to $\angle APB$ as P takes different positions around the arc?'

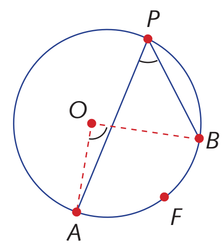
We will begin our study by focusing on $\angle APB$ in relation to $\angle AOB$, the angle subtended at the centre of the circle by the arc AFB. There are four cases to consider.



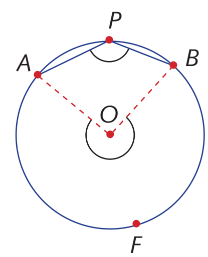
Case 1



Case 2



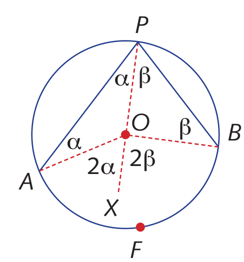
Case 3



Case 4

In the first three cases, AFB is a minor arc and $\angle APB$ is acute. However, in the fourth case, AFB is a major arc and $\angle APB$ is obtuse.

Case 1



Suppose that AFB is a minor arc, and A, B and P are located as in the diagram. We draw all three radii, AO, BO and PO, and produce PO to X.

Since AO and PO are radii, ΔAOP is isosceles. Let the equal angles be α.

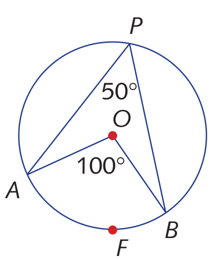
Similarly, ΔBOP is isosceles. Let the equal angles be β.

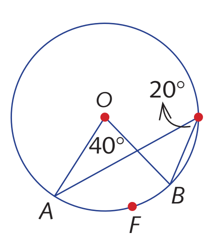
Next, using the fact that an exterior angle of a triangle is equal to the sum of the two opposite interior angles, $\angle AOX = 2\alpha$ and $\angle BOX = 2\beta$.

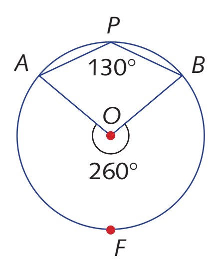
Hence, $\angle AOB = 2\alpha + 2\beta$ and $\angle APB = \alpha + \beta$. The following result has now been proved for case 1.

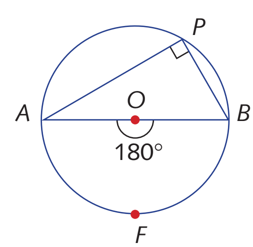
Theorem: The angle at the centre subtended by an arc of a circle is twice an angle at the circumference subtended by the same arc.

The proof for Case 4 is the same as the above proof for Case 1. It relates the obtuse angle $\angle APB$ to the reflex angle $\angle AOB$. The other two cases will be dealt with in question **7** of Exercise 13A. This will complete the proof of the theorem. Some examples of the relationship between angles at the circumference and at the centre, when subtended by a common arc, are illustrated in the following diagrams.









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Semicircles

As we can see from the fourth example from the previous page, when the arc is a semicircle, the angle at the centre is $180°$ and the angle at the circumference is $90°$. You will recognise that this is precisely the situation covered by Thales' theorem, which is thus a **special case** of our new theorem.

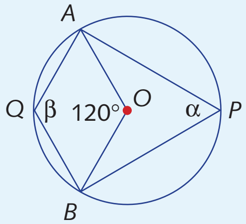
Thus, the two theorems are an excellent example of a **theorem and its generalisation**. This situation occurs routinely throughout mathematics. For example, the cosine rule can be thought of as a generalisation of Pythagoras' theorem.

Angles at the centre and the circumference

The angle at the centre subtended by an arc of a circle is twice an angle at the circumference subtended by the same arc.

Example 2

Find α and β in the diagram shown, where O is the centre of the circle.



Solution

$\alpha = 60°$ (angle at the centre is half the angle at the circumference on the same arc AQB)

Next, reflex $\angle AOB = 240°$ (angles in a revolution at O)

so $\beta = 120°$ (angle at the centre is half the angle at the circumference on the same arc APB)

Exercise 13A

Note: Points labelled O in this exercise are always centres of circles.

1 **a** **i** Use compasses to draw a large circle with centre O, and draw a diameter AOB.

ii Draw an angle $\angle APB$ in one of the semicircles. What is its size?

b **i** Draw another large circle, and draw a chord AB that is not a diameter.

ii Draw the angle at the centre and an angle at the circumference subtended by the minor arc AB. How are these two angles related?

iii Mark the angle at the centre on the major arc AB and draw an angle at the circumference subtended by this major arc. How are these two angles related?

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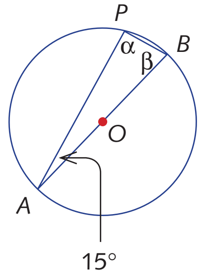
Example 1

2 Find the values of α, β, γ and θ, giving reasons.

a



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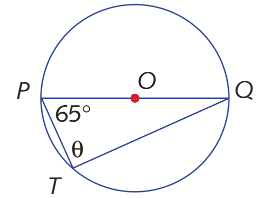

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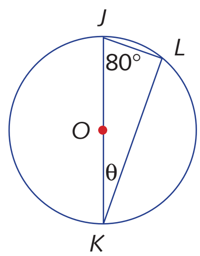

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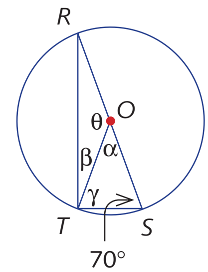

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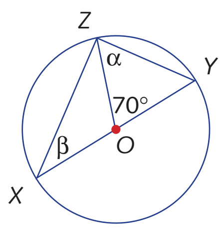

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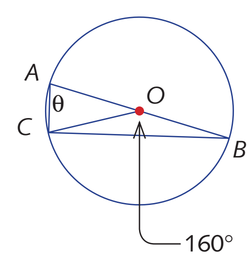
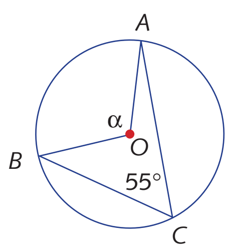


Example 2

3 Find the values of α, β, γ and θ, giving reasons.

a



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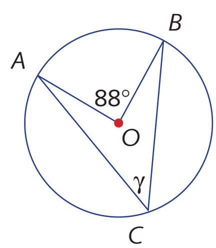

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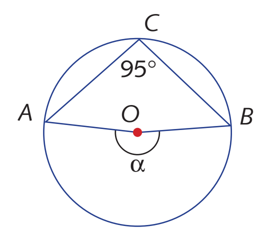

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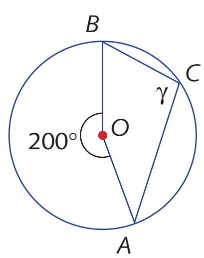

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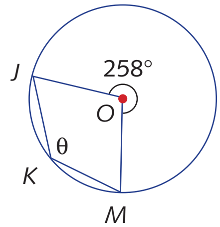

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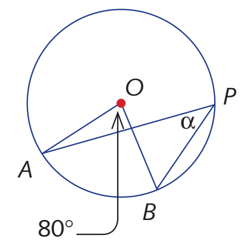

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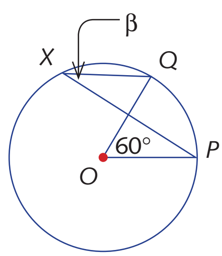

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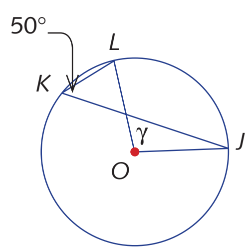

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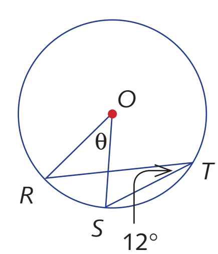

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Example 2

4 Find the values of α, β, γ and θ, giving reasons.

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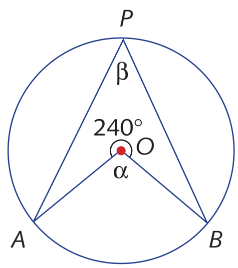

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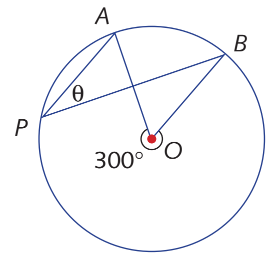

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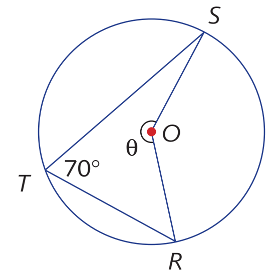

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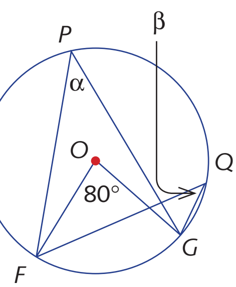

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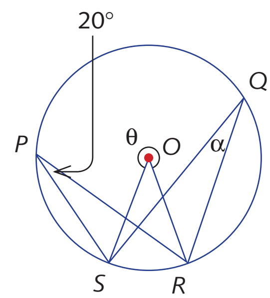

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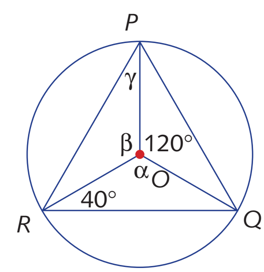

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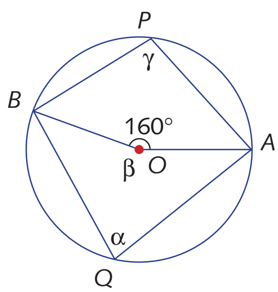

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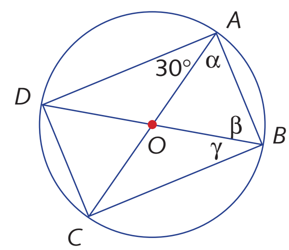

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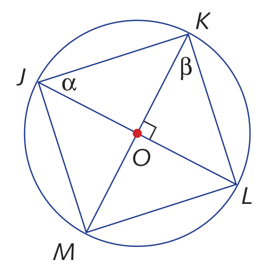

5 Find the values of α, β and γ, giving reasons.

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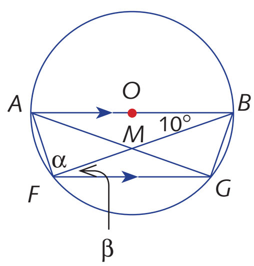

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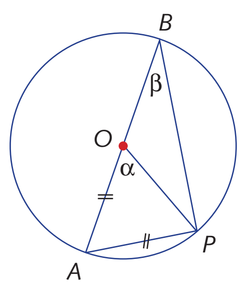

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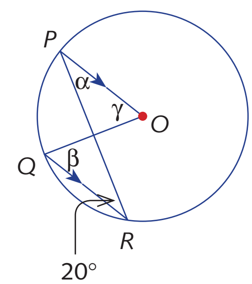

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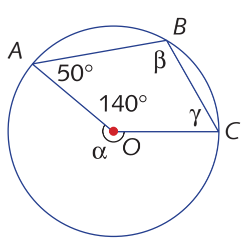

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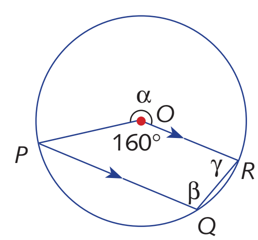

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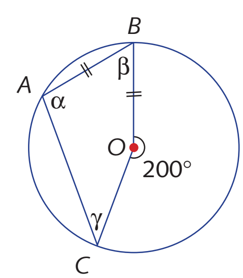
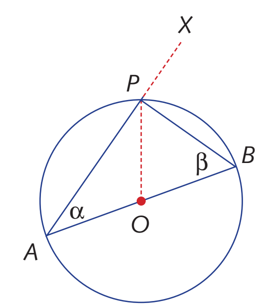


6 Thales' theorem states that: *An angle in a semicircle is a right angle*. This question develops two other proofs of Thales' theorem. We must prove, in each part, that $\angle APB = 90°$.



a (*Euclid's proof*) Join PO, and produce AP to X.

Let $\angle A = \alpha$ and $\angle B = \beta$.

i Prove that $\angle APB = \alpha + \beta$, and that $\angle XPB = \alpha + \beta$.

ii Hence, prove that $\alpha + \beta = 90°$.

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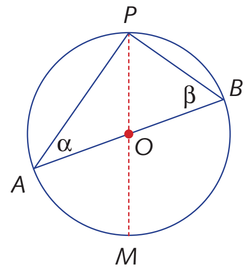
b Join PO and produce it to M.

i Prove that $\angle AOM = 2\alpha$ and $\angle BOM = 2\beta$.

ii Hence, prove that $2\alpha + 2\beta = 180°$.

iii Deduce that $\alpha + \beta = 90°$.



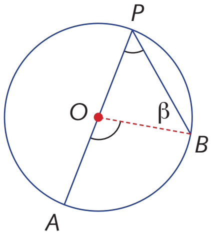
7 Prove that: *An angle at the centre subtended by an arc is twice an angle at the circumference subtended by the same arc.*

We proved Case 1 of this and noted Case 4 follows in the same manner. We pointed out that there are two other cases to consider.

a Using the diagram to the right:

i prove that $\angle APB = \beta$

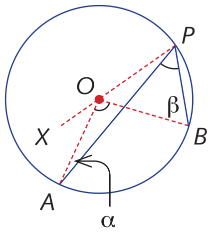
ii prove that $\angle AOB = 2\beta$



b Using the diagram to the right:

i prove that $\angle APB = \beta - \alpha$

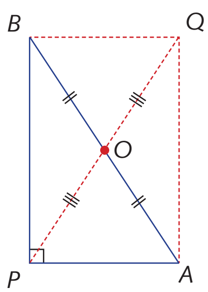
ii prove that $\angle AOB = 2(\beta - \alpha)$



8 The converse of Thales' theorem is established by proving the following result:

The midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle.

Let ΔABP be right-angled at P, and let O be the midpoint of the hypotenuse AB. Draw PO and produce it to Q so that $PO = OQ$. Draw AQ and BQ.



a Explain why $APBQ$ is a parallelogram.

b Hence, explain why $APBQ$ is a rectangle.

c Hence, explain why $AO = BO = PO$ and why the circle with diameter AB passes through P.

9 (*An application of the angle at the centre and circumference theorem*) A horse is travelling around a circular track at a constant speed. A punter standing at the very edge of the track is following him with binoculars. Explain why the punter's binoculars are rotating at a constant rate.

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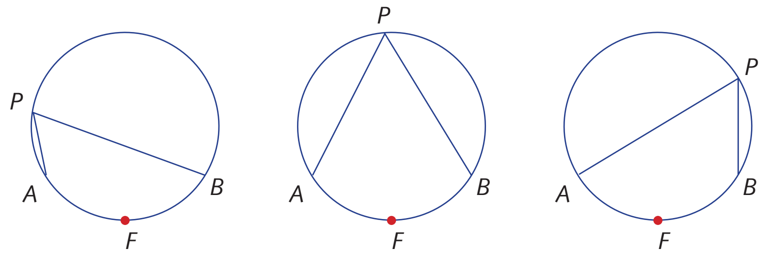
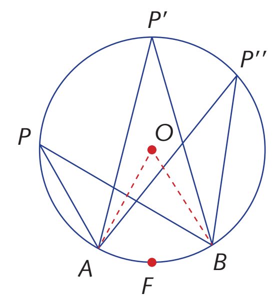
13B Angles at the circumference and cyclic quadrilaterals

Let us look at three angles at the circumference, all subtended by the same arc AFB.



We know already that all three angles are half the angle $\angle AOB$ at the centre subtended by this same arc.

It follows immediately that all three angles are equal. This result is important enough to state as a separate theorem, in the box below.



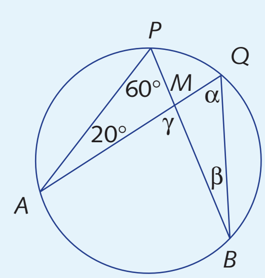
Angles at the circumference

Angles at the circumference of a circle subtended by the same arc are equal.

This is often stated as 'Angles in the same segment are equal'.

Example 3

Find α, β and γ in the diagram to the right.



Solution

First, $\alpha = 60°$ (angles on the same arc AB)

Second, $\beta = 20°$ (angles on the same arc PQ)

Third, $\gamma = 80°$ (exterior angle of ΔAPM)

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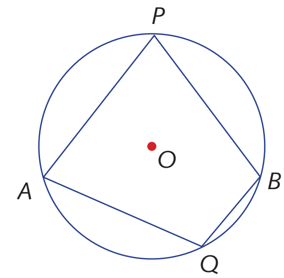
Cyclic quadrilaterals

A **cyclic quadrilateral** is a quadrilateral whose vertices all lie on a circle. We also say that the points A, B, P and Q are **concyclic**.

The opposite angles $\angle P$ and $\angle Q$ of the cyclic quadrilateral $APBQ$ are closely related.



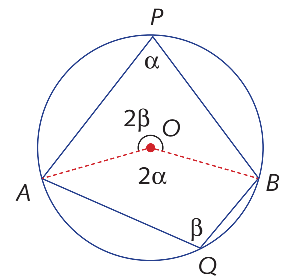
The key to finding the relationship is to draw the radii AO and BO.

First, $\angle P$ is half the angle $\angle AOB$ at the centre on the same arc AQB; we have marked these angles α and 2α.

Second, $\angle Q$ is half the reflex angle $\angle AOB$ at the centre on the same arc APB; we have marked these angles β and 2β.

$2\alpha + 2\beta = 360°$ (angles in a revolution at O)

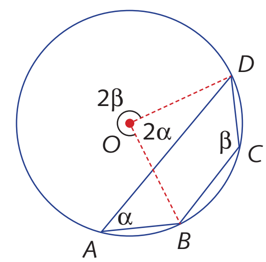
so $\alpha + \beta = 180°$



Hence, the opposite angles $\angle P$ and $\angle Q$ are supplementary.

The diagram could also have been drawn as shown, but the proof is unchanged.

We usually state this as a result about cyclic quadrilaterals.



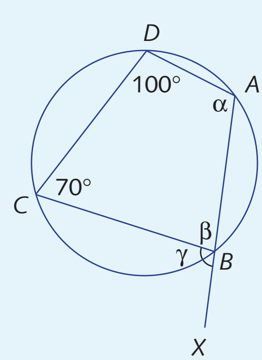
Cyclic quadrilaterals

The opposite angles of a cyclic quadrilateral are supplementary.

An interesting alternative proof is given as question **8** in Exercise 13B. Every cyclic quadrilateral is convex because none of its angles are reflex, but not every convex quadrilateral is cyclic.

Example 4

Find α, β and γ in the diagram shown.



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Solution

$\alpha + 70° = 180°$ (opposite angles of cyclic quadrilateral $ABCD$)

$\alpha = 110°$

$\beta + 100° = 180°$ (opposite angles of cyclic quadrilateral $ABCD$)

$\beta = 80°$

$\gamma + \beta = 180°$ (straight angle at B)

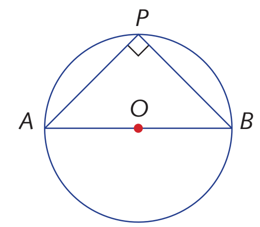
$\gamma = 100°$

The converse of Thales' theorem

We began this chapter by proving Thales' theorem:

An angle in a semicircle is a right angle.

Thales' theorem has an important converse, which was proved in question **8** of exercise 13A. It elegantly uses the diagonal properties of rectangles.

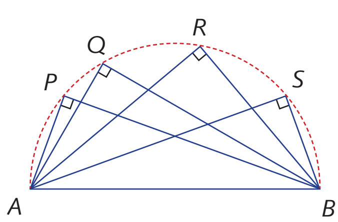


Converse of Thales' theorem

If an interval AB subtends a right angle at a point P, then P lies on the circle with diameter AB.

Here is a diagram that illustrates the converse of Thales' theorem very nicely. Suppose that AB is a line interval. A person walks from A to B in a curved path $APQRSB$ so that AB always subtends a right angle at his position.

The converse of Thales' theorem tells us that his path is a semicircle.



Exercise 13B

Note: Points labelled O in this exercise are always centres of circles.

1 **a** Draw a large circle, and draw a chord AB that is not a diameter.

b Draw two angles at the circumference standing on the minor arc AB. How are these two angles related?

c Draw two angles at the circumference standing on the major arc AB. How are these two angles related?

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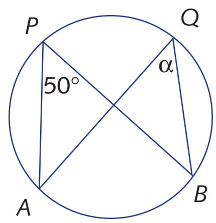
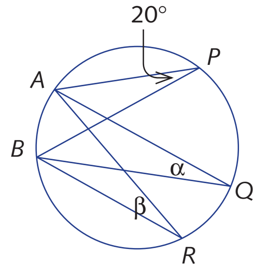
Example 3

2 Find the values of α, β and θ, giving reasons.

a



b



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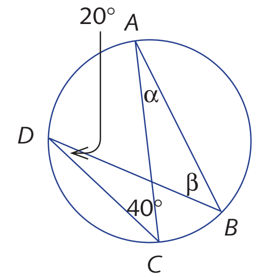

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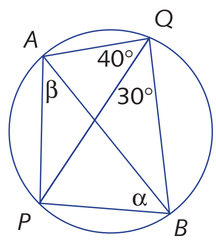

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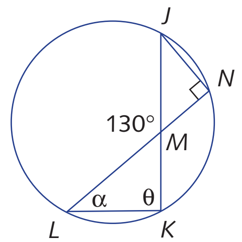

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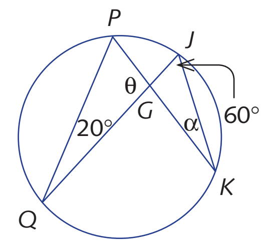
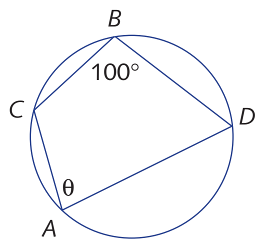


Example 4

3 Find the values of α, β, γ and θ, giving reasons.

a



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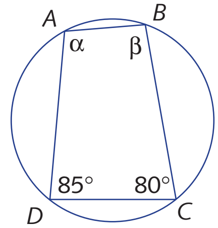

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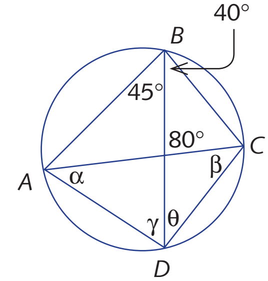

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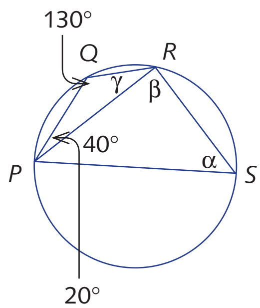

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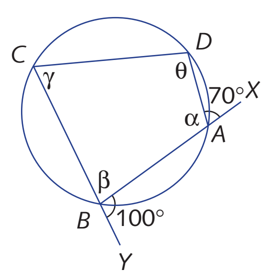

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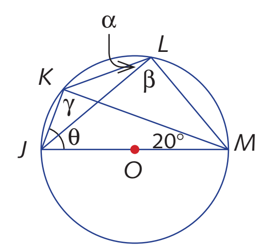
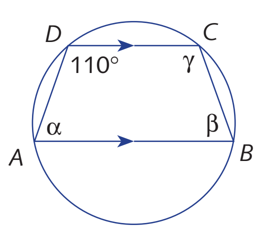


4 Find the values of α, β, γ and θ, giving reasons.

a



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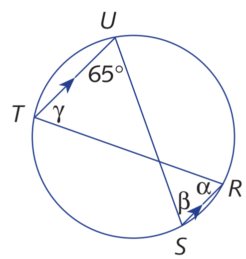

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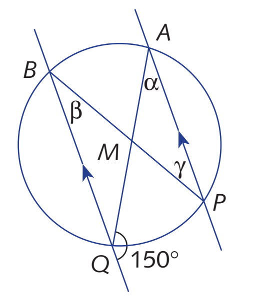

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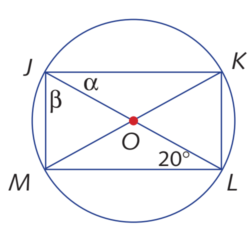

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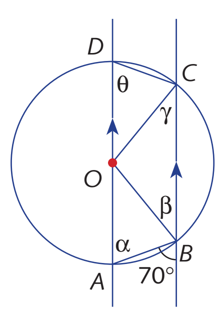

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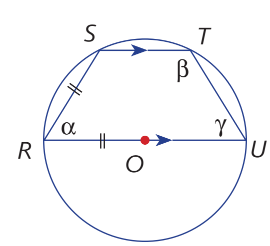

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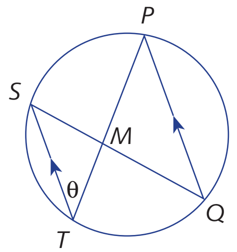
5 **a**

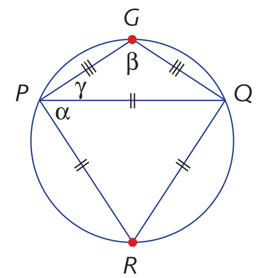


i Prove that

$\angle P = \angle Q = \angle S = \angle T$.

ii Prove that $PT = SQ$.

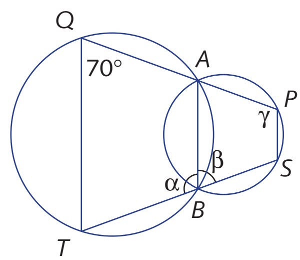
b



i Find α, β and γ.

ii Prove that $PQ \perp GR$.

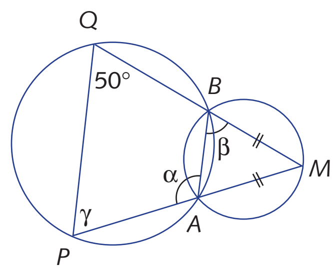
6 **a**



i Find α, β and γ.

ii Prove that $PS \parallel QT$.

b



i Find α, β and γ.

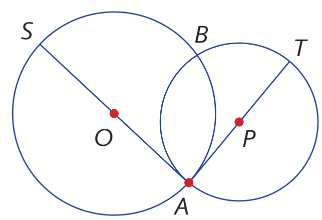
ii Prove that $AB \parallel PQ$.

iii Prove that $AP = BQ$.

7 The centres of the circles are O and P.

a **i** Find $\angle ABS$ and $\angle ABT$.

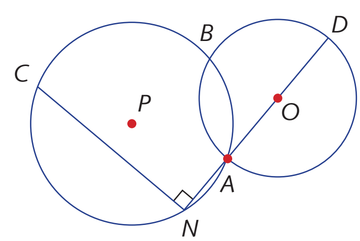
ii Hence, prove that S, B and T are collinear.



b **i** Find $\angle ABC$ and $\angle ABD$.

ii Hence, prove that C, B and D are collinear.

iii Why is AC a diameter?



8 Here is an alternative proof that: *The opposite angles of a cyclic quadrilateral are supplementary*.

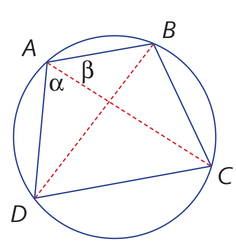
In the cyclic quadrilateral $ABCD$, draw the diagonals AC and BD.

Let $\alpha = \angle DAC$ and $\beta = \angle BAC$.

a Prove that $\angle DBC = \alpha$ and $\angle BDC = \beta$.

b Hence, prove that $\angle DCB = 180° - (\alpha + \beta)$.

c Deduce that $\angle DAB + \angle DCB = 180°$.



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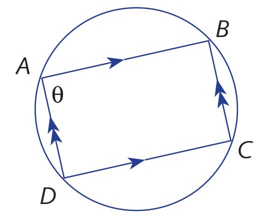
9 **a** Prove that: *A cyclic parallelogram is a rectangle*.

In the cyclic parallelogram $ABCD$, let $\angle A = \theta$.

i Give reasons why $\angle C = 180° - \theta$ and why $\angle C = \theta$.

ii Hence, prove that $ABCD$ is a rectangle.

b Use part **a** to prove that: *A cyclic rhombus is a square*.

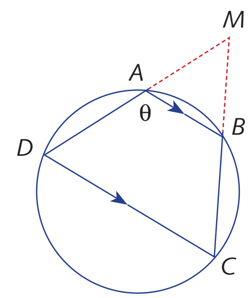


10 Prove that: *In a cyclic trapezium that is not a parallelogram, the non-parallel sides have equal length*.

Let $ABCD$ be a cyclic trapezium with $AB \parallel DC$ and $AD \nparallel BC$. Suppose DA meets CB at M and let $\angle DAB = \theta$.

a Prove that ΔABM and ΔDCM are isosceles.

b Hence, prove that $AD = BC$.

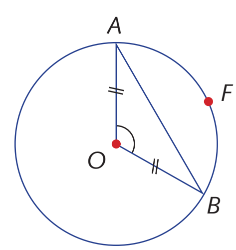


13C Chords and angles at the centre

Take a minor arc AFB of a circle and join the radii AO and BO. The angle $\angle AOB$ at the centre is called the **angle subtended at the centre** by the arc AFB. It is also called the angle at the centre subtended by the chord AB.

$OA = OB$ (radii of a circle)

Therefore, ΔAOB is isosceles. This is the key idea in the results of this section.



Equal chords and equal angles at the centre

Suppose now that two chords each subtend an angle at the centre of the circle. Two results about this situation can be proved.

Theorem: Chords of equal length subtend equal angles at the centre of the circle.

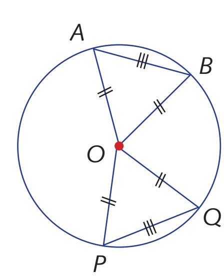
Proof: In the diagram, AB and PQ are chords of equal length.

$OA = OB = OP = OQ$ (radii of a circle)

From the diagram,

$\Delta AOB \equiv \Delta POQ$ (SSS)

Hence, $\angle AOB = \angle POQ$ (matching angles of congruent triangles)



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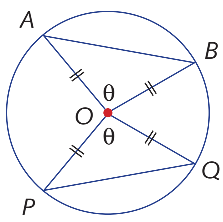
Theorem: Conversely, chords subtending equal angles at the centre have equal length.

Proof: In the diagram, AB and PQ subtend equal angles at O,

so $\Delta AOB \equiv \Delta POQ$ (SAS)

Hence, $AB = PQ$ (matching sides of congruent triangles)



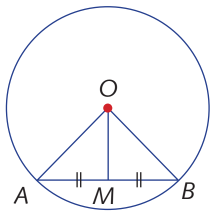
Chords and angles at the centre

- Chords of equal length subtend equal angles at the centre of a circle.
- Conversely, chords subtending equal angles at the centre of a circle have equal length.

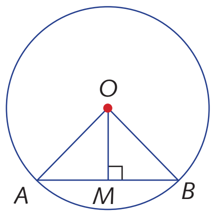
The midpoint of a chord

Three theorems about the midpoint of a chord are stated below. The proofs are dealt with in question **6** of Exercise 13C.

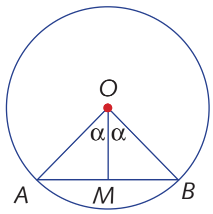
Theorem: The interval joining the midpoint of a chord to the centre of a circle is perpendicular to the chord, and bisects the angle at the centre subtended by the chord.



Theorem: The perpendicular from the centre of a circle to a chord bisects the chord, and bisects the angle at the centre subtended by the chord.



Theorem: The bisector of the angle at the centre of a circle subtended by a chord bisects the chord, and is perpendicular to it.



Chords and calculations

The circle theorems stated above can be used in conjuction with Pythagoras' theorem and trigonometry to calculate lengths and angles.

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Example 5

A chord of length 12 cm is drawn in a circle of radius 8 cm.

a How far is the chord from the centre (that is, the perpendicular distance from the centre to the chord)?

b What angle, correct to one decimal place, does the chord subtend at the centre?

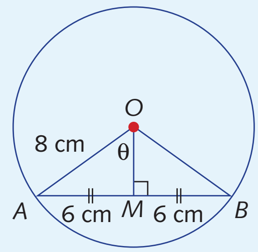
Solution

a Draw the perpendicular OM from O to the chord. By a midpoint of the chord theorem, M is the midpoint of AB, so $AM = 6$.

Hence, $OM^2 = 8^2 - 6^2$ (Pythagoras' theorem)

$$OM = \sqrt{28}$$

$$= 2\sqrt{7} \text{ cm}$$



b Let $\theta = \angle AOM$

Then, $\sin\theta = \dfrac{6}{8}$

$$\theta \approx 48.59°$$

So, $\angle AOB = 2\theta \approx 97.2°$

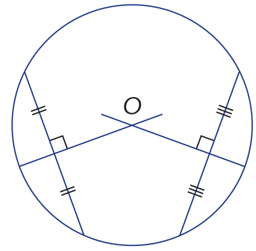
The midpoint of a chord

- The interval joining the midpoint of a chord to the centre of a circle is perpendicular to the chord, and bisects the angle at the centre subtended by the chord.
- The perpendicular from the centre of a circle to a chord bisects the chord, and bisects the angle at the centre subtended by the chord.
- The bisector of the angle at the centre of a circle subtended by a chord bisects the chord, and is perpendicular to it.

Finding the centre of a circle

Suppose that we have a circle. How do we find its centre?

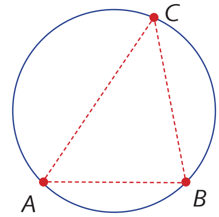
The first midpoint-of-a-chord theorem above tells us that the centre lies on the perpendicular bisector of every chord. Thus, if we draw two chords that are not parallel, and construct their perpendicular bisectors, the point of intersection of the bisectors will be the centre of the circle.



The circumcircle of a triangle

Here is an important fact about circles.

Suppose that three points A, B and C form a triangle, meaning that they are not collinear. Then there is a circle passing through all three points. The circle is called the **circumcircle** of ΔABC, and its centre is called the **circumcentre** of the triangle.



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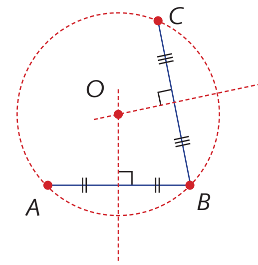
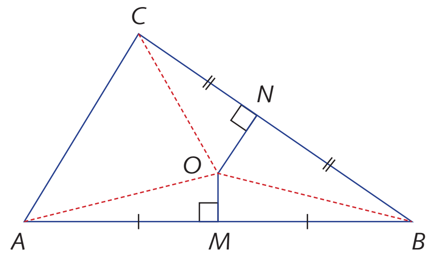
Here is a simple construction of the circumcentre and circumcircle.

Construct the perpendicular bisectors of two sides AB and BC, and let them meet at O. Then O is the circumcentre of ΔABC, and we can use it to draw the circumcircle through A, B and C.



Here is the theorem that justifies all the previous remarks.

Theorem: The intersection of the perpendicular bisectors of two sides of a triangle is the centre of a circle passing through all three verticles.



Proof: Let ABC be a triangle.

Let M be the midpoint of AB, and let N be the midpoint of BC.

Let the perpendicular bisectors of AB and BC meet at O, and join AO, BO and CO.

Then $\Delta AOM \equiv \Delta BOM$ (SAS)

so $AO = BO$ (matching sides of congruent triangles),

and $\Delta CON \equiv \Delta BON$ (SAS)

so $CO = BO$ (matching sides of congruent triangles),

so $AO = BO = CO$

Hence, O is equidistant from A, B and C, so the circle with centre O and radius AO passes through A, B and C.

The centre of a circle and circumcentre of a triangle

- To find the centre of a given circle, construct the perpendicular bisectors of two non-parallel chords, and take their point of intersection.
- To find the circumcentre of a given triangle, construct the perpendicular bisectors of two sides, and take their point of intersection.

Exercise 13C

Note: Points labelled O in this exercise are always centres of circles.

1 **a** Draw a large circle (and ignore the fact that you may be able to see the mark that the compasses made at the centre). Draw two non-parallel chords AB and PQ, then construct their perpendicular bisectors. The point where the bisectors intersect is the centre of the circle.

b Draw a large triangle ABC.

i Construct the perpendicular bisectors of two sides, and let them intersect at O. Construct the circle with circumcentre O passing through all three vertices of the triangle.

ii Construct the perpendicular bisector of the third side. It should also pass through O.

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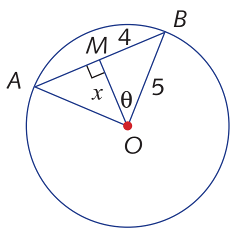
Example 5

2 Find the exact value of x, as a surd if necessary. Then use trigonometry to find the value of θ, correct to one decimal place.

a



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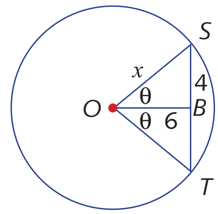

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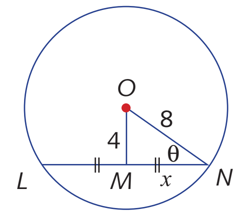


3 **a** A chord subtends an angle of $90°$ at the centre of a circle of radius 12 cm.

i How long is the chord?

ii How far is the midpoint of the chord from the centre?

b In a circle of radius 20 cm, the midpoint of a chord is 16 cm from the centre.

i How long is the chord?

ii What angle does the chord subtend at the centre, correct to one decimal place?

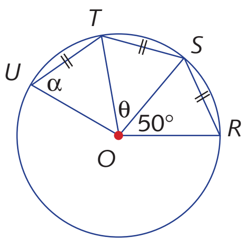
c In a circle of radius 10 cm, a chord has length 16 cm.

i What is the perpendicular distance from the chord to the centre?

ii What angle does the chord subtend at the centre, correct to one decimal place?

4 Find the values of α, β, γ and θ, giving reasons.

a



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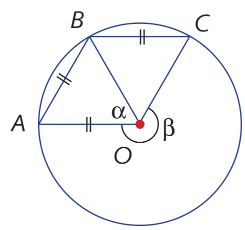

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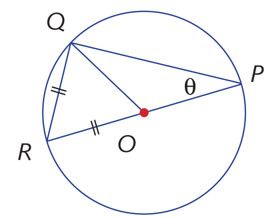

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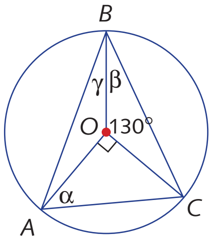

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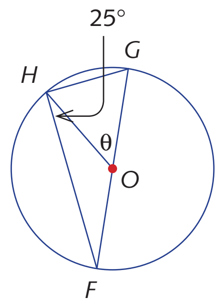

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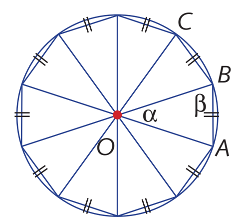
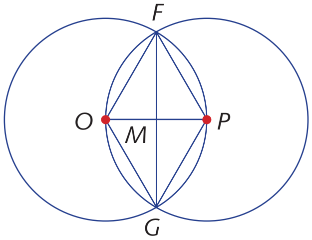


5 Let two circles of radius 1 and centres O and P each pass through the centre of the other, and intersect at F and G.

Let FG meet OP at M.

a Find $\angle FPO$, $\angle FGO$ and $\angle FMO$.

b Find the length of the common chord FG.



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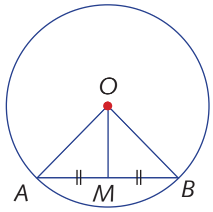
6 This question leads you through the proofs of the three theorems in the text about the midpoint of a chord.

a Prove that: *The line joining the midpoint of a chord to the centre is perpendicular to the chord, and bisects the angle at the centre subtended by the chord.*

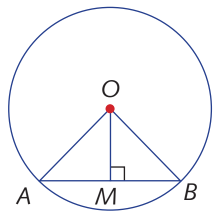


Let M be the midpoint of the chord AB.

i Prove that $\Delta AOM \equiv \Delta BOM$.

ii Hence, prove that $OM \perp AB$ and that OM bisects $\angle AOB$.

b Prove that: *The perpendicular from the centre to a chord bisects the chord, and bisects the angle at the centre subtended by the chord.*

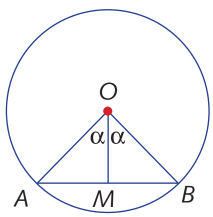


Let M be the foot of the perpendicular from O to chord AB.

i Prove that $\Delta AOM \equiv \Delta BOM$.

ii Hence, prove that M is the midpoint of AB and that OM bisects $\angle AOB$.

c Prove that: *The bisector of the angle at the centre subtended by a chord bisects the chord, and is perpendicular to it.*

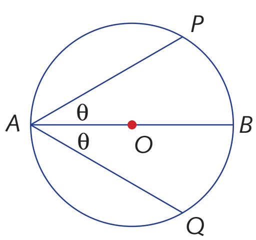


Let the bisector of $\angle AOB$ meet the chord AB at M.

i Prove that $\Delta AOM \equiv \Delta BOM$.

ii Hence, prove that M is the midpoint of AB and that $OM \perp AB$.

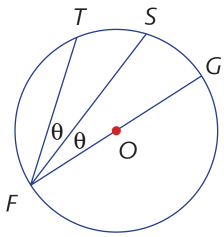
7 a



i Prove that $\Delta AOP \equiv \Delta AOQ$.

ii Prove that $AP = AQ$.

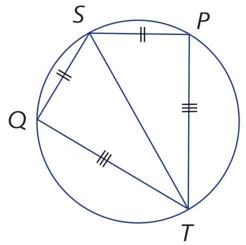
b



i Prove that $\angle FSO = \angle TFS$.

ii Prove that $FT \parallel OS$.

c



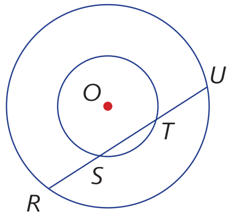
i Prove that $\Delta PST \equiv \Delta QST$.

ii Prove that $\angle P = \angle Q$.

iii Prove that $\angle P + \angle Q = 180°$.

iv Prove that ST is a diameter.

d



i Prove that $\angle OST = \angle OTS$.

ii Prove that $\angle ORU = \angle OUR$.

iii Prove that $\Delta ORT \equiv \Delta OUS$.

iv Prove that $RS = TU$.

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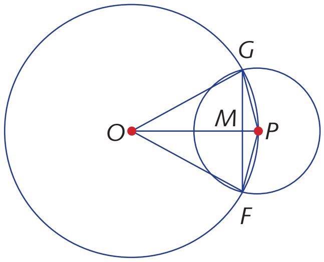
8 Prove that: *When two circles intersect, the line joining their centres is the perpendicular bisector of their common chord.*



Let two circles with centres O and P intersect at F and G. Let OP meet the common chord FG at M.

a Prove that $\Delta FOP \equiv \Delta GOP$.

b Prove that $\Delta FOM \equiv \Delta GOM$.

c Hence, prove that $FM = GM$ and $OP \perp FG$.

9 Let AB be an interval with midpoint M, and let P be a point in the plane not on AB.

a Prove that if P is equidistant from A and B, then P lies on the perpendicular bisector of AB.

b Conversely, prove that if P lies on the perpendicular bisector of AB, then P is equidistant from A and B.

c Use parts **a** and **b** to prove that a circle has only one centre.

d For these questions you will need to think in three dimensions.

i A circle is drawn on a piece of paper that lies flat on the table. Is there any other point in three-dimensional space, other than the centre of the circle, that is equidistant from all the points on the circle?

ii What geometrical object is formed by taking, in three dimensions, the set of all points that are a fixed distance from a given point?

iii What geometrical object is formed by taking, in three dimensions, the set of all points that are a fixed distance from a given line?

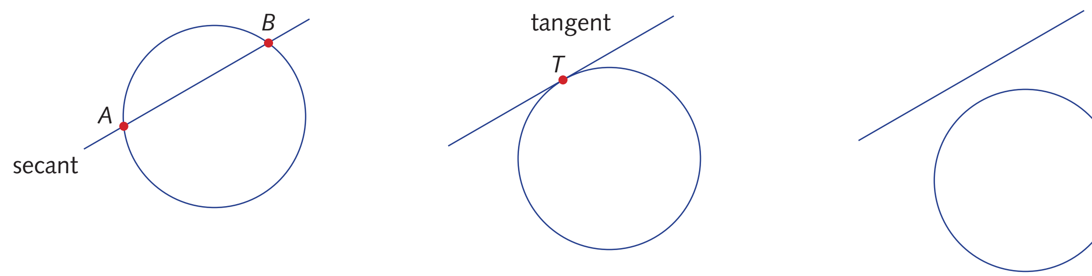
iv What geometrical object is formed by taking, in three dimensions, the set of all points that are a fixed distance from a given interval?

v What geometrical object is formed by taking, in three dimensions, the set of all points that are equidistant from the endpoints of an interval?

vi How could you find the centre of a sphere?

13D Tangents and radii

The diagrams show that a line can intersect a circle at two points, one point or no points.



two points one point no points

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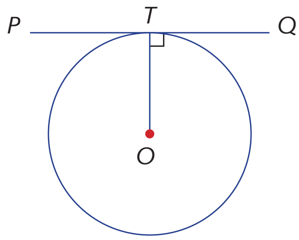
- A line that intersects a circle at two points is called a **secant**, because it *cuts* the circle into two pieces. (The word *secant* comes from Latin and means 'cutting'.)
- A line that intersects a circle at just one point is called a **tangent**. It *touches* the circle at that *point of contact*, but does not pass inside it. (The word *tangent* also comes from Latin and means 'touching'.)

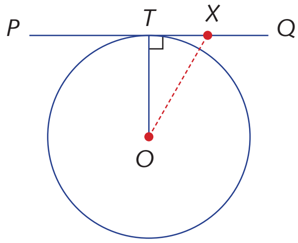
Constructing a tangent



In the diagram, OT is a radius of a circle. The line PTQ is perpendicular to the radius OT.

Could PQ intersect the circle at a second point (other than T)? The symmetry of the diagram about the line OT suggests that it cannot, and here is the proof.

Theorem: The line through a point on a circle perpendicular to the radius at that point is the tangent at that point.



Proof: Let OT be a radius, and let PTQ be perpendicular to OT. Let X be a point other than T on PTQ.

Then $OX^2 = OT^2 + TX^2$ (Pythagoras' theorem)

$OX^2 > OT^2$ since TX is non-zero,

so $OX > OT$, and OT is the radius of the circle

and thus X lies outside the circle.

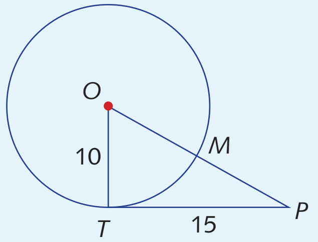
Hence, PTQ intersects the circle at only the one point, T, and so PTQ is a tangent to the circle.

Example 6

In the diagram, TP is a tangent to the circle with centre O.

a Find OP and MP.

b Find $\angle TOP$, correct to one decimal place.



Solution

a We know that $OT \perp TP$ (radius and tangent),

so $OP^2 = 10^2 + 15^2$ (Pythagoras' theorem)

$= 325$

$OP = 5\sqrt{13}$

Hence, $MP = 5\sqrt{13} - 10$

b Let $\theta = \angle TOP$

Then, $\tan\theta = \dfrac{15}{10}$

$\theta \approx 56.3°$

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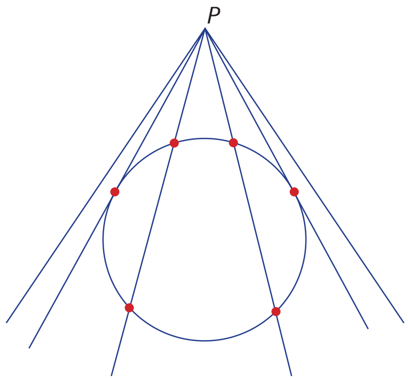
Tangents from an external point

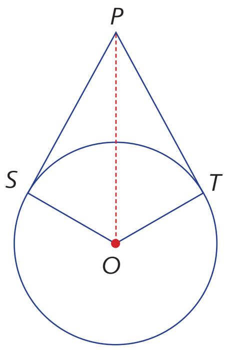
Let P be a point outside a circle. The diagram shows how different lines through P intersect the circle at two, one or no points. You can see that there are clearly exactly two tangents to the circle from P.

We now prove that these two tangents from the point P to the circle have equal length.



Theorem: The tangents to a circle from a point outside have equal length.

Proof: Let P be a point outside the circle with centre O.
Let the tangents from P touch the circle at S and T.



In the triangles OPS and OPT,

$$OP = OP \text{ (common)}$$

$$OS = OT \text{ (radii)}$$

$$\angle PSO = \angle PTO = 90^\circ \text{ (radius and tangent)}$$

so $\Delta OPS \equiv \Delta OPT$ (RHS)

Hence, $PS = PT$ (matching sides of congruent triangles)

Alternative proof: ΔPST and ΔPTO are right-angles

Therefore, $PS^2 = PO^2 - OS^2$ (Pythagoras' theorem)

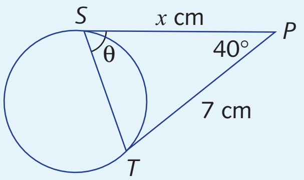
$= PO^2 - OT^2$ (radii of a circle)

$= PT^2$ (Pythagoras' theorem)

Example 7

The intervals PS and PT are tangents.

Find θ and x.



Solution

First, $x = 7$ (tangents from an external point)

Hence, $\angle T = \theta$ (base angles of isosceles ΔPST)

so $\theta = 70^\circ$ (angle sum of ΔPST)

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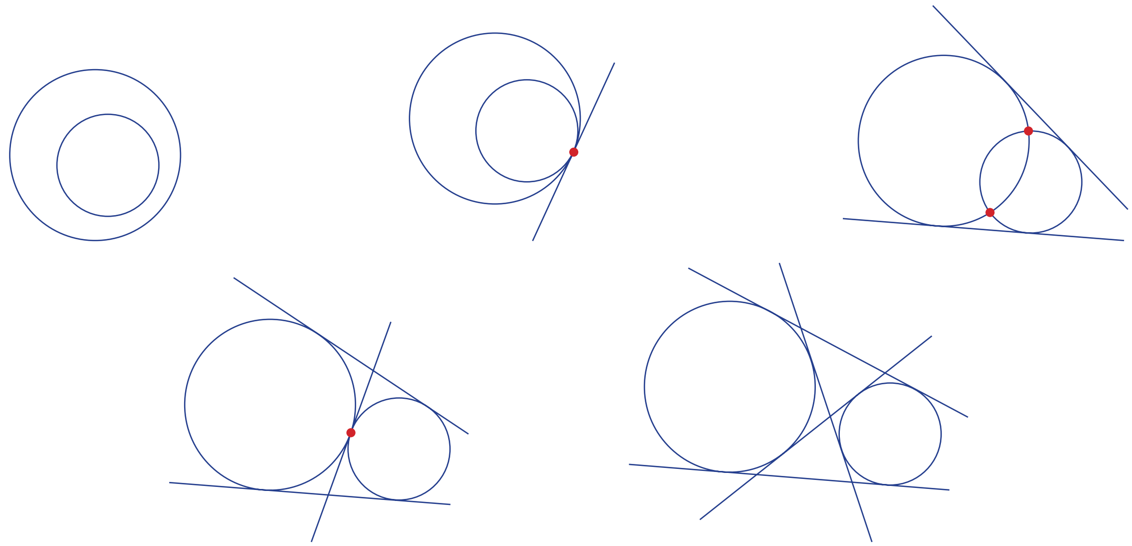
Common tangents and touching circles

The five diagrams below show all the ways in which two circles of different radii can intersect. Start with the smaller circle inside the larger, and move it slowly to the right. Notice that the two circles can intersect at two, one or no points.



The various lines are all the **common tangents** to the two circles. There are 0, 1, 2, 3 and 4 common tangents in the five successive diagrams.

In the second and fourth diagrams, the two circles **touch** each other, and they have a **common tangent at the point of contact**.

Tangents to a circle

Tangent and radius

- The line through a point on a circle perpendicular to the radius at that point is the tangent at that point.

Tangents from outside the circle

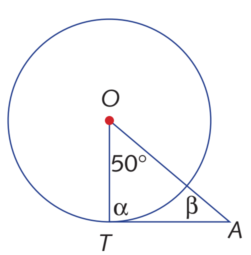
- The tangents to a circle from a point outside have equal length.

Exercise 13D

Note: Points labelled O in this exercise are always centres of circles.

1 Find the values of α, β, γ and θ, giving reasons. In each diagram, a tangent is drawn at T.

a



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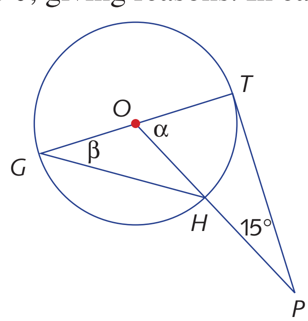

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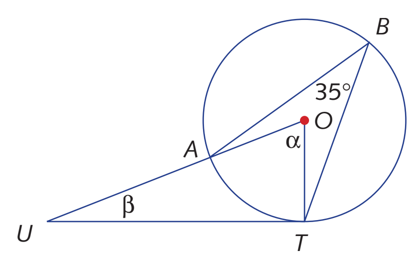

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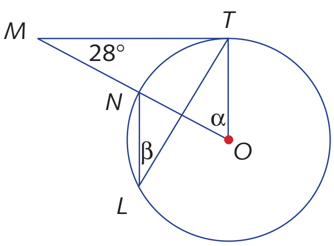

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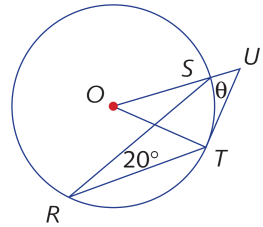

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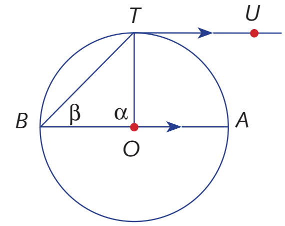

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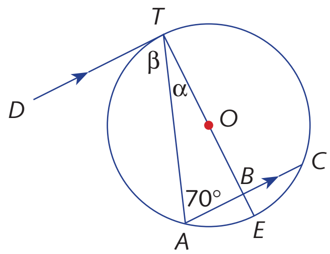

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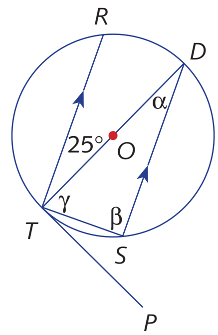

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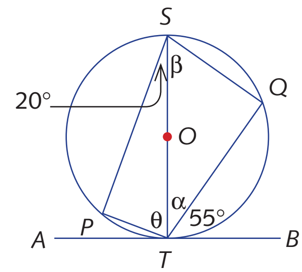

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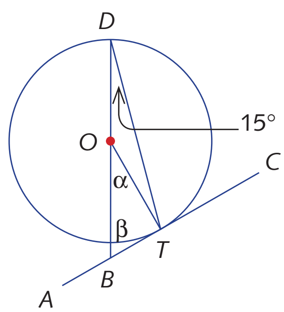

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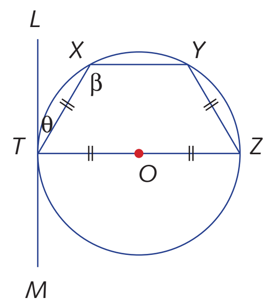

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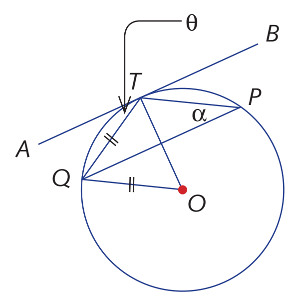
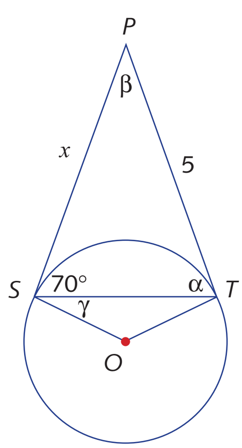


Example 7

2 Find the values of α, β, γ and θ, and the values of x, y and z. In the diagrams, tangents are drawn to the circle at S, T and U.

a



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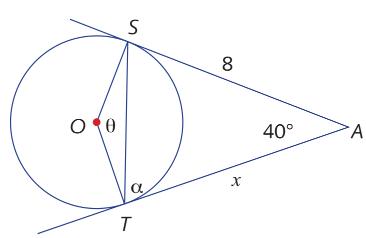

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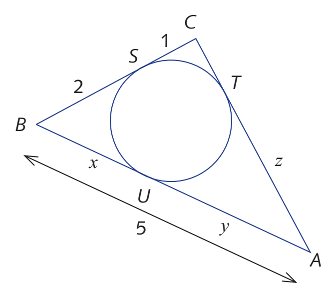

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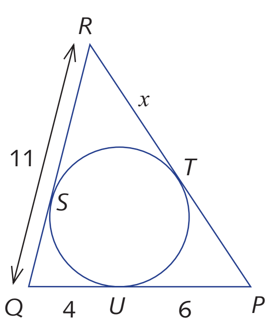

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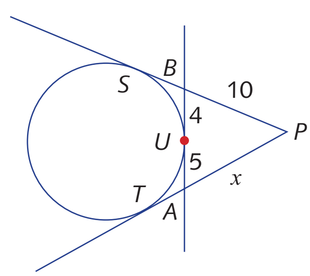

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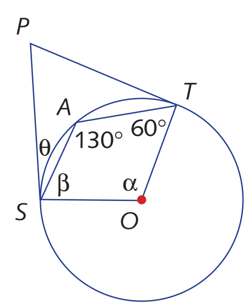

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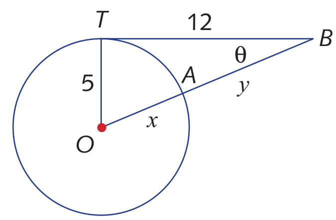
Example 6

3 Find the values of x, y and θ correct to two decimal places, where tangents are drawn at S and T.

a



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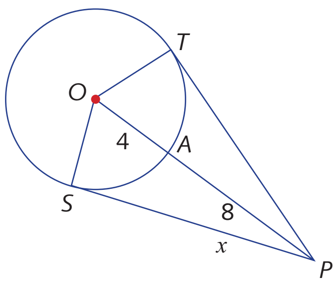

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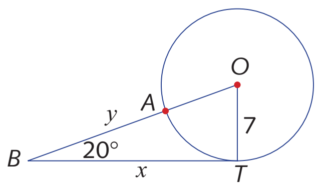
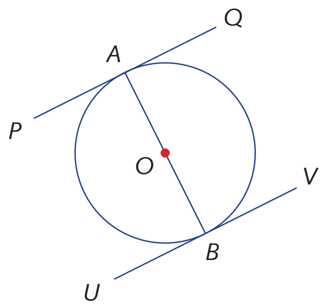
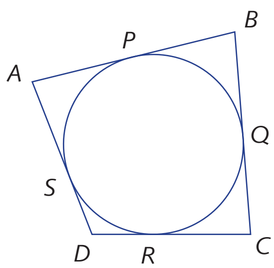


4 Prove that: *The tangents at the endpoints of a diameter are parallel.*

Let PAQ and UBV be the tangents at the endpoints of a diameter AOB. Prove that $PQ \parallel UV$.



5 The tangents at the four points P, Q, R and S on a circle form a quadrilateral $ABCD$. Prove that $AB + CD = AD + BC$.

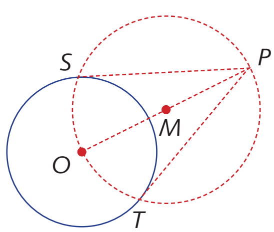


6 This question describes the method of construction of tangents to a circle from an external point P.

a Draw a circle with centre O and choose a point, P, outside the circle. Let M be the midpoint of OP, and hence draw the circle with diameter OP. Let the circles intersect at S and T, and join PS and PT.

b Prove that PS and PT are tangents to the original circle.

c Deduce that $PS = PT$.



7 Let PS and PT be the two tangents to a circle with centre O from a point, P, outside the circle.

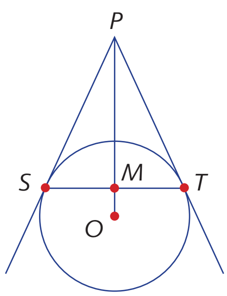
a **i** Prove that $\Delta PSO \equiv \Delta PTO$.

ii Hence, prove that the tangents have equal length, and that OP bisects the angle between the tangents and bisects the angle between the radii at OS and OT.

b Join the chord ST and let it meet PO at M.

i Prove that $\Delta SPM \equiv \Delta TPM$.

ii Hence, prove that OP is the perpendicular bisector of ST.



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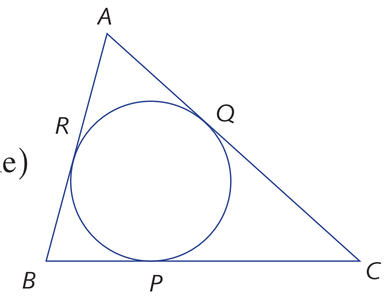
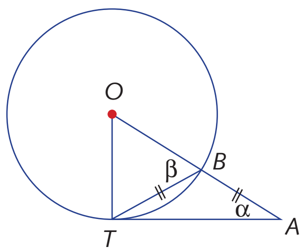
8 The circle in the diagram is called the **incircle** of triangle ABC. It touches the three sides of ΔABC at P, Q and R. Prove that:

Area of triangle $= \frac{1}{2} \times (\text{perimeter of triangle}) \times (\text{radius of circle})$

You will need to join the radii OP, OQ, and OR and the intervals OA, OB and OC, where O is the centre of the incircle.



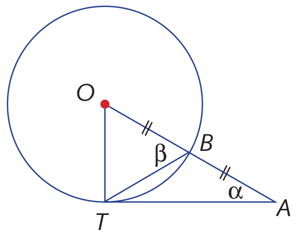
9 **a**



i Prove that $\alpha = 30°$.

ii Find β.

b



i Prove that $\sin \alpha = \frac{1}{2}$.

ii Prove that $\beta = 60°$.

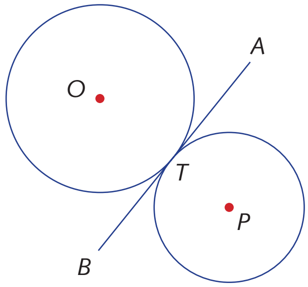
10 Prove that: *When two circles touch, their centres and their point of contact are collinear.*

a Let two circles with centres O and P touch externally at T. Let ATB be the common tangent at T.

i Find $\angle ATO$ and $\angle ATP$.

ii Hence, prove that O, T and P are collinear.

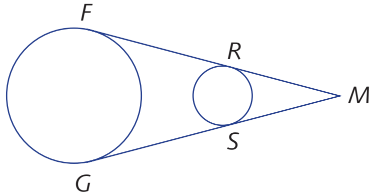
b Draw a diagram of two circles touching internally, and prove the theorem in this case.



11 **a** Each tangent, FR and GS, in the diagram is called a **direct common tangent** because the two circles lie on the same side of the tangent. Produce the two tangents to meet at M.

i Prove that $MF = MG$ and $MR = MS$.

ii Hence, prove that $FR = GS$.

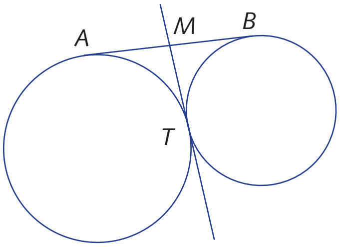


b Draw a diagram showing **indirect common tangents**, and prove that they also have equal length. (*Note*: Indirect common tangents cross over, and intersect between the two circles.)

12 Let AB be a direct common tangent of two circles touching externally. Let the common tangent at the point of contact, T, meet AB at M.

a Prove that $MA = MB = MT$.

b Hence, deduce that $\angle ATB$ is a right angle.



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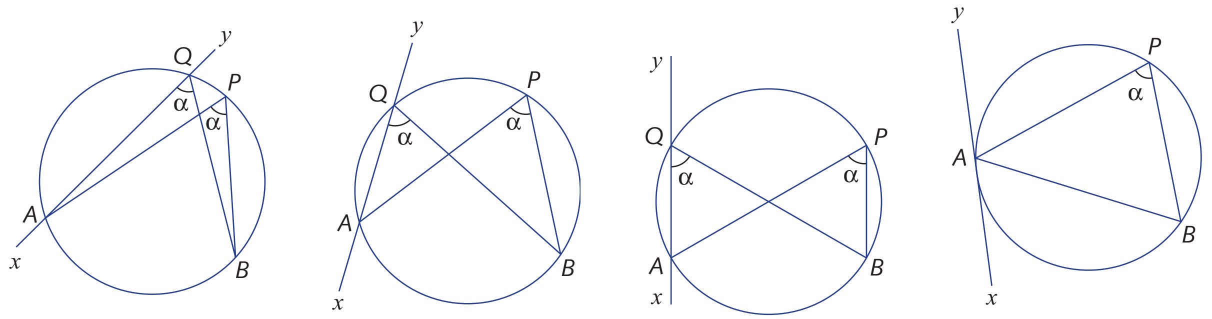
13E The alternate segment theorem

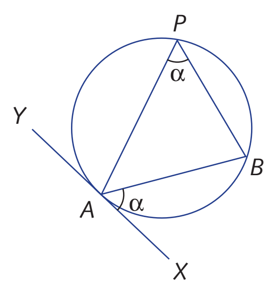
Consider the secant XY that intersects a circle at points A and Q. Consider also points P and B and the angles subtended at P and Q by the arc AB.



As you can see from the images above, as Q approaches A along the circumference, $\angle XQB$ and $\angle P$ remain equal. It therefore seems reasonable to suppose that as Q coincides with A, and XY becomes tangent to the circle at A, $\angle XAB$ and $\angle P$ are equal.

This is in fact the case, and is the **alternate segment theorem**.

The **alternate segment** ('alternate' here simply means 'other') is the segment of the circle on the other side of the chord AB from $\angle XAB$. The angle $\angle P$ is an *angle in the alternate segment*.



Theorem: The angle between a tangent and a chord is equal to any angle in the alternate segment.

Proof: Let AB be a chord of a circle and let XAY be a tangent at A. Let P be a point on the circle on the other side of the chord AB from $\angle XAB$. Let $\angle P = \theta$.

We must prove that $\angle XAB = \angle P$.

Draw the diameter AON, and join BN.

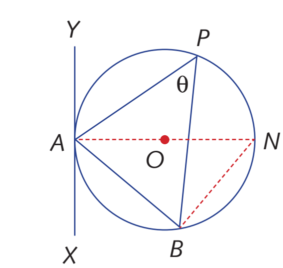
Then $\angle N = \theta$ (angles on the same arc AB)

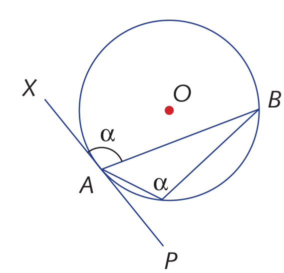
and $\angle NBA = 90^\circ$ (angle in the semicircle NBA)

and $\angle NAX = 90^\circ$ (radius and tangent)

Hence, $\angle NAB = 90^\circ - \theta$ (angle sum of $\Delta\, NAB$)

so $\angle XAB = \theta$ (adjacent angles in a right angle)





Note: This proof is only valid when $\angle XAB$ is acute. In the exercises we will prove the result when $\angle XAB$ is obtuse.

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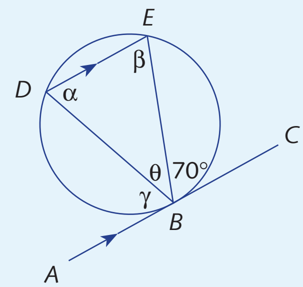
Example 8

Find α, β, γ and θ in the figure shown to the right.



Solution

$\alpha = 70°$ (alternate segment theorem)

$\beta = 70°$ (alternate angles, $DE \parallel AC$)

$\gamma = 70°$ (alternate angles, $DE \parallel AC$)

$\theta = 40°$ (angles in a straight angle at B)

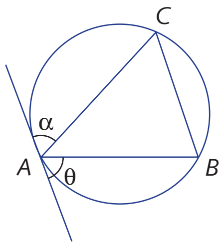
The alternate segment theorem

The angle between a tangent and a chord is equal to any angle in the alternate segment.

Exercise 13E

Note: Points labelled O in this exercise are always centres of circles.

1 Draw a large circle and a chord AB. At one end of the chord, draw the tangent to the circle.



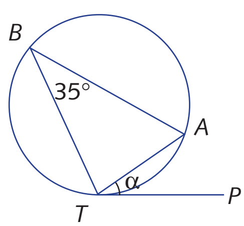
a Mark one of the angles θ between the tangent and the chord AB, then draw any angle in the alternate segment. How are these two angles related?

b Mark the angle α between the tangent and the chord AC. Which angle is equal to α?

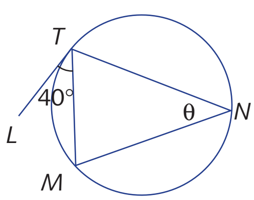
Example 8

2 Find the values of α, β, γ and θ, giving reasons. In each diagram, a tangent is drawn at T.

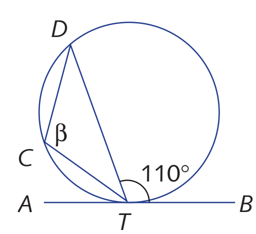
a



b



c



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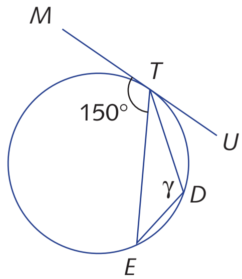
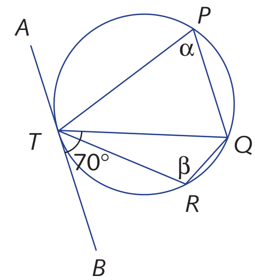
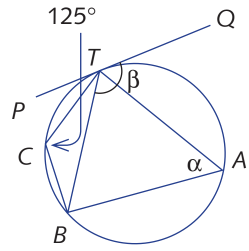
d



e

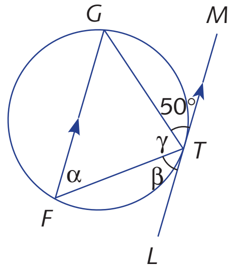


f



3 Find the values of α, β, γ and θ, giving reasons. In the diagrams, tangents are drawn at S, T and U.

a



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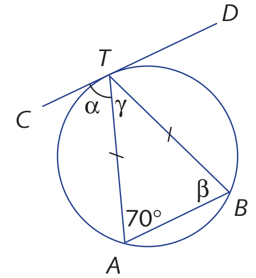

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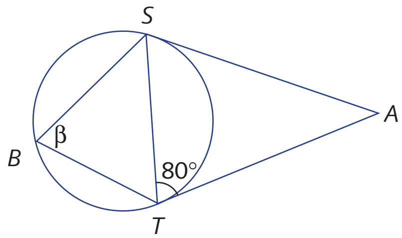

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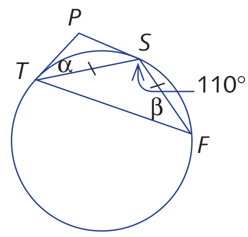

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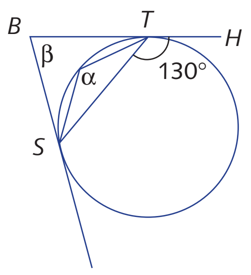

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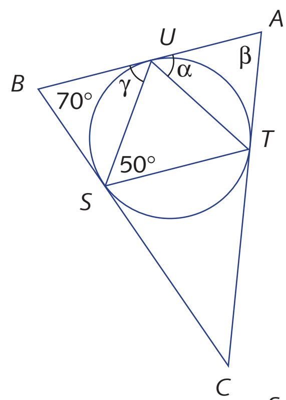
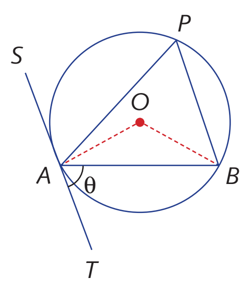


4 Here is a different proof of the alternate segment theorem.



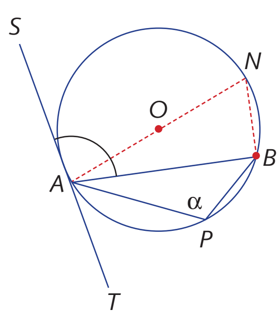
Let AB be a chord of a circle, and let SAT be the tangent at A. Let $\theta = \angle BAT$ be an acute angle. We must prove that $\angle APB = \theta$.

a Join the radii OA and OB. What is the size of $\angle BAO$?

b What is the size of $\angle AOB$?

c Hence, prove that $\angle APB = \theta$.

5 Show that the alternate segment theorem holds when the angle between the tangent and the chord is obtuse.



Let $\angle P = \alpha$. We must prove $\angle SAB = \alpha$.

a Construct diameter AN and chord NB. Show that $\angle ANB = 180° - \alpha$.

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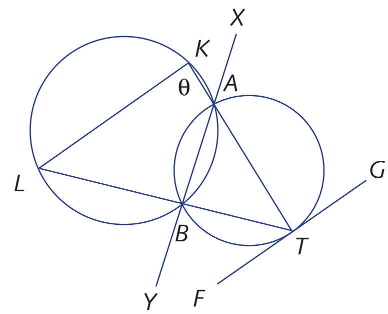
b Show that $\angle NAB = \alpha - 90°$.

c Using $\angle SAB = \angle SAN + \angle NAB$, show that $\angle SAB = \alpha$.

d Use the technique used in question **4** to prove the alternate segment theorem for an obtuse angle.

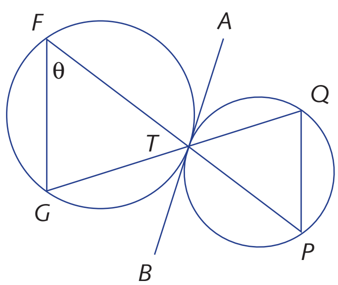
6 Choose T on the smaller circle. Let FTG be the tangent at T and construct lines KAT and TBL as shown in the diagram. Let $\angle LKT = \theta$.



a Prove that $\angle GTA = \theta$.

b Hence, prove that $LK \parallel FG$.

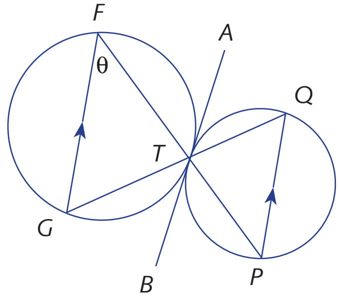
7 The two circles in the diagram touch externally at T, with common tangent ATB at the point of contact, and FTP and GTQ are straight lines.



a Let $\angle F = \theta$. Prove that $\angle GTB = \theta$ and $\angle QTA = \theta$.

b Hence, prove that $FG \parallel QP$.

8 The two circles in the diagram touch externally at T, with common tangent ATB at the point of contact. Suppose P, T and F are collinear and $GF \parallel QP$.



a Let $\theta = \angle F$. Prove that $\angle P = \theta$.

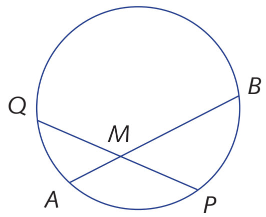
b Hence, prove that the points G, T and Q are collinear.

13F Similarity and circles

Intersecting chords

Take a point, M, inside a circle, and draw two chords, AMB and QMP, through M. Each chord is thus divided into two subintervals called **intercepts**. In the following, we shall prove that:

$$AM \times BM = PM \times QM$$



This is a very interesting and useful result called the *intersecting chord theorem*.

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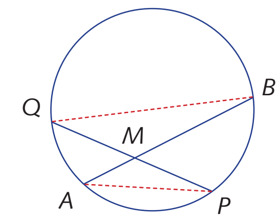
Theorem: When two chords of a circle intersect, the product of the intercepts on one chord equals the product of the intercepts on the other chord.

Proof: Draw the intervals AP and BQ to make two triangles AMP and QMB.



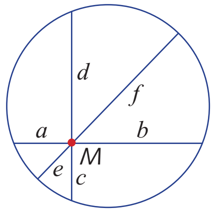
ΔAMP is similar to ΔQMB (AAA).

Hence,

$\frac{AM}{QM} = \frac{PM}{BM}$ (matching sides of similar triangles)

so, $AM \times BM = PM \times QM$.

Note: If we have a family of chords passing through a point, we can apply the theorem to see that $ab = cd = ef$.



Intercepts

A point M on an interval AB divides that interval into two subintervals AM and MB, called intercepts.

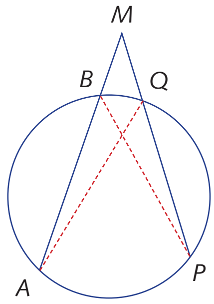
For the next two theorems, we will need to apply this definition to the situation where the dividing point M is still on the line AB, but is outside the interval AB.

The intercepts are still AM and BM. Everything works in exactly the same way provided that both intercepts are measured from M.

Secants from an external point

Now take a point, M, outside a circle, and draw two secants, MBA and MQP, to the circle from M. Provided that we continue to take our lengths from M to the circle, the statement of the result is the same. That is:

$$AM \times BM = PM \times QM$$



Theorem: When two secants intersect outside a circle, the product of the intercepts on one secant equals the product of the intercepts on the other secant.

The proof by similarity is practically the same as when M is inside the circle, and we will address this in question **2** of Exercise 13F.

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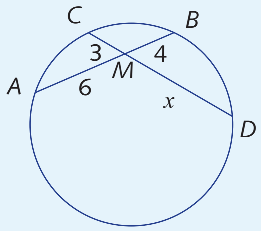
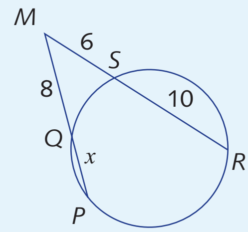
Example 9

Find x in each diagram.

a



b



Solution

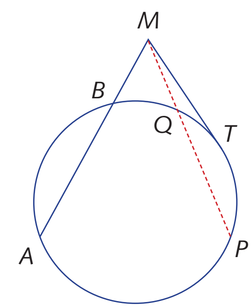
a Using intersecting chords:

$$3 \times x = 6 \times 4$$
$$x = 8$$

b Using secants from an external point:

$$8 \times (8 + x) = 6 \times (6 + 10)$$
$$8(8 + x) = 96$$
$$8 + x = 12$$
$$x = 4$$

Tangent and secant from an external point



As the point Q moves towards T, the line MQP becomes the tangent at T. Thus, the previous product $PM \times QM$ has become the square TM^2. That is:

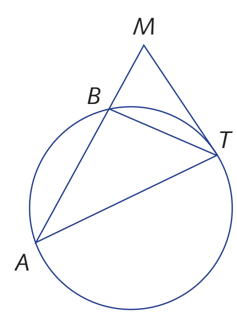
$$AM \times BM = TM^2$$

We therefore have a new theorem. For completeness, we give another proof.

Theorem: When a secant and a tangent to a circle intersect, the product of the intercepts on the secant equals the square of the tangent.

That is, $AM \times BM = TM^2$

Proof: Let M be a point external to a circle.
Let TM be a tangent from M. Suppose a secant from M cuts the circle at A and B.



Draw the intervals AT and BT, and look at the two triangles AMT and TMB.

$$\angle AMT = \angle TMB \text{ (common angle)}$$

$$\angle MAT = \angle MTB \text{ (alternate segment theorem)}$$

so ΔAMT is similar to ΔTMB (AAA).

Hence, $\dfrac{AM}{TM} = \dfrac{TM}{BM}$ (matching sides of similar triangles)

so $AM \times BM = TM^2$

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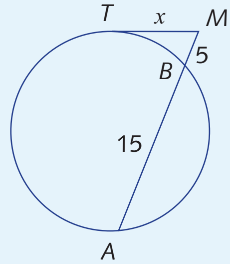
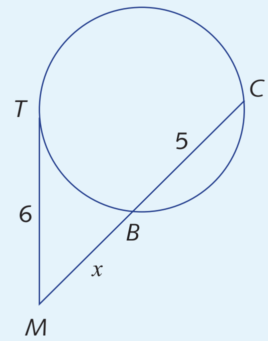
Example 10

Find x in each diagram, given that MT is a tangent to the circle.

a



b



Solution

We use the tangent and secant theorem in each part.

a $x^2 = 5 \times (5 + 15)$

$x^2 = 100$

$x = 10$ (since x is positive)

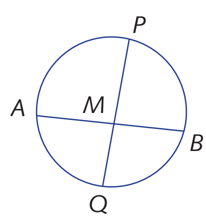
b $x \times (x + 5) = 6^2$

$x^2 \times 5x - 36 = 0$

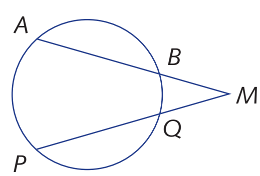
$(x + 9)(x - 4) = 0$

$x = 4$ (since x is positive)

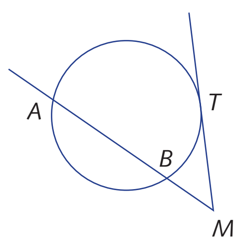
Chords, secants and tangents



$AM \times BM = PM \times QM$



$AM \times BM = PM \times QM$



$AM \times BM = TM^2$

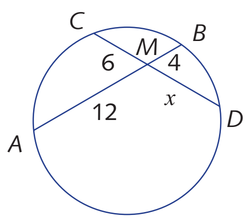
Exercise 13F

Note: Points labelled O in this exercise are always centres of circles.

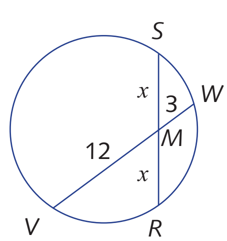
Example 9, 10

1 In each diagram, find the value of x, giving reasons. Tangents are drawn at the point T.

a



b



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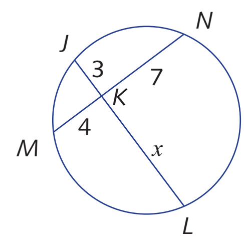

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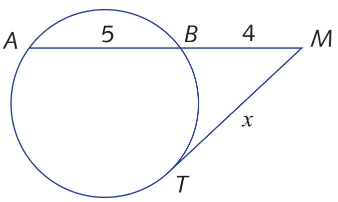

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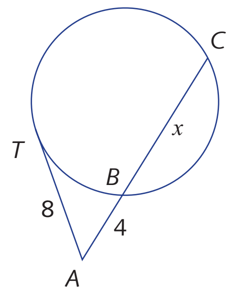

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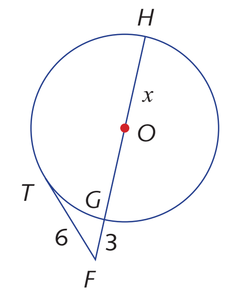

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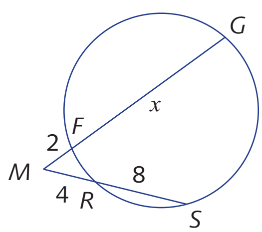

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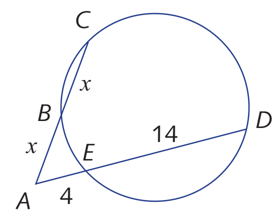

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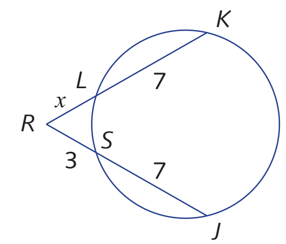

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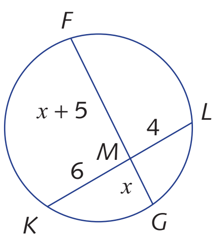

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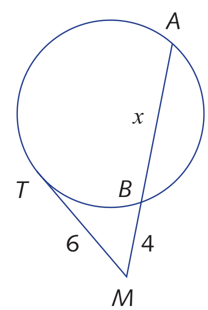

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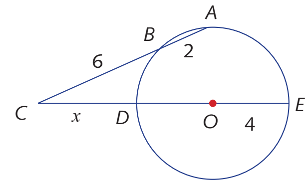
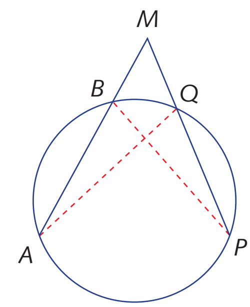


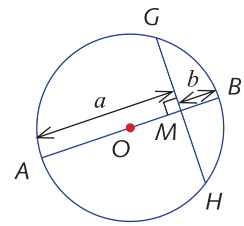
2 Let secants from a point, M, external to the circle cut the circle at points A and B and P and Q, as shown in the diagram. Prove that $AM \times BM = PM \times QM$.



This is the proof of the theorem stated on page 410.

3 Prove the result of Question **2** by drawing a tangent MT and using the 'tangent and secant' theorem.

4 Let AOB be a diameter of a circle, and let GH be a chord perpendicular to AB, meeting AB at M.



a Why is M the midpoint of GH?

b Let $g = GM$, $a = AM$ and $b = BM$. Prove that $g^2 = ab$.

c Explain why the radius of the circle is $\frac{a+b}{2}$.

d Prove that $\sqrt{ab} \le \frac{a+b}{2}$.

This is the well known Arithmetic mean–Geometric mean inequality.

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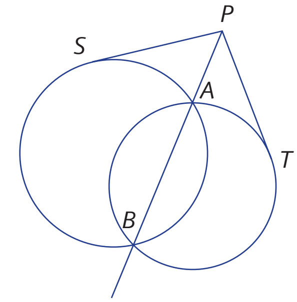
5 Let P be a point on the common secant AB of two intersecting circles. Let PS and PT be tangents from P, one to each circle. Prove that $PS = PT$.



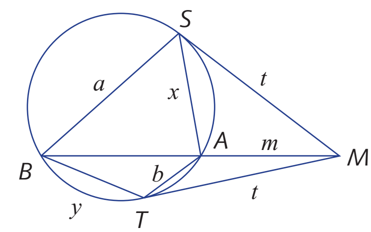
6 In the diagram, MS and MT are tangents from an external point, M.

a Prove that ΔMSA is similar to ΔMBS.

b Hence, prove that $\frac{a}{x} = \frac{t}{m}$.

c Similarly, prove that $\frac{y}{b} = \frac{t}{m}$.

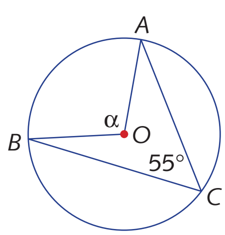
d Hence, prove that $ab = xy$.



Review exercise

1 Find the values of the pronumerals.

a



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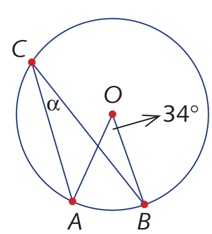

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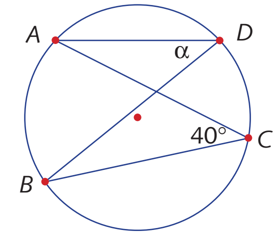

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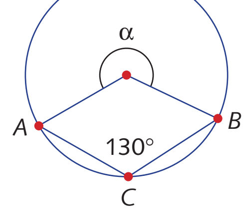

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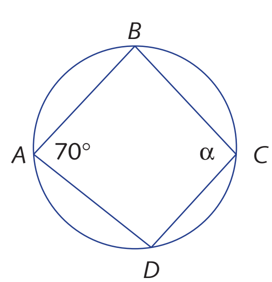

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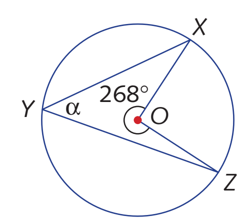
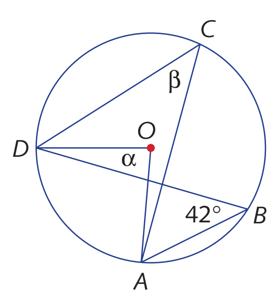


2 Find the values of the pronumerals.

a



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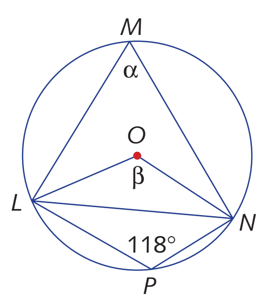

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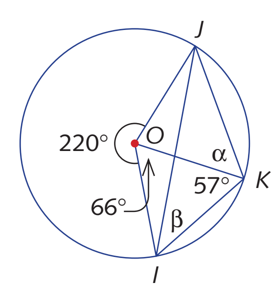
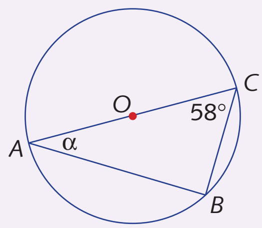


3 Find the values of the pronumerals.

a



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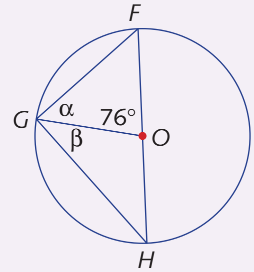

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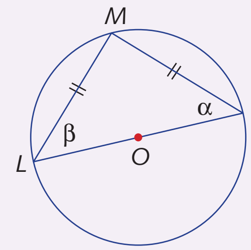


4 $ABCD$ is a cyclic quadrilateral. Its diagonals AC and BD intersect at P. Prove that ΔAPD is similar to ΔBPC.

5 The quadrilateral $ABCD$ has its vertices on a circle with centre O. The diagonal AB is a diameter of the circle and $AC = BD$. Prove that $AD = BC$.

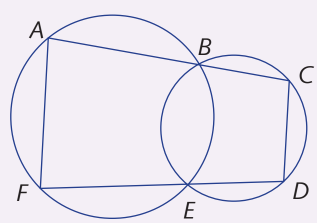
6 $ABCD$ is a cyclic quadrilateral with AD parallel to BC. The diagonals AC and BD intersect at P. Prove that $\angle APB = 2\angle ACB$.

7 $ABCD$ is a cyclic quadrilateral. Chord AB is produced and a point E is marked on the line AB so that B is between A and E. Prove that $\angle EBC = \angle ADC$.

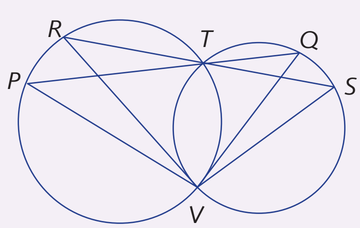
8 $PQRS$ is a cyclic quadrilateral. The diagonal PR bisects both $\angle SPQ$ and $\angle SRQ$. Prove that $\angle PQR$ is a right angle.

9 In the diagram opposite, the two circles intersect at B and E.

Prove that AF is parallel to CD.



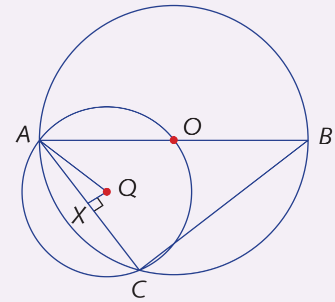
10 Two circles intersect at T and V. The intervals PTQ and RTS are drawn as shown. Prove that $\angle PVR = \angle QVS$.



11 In the figure, AOB is the diameter of the circle ABC with centre O. The point Q is the centre of another circle that passes through the points A, O and C, and $QX \perp AC$.

a Prove that $\angle AQX = 2\angle ABC$.

b Show that $AB^2 = BC^2 + 4\left(AQ^2 - XQ^2\right)$.



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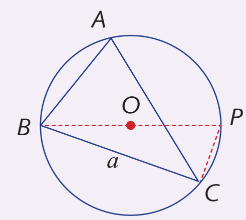
Challenge exercise

1 Here is a form of the sine rule that shows its connection with the circumcircle of a triangle.

In any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$,

where R is the radius of the circumcircle of triangle ABC. Assume that A is acute.

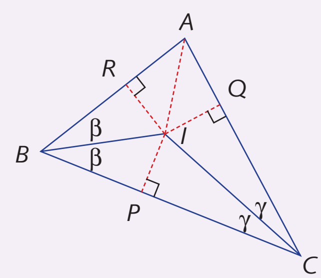


Draw the circumcircle of ΔABC, and let O be the circumcentre.

a Hence, prove that $\dfrac{a}{\sin A} = 2R$.

b Prove the result when $\angle A$ is obtuse.

2 Prove that: *In any triangle ABC, the bisectors of the vertex angles are concurrent, and the resulting* **incentre** *is the centre of a circle that touches all sides of the triangle.*



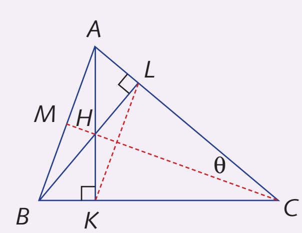
Let the angle bisectors of $\angle B$ and $\angle C$ meet at I, and draw IA. Draw the perpendiculars IP, IQ and IR to the sides BC, CA and AB, respectively.

a Use congruence to prove that $IR = IP$, and that $IP = IQ$.

b Use congruence to prove that IA bisects $\angle A$.

c Why does the circle with centre I and radius IR touch all three sides of the triangle?

3 Prove that: *In any triangle ABC, the altitudes are concurrent (their intersection is called the* **orthocentre** *of the triangle).*



Let the altitudes AK and BL meet at H. Draw CH and produce it to meet AB at M.

a Prove that C, K, H and L are concyclic.

b Let $\angle ACM = \theta$. Prove that $\angle AKL = \theta$.

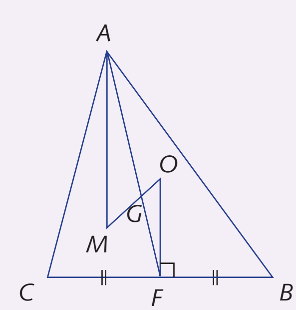
c Prove that B, K, L and A are concyclic, and hence prove that $\angle ABL = \theta$.

d Hence, prove that CM is an altitude of the triangle. That is, we have proved that the three altitudes are concurrent.

4 Euler discovered a wonderful theorem that the Greeks had missed:

The orthocentre, the centroid and the circumcentre of a triangle are collinear, with the centroid dividing the interval joining the orthocentre and circumcentre in the ratio 2 : 1.

(This line is called the **Euler line**.)



Note: The **centroid** is the point of intersection of the **medians** of a triangle, which are the lines drawn from any vertex of a triangle to the midpoint of the opposite side.

Let O and G be the circumcentre and centroid, respectively, of ΔABC. Draw OG and produce it to a point, M, such that $OG : GM = 1 : 2$.

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a Let F be the midpoint of BC. Use the fact that the centroid, G, divides the median AF in the ratio 2 : 1 to prove that ΔGOF is similar to ΔGMA.

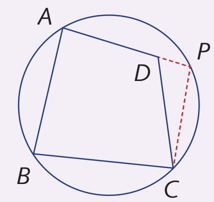
b Hence, prove that M lies on the altitude from A.

c Show that point M is the point H constructed in Question **3**.

5 Prove the following converse of the cyclic quadilateral theorem:

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Let the opposite angles of the quadrilateral $ABCD$ be supplementary. Draw the circle through the points A, B and C. Let AD, produced if necessary, meet the circle at P, and draw PC.



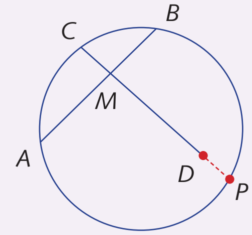
a Prove that $\angle P = \angle D$.

b Hence, prove that the points P and D coincide.

6 Prove the following converse of the intersecting chords theorem:

Suppose that two intervals AB and CD intersect at M, and that $AM \times BM = CM \times DM$. Then the points A, B, C and D are concyclic.

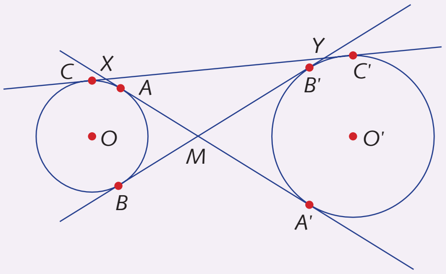
Draw the circle through the points A, B and C. Let CD, produced if necessary, meet the circle at P.



a Prove that $PM \times CM = AM \times BM$.

b Hence, prove that the points P and D coincide.

7 Take two non-intersecting circles in the plane with centres O and O'. Draw two indirect common tangents AA' and BB', and one direct tangent CC', where A, B and C lie on the first circle, and A', B' and C' lie on the second circle. Produce AA' and BB' to meet CC' at X and Y.



a Prove that $AA' = BB'$.

b Prove that $AA' = XY$.

c Describe what happens when the two circles are touching each other externally.

8 Two circles intersect at A and B. A straight line passing through A meets the two circles respectively at C and D.

a Show that any two triangles CBD formed in this way are similar.

b Which of these triangles has the larger area?

9 Two circles touch externally at P, and a common tangent touches them at A and B. Let the common tangent at P meet AB at C.

a Show that C is the midpoint of AB.

b A line passing through P meets the two circles at D and E. Draw the tangents to each circle at D and at E. Show that the tangents are parallel.

10 If ΔABC has side lengths a, b and c, prove that:

$$\frac{2 \times (\text{Area of } \Delta ABC)}{abc} = \frac{\sin\ C}{c} = \frac{\sin\ A}{a} = \frac{\sin\ B}{b}$$

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