CHAPTER

14

Number and Algebra

Indices, exponentials and logarithms – part 2

In Chapter 9, starting with integer powers of numbers, we developed the ideas of the exponential function and the logarithmic function. We learned basic properties, such as:

$$2^x 2^y = 2^{x+y} \quad \text{and} \quad \log_2 (xy) = \log_2 x + \log_2 y$$

In this chapter, we will investigate the change of base formula and meet a range of new applications, especially applications to science.

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14A Logarithm rules

In Section 9G, we introduced logarithms. Logarithms are closely related to indices. Recall that the logarithm of a number to base a is the **index** to which a is raised to give this number. For example:

$3^4 = 81$ is equivalent to $\log_3 81 = 4$

$10^6 = 1\,000\,000$ is equivalent to $\log_{10} 1\,000\,000 = 6$

$5^{-3} = \frac{1}{125}$ is equivalent to $\log_5 \frac{1}{125} = -3$

$16^{\frac{3}{4}} = 8$ is equivalent to $\log_{16} 8 = \frac{3}{4}$

In general, the **logarithmic function** is defined as follows:

If $a > 0$, $a \neq 1$ and $y = a^x$, then $\log_a y = x$

Logarithms obey a number of important laws. Each one comes from a property of indices.

Index laws

If a and b are positive numbers and x and y are rational numbers, then:

Index law 1 $a^x a^y = a^{x+y}$ Index law 2 $\frac{a^x}{a^y} = a^{x-y}$

Index law 3 $(a^x)^y = a^{xy}$ Index law 4 $(ab)^x = a^x b^x$

Index law 5 $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

The first three index laws have a direct correspondence to the first three logarithmic laws, which are developed below.

Suppose $a > 0$ and $a \neq 1$ for the rest of this section.

Logarithmic Law 1 If x and y are positive numbers, then $\log_a xy = \log_a x + \log_a y$. That is, the logarithm of a product is the sum of the logarithms.

Suppose that $\log_a x = c$ and $\log_a y = d$

That is, $x = a^c$ and $y = a^d$

Then $xy = a^c \times a^d$

$= a^{c+d}$ (by Index law 1)

So $\log_a xy = \log_a a^{c+d}$

$= c + d$

$= \log_a x + \log_a y$

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Logarithmic Law 2 If x and y are positive numbers, then $\log_a \frac{x}{y} = \log_a x - \log_a y$.
That is, the logarithm of a quotient is the difference of their logarithms.

Suppose that $\log_a x = c$ and $\log_a y = d$

That is, $x = a^c$ and $y = a^d$

Then $\frac{x}{y} = \frac{a^c}{a^d}$

$= a^{c-d}$ (by Index law 2)

So $\log_a \frac{x}{y} = \log_a a^{c-d}$

$= c - d$

$= \log_a x - \log_a y$

Logarithmic Law 3 If x is a positive number and n is any rational number, then $\log_a(x^n) = n\log_a x$.

This follows from index law 3. Suppose that $\log_a x = c$. That is, $x = a^c$.

Then $x^n = \left(a^c\right)^n = a^{cn}$ (by Index law 3)

So $\log_a(x^n) = \log_a(a^{cn})$

Hence, $\log_a(x^n) = cn$

$= n \log_a x$, as required

Logarithmic Law 4 If x is a positive number, then $\log_a \frac{1}{x} = -\log_a x$.

This follows from logarithm law 3.

$\log_a \frac{1}{x} = \log_a x^{-1}$ (definition)

$= -\log_a x$ (logarithm law 3)

Logarithmic Law 5 $\log_a 1 = 0$ and $\log_a a = 1$

Let the base a be a positive number, with $a \neq 1$.

Since $a^0 = 1$, we have $\log_a 1 = 0$.

Similarly, since $a^1 = a$, we have $\log_a a = 1$.

Example 1

Write each statement in logarithmic form.

a $2^4 = 16$ **b** $5^3 = 125$ **c** $10^{-3} = 0.001$ **d** $2^{-4} = \frac{1}{16}$

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Solution

a $2^4 = 16$ so $\log_2 16 = 4$

b $5^3 = 125$ so $\log_5 125 = 3$

c $10^{-3} = 0.001$ so $\log_{10} 0.001 = -3$

d $2^{-4} = \frac{1}{16}$ so $\log_2 \frac{1}{16} = -4$

Example 2

Evaluate each logarithm.

a $\log_2 256$

b $\log_2 \sqrt[3]{2}$

c $\log_3 81$

d $\log_9 81$

e $\log_5 \frac{1}{5}$

f $\log_7 \frac{1}{49}$

Solution

Method 1

a $256 = 2^8$, so $\log_2 256 = 8$

b $\sqrt[3]{2} = 2^{\frac{1}{3}}$, so $\log_2 \sqrt[3]{2} = \frac{1}{3}$

c $81 = 3^4$, so $\log_3 81 = 4$

d $81 = 9^2$, so $\log_9 81 = 2$

e $\log_5 \frac{1}{5} = \log_5 5^{-1}$

$= -1$

f $\log_7 \frac{1}{49} = \log_7 7^{-2}$

$= -2$

Method 2

The following method introduces a pronumeral x.

a Let $x = \log_2 256$

so $2^x = 256 = 2^8$

$x = 8$

b Let $x = \log_2 \sqrt[3]{2}$

so $2^x = \sqrt[3]{2} = 2^{\frac{1}{3}}$

$x = \frac{1}{3}$

c Let $x = \log_3 81$

so $3^x = 81 = 3^4$

$x = 4$

d Let $x = \log_9 81$

so $9^x = 81 = 9^2$

$x = 2$

Example 3

Solve each logarithmic equation.

a $\log_2 x = 5$

b $\log_7 (x - 1) = 2$

c $\log_x 64 = 6$

d $\log_x \frac{1}{25} = -2$

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Solution

a $\log_2 x = 5$

so $x = 2^5$

$= 32$

b $\log_7 (x - 1) = 2$

so $x - 1 = 7^2$

$= 49$

$x = 50$

c $\log_x 64 = 6$

so $x^6 = 64$

$x^6 = 2^6$

$x = 2$, since $x > 0$

d $\log_x \frac{1}{25} = -2$

so $x^{-2} = \frac{1}{25}$

$x^2 = 25$

$x = 5$, since $x > 0$

Example 4

Write each statement in logarithmic form.

a $y = b^x$ **b** $a^x = N$ **c** $7^0 = 1$ **d** $3\sqrt{3} = 3^{\frac{3}{2}}$

Solution

a $y = b^x$ becomes $x = \log_b y$

b $a^x = N$ becomes $x = \log_a N$

c $7^0 = 1$ becomes $\log_7 1 = 0$

d $3\sqrt{3} = 3^{\frac{3}{2}}$ becomes $\log_3 3\sqrt{3} = \frac{3}{2}$

Example 5

Given $\log_7 2 = \alpha$, $\log_7 3 = \beta$ and $\log_7 5 = \gamma$, express each in terms of α, β and γ.

a $\log_7 6$ **b** $\log_7 75$ **c** $\log_7 \frac{15}{2}$

Solution

a $\log_7 6 = \log_7 (2 \times 3)$

$= \log_7 2 + \log_7 3$

$= \alpha + \beta$

b $\log_7 75 = \log_7 (3 \times 25)$

$= \log_7 3 + \log_7 5^2$

$= \log_7 3 + 2 \log_7 5$

$= \beta + 2\gamma$

c $\log_7 \frac{15}{2} = \log_7 15 - \log_7 2$

$= \log_7 (3 \times 5) - \log_7 2$

$= \log_7 3 + \log_7 5 - \log_7 2$

$= \beta + \gamma - \alpha$

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Exercise 14A

Example 2

1 Calculate each logarithm.

a $\log_2 8$ **b** $\log_3 27$ **c** $\log_2 2048$ **d** $\log_7 1$

e $\log_5 625$ **f** $\log_7 343$ **g** $\log_{10} 10\,000$ **h** $\log_{10} 1\,000\,000$

2 Calculate:

a $\log_2 \frac{1}{16}$ **b** $\log_3 \frac{1}{27}$ **c** $\log_{10} \frac{1}{10}$ **d** $\log_{10} 0.01$

e $\log_5 \frac{1}{125}$ **f** $\log_6 \frac{1}{36}$ **g** $\log_2 \frac{1}{1024}$ **h** $\log_{10} 0.0001$

3 Evaluate:

a $\log_2 2\sqrt{2}$ **b** $\log_3 9\sqrt{3}$ **c** $\log_6 36\sqrt{6}$ **d** $\log_2 4\sqrt{2}$

e $\log_3(27\sqrt{3})$ **f** $\log_{10}\left(\frac{1}{100\sqrt{10}}\right)$ **g** $\log_5(5^2 \times \sqrt[3]{5})$ **h** $\log_8 \sqrt{2}$

Example 3a, b

4 Solve each equation for x.

a $\log_2 x = 5$ **b** $\log_3 x = 6$ **c** $\log_{10} x = 3$

d $\log_{10} x = -3$ **e** $\log_{10} x = -4$ **f** $\log_5 x = 4$

g $\log_2(x - 3) = 1$ **h** $\log_2(x + 4) = 6$ **i** $\log_2(x - 5) = 3$

Example 3c, d

5 Solve each equation.

a $\log_x 81 = 2$ **b** $\log_x 8 = 6$ **c** $\log_x 1024 = 5$

d $\log_x 1024 = 10$ **e** $\log_x 9 = 2$ **f** $\log_x 1000 = 3$

Example 1, 4

6 Write each statement in logarithmic form.

a $2 = \left(\sqrt{2}\right)^2$ **b** $0.001 = 10^{-3}$ **c** $\left(\frac{1}{2}\right)^{-1} = 2$ **d** $1024 = 32^2$

e $10^x = N$ **f** $5\sqrt{2} = 5^{\frac{3}{2}}$ **g** $5^0 = 1$ **h** $13^1 = 13$

7 Write each statement in exponential form.

a $\log_2 32 = 5$ **b** $\log_3 81 = 4$ **c** $\log_{10} 0.001 = -3$

d $\log_3 27\sqrt{3} = \frac{7}{2}$ **e** $\log_b y = x$ **f** $\log_a N = x$

8 Simplify:

a $\log_3 7 + \log_3 5$ **b** $\log_2 3 + \log_2 5$ **c** $\log_2 9 + \log_2 7$

d $\log_{10} 5 + \log_{10} 20$ **e** $\log_6 4 + \log_6 9$ **f** $\log_3 7 + \log_3 \frac{1}{7}$

9 Simplify:

a $\log_3 100 - \log_3 10$ **b** $\log_7 20 - \log_7 10$ **c** $\log_7 21 - \log_7 3$

d $\log_3 17 - \log_3 51$ **e** $\log_5 100 - \log_5 10$ **f** $\log_5 10 - \log_5 2$

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10 Simplify:

a $\log_2 3 + \log_2 5 + \log_2 7$

b $\log_3 100 - \log_3 10 - \log_3 2$

c $\log_5 7 + \log_5 343 - 2\log_5 49$

d $\log_7 25 + \log_7 3 - \log_7 75$

Example 5

11 Given that $\log_{10} 2 = \alpha$, $\log_{10} 3 = \beta$, $\log_{10} 5 = \gamma$ and $\log_{10} 7 = \delta$, express in terms of α, β, γ and δ:

a $\log_{10} 12$

b $\log_{10} 75$

c $\log_{10} 210$

d $\log_{10} 6\,000\,000$

e $\log_{10} 1875$

f $\log_{10} 1050$

g $\log_{10}(2^a 3^b 5^c 7^d)$

h What does $\alpha + \gamma$ equal?

12 Find a relation between x and y that does not involve logarithms.

a $\log_3 x + \log_3 y = \log_3(x + y)$

b $2\log_{10} x - 3\log_{10} y = -1$

c $\log_5 y = 3 + 2\log_5 x$

d $\log_7(1 + y) - \log_7(1 - y) = x$

13 $V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r. Express $\log_2 V$ in terms of $\log_2 r$.

14 If $y = a \times 10^{bx}$, express x in terms of the other pronumerals.

15 Solve $\log_{10} A = bt + \log_{10} P$ for A.

14B Change of base

In Section 14A we studied logarithms to one base (which was a positive number other than 1) and their relationships, such as:

$$\log_a x + \log_a y = \log_a xy$$

Often we need to work with different bases and, in particular, calculate quantities such as $\log_5 8$, which is clearly between 1 and 2. It is of immediate concern that some calculators do not have the capacity to calculate $\log_5 8$ directly, but they can calculate $\log_{10} 8$ and $\log_{10} 5$.

We will show that $\log_5 8 = \dfrac{\log_{10} 8}{\log_{10} 5} \approx 1.2920$.

This is a special case of the **change of base formula**:

$$\log_b c = \frac{\log_a c}{\log_a b}$$

where a, b and c are positive numbers, $a \neq 1$ and $b \neq 1$.

The change of base formula is very important in later mathematics.

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Proof 1

Let $x = \log_b c$ so, $b^x = c$

Taking logarithms to base a of both sides:

$\log_a b^x = \log_a c$

$x \log_a b = \log_a c$ (Logarithm law 3)

$$x = \frac{\log_a c}{\log_a b}$$

That is, $\log_b c = \dfrac{\log_a c}{\log_a b}$

Proof 2

If $\log_a b = e$, then $a^e = b$

Similarly, if $\log_b c = f$, then $b^f = c$

Hence, $c = b^f = (a^e)^f = a^{ef}$

So $\log_a c = ef = \log_a b \times \log_b c$

and $\log_b c = \dfrac{\log_a c}{\log_a b}$

Change of base formula

- If a, b and c are positive numbers, $a \neq 1$ and $b \neq 1$ then:

  $\log_b c = \dfrac{\log_a c}{\log_a b}$

- This formula can also be written as:

  $\log_a c = \log_a b \times \log_b c$

These formulas are called 'change of base' formulas, since they allow the calculation of logarithms to the base b from knowledge of logarithms to the base a.

Example 6

By changing to base 2, calculate $\log_{16} 8$.

Solution

$\log_2 8 = 3$ and $\log_2 16 = 4$,

$$\text{hence, } \log_{16} 8 = \frac{\log_2 8}{\log_2 16} = \frac{3}{4}$$

So $\log_{16} 8 = \dfrac{3}{4}$

As a check, $16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$

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Example 7

Calculate $\log_7 8$, correct to four decimal places, using base 10 logarithms

Solution

Changing from base 7 to base 10:

$$\log_7 8 = \frac{\log_{10} 8}{\log_{10} 7}$$
$$\approx 1.0686$$

As a check, $7^{1.0686} \approx 7.9997$ with a calculator.

Example 8

If $3^x = 7$, calculate x, correct to four decimal places.

Solution

$$x = \log_3 7 = \frac{\log_{10} 7}{\log_{10} 3}$$
$$\approx 1.7712$$

Example 9

Suppose that $a > 0$. Find the exact value of $\log_{a^2} a^3$.

Solution

$$\log_{a^2} a^3 = \frac{\log_a a^3}{\log_a a^2}$$
$$= \frac{3\log_a a}{2\log_a a}$$
$$= \frac{3}{2}$$

As a check, $(a^2)^{\frac{3}{2}} = a^3$

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Exercise 14B

In this exercise, a, b and c are positive and not equal to 1.

Example 6

1 **a** By changing to base 3, calculate $\log_9 243$.

b By changing to base 2, calculate $\log_8 32$.

Example 7

2 Use the change of base formula to convert to base 10 and calculate these logorithms, correct to four decimal places.

a $\log_7 9$ **b** $\log_5 3$ **c** $\log_3 5$

d $\log_3 13$ **e** $\log_{19} 17$ **f** $\log_7 \frac{1}{4}$

Example 8

3 Solve for x, correct to four decimal places.

a $2^x = 5$ **b** $3^x = 18$ **c** $5^x = 2$

d $5^x = 17$ **e** $2^{-x} = 7$ **f** $3^{-x} = 5$

4 Solve for x, correct to four decimal places.

a $(0.01)^x = 7$ **b** $5^{1-2x} = 3$ **c** $4^{2x-1} = 7^{x-3}$ **d** $3^{3x-3} = 5^{5x-5}$

5 Simplify:

a $(\log_a b)(\log_b a)$ **b** $(\log_a b)(\log_b c)(\log_c a)$

Example 9

6 Change to base a and simplify.

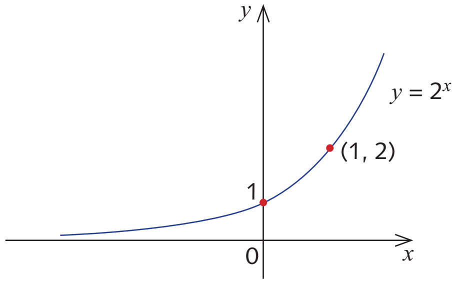
a $\log_{a^2} a^3$ **b** $\log_{a^2} a^7$ **c** $\log_{a^3} a^5$ **d** $\log_{\sqrt[3]{a}} \sqrt[11]{a}$

e $\log_a a^8 - \log_a a^7 + \log_a a^{11}$ **f** $\log_{\sqrt{a}} \sqrt[3]{a} + \log_{\sqrt[3]{a}} \sqrt[4]{a} + \log_{\sqrt[4]{a}} \sqrt[5]{a}$

14C Graphs of exponential and logarithm functions

We saw the basic shape of the graph of an exponential function in Chapter 9.

For example, $y = 2^x$ is graphed to the right.



The graph has the following features:

- The y-intercept is 1.
- There is no x-intercept.
- The y-values are always positive.
- As x takes large positive values, 2^x becomes very large.
- As x takes large negative values, 2^x becomes very small.
- The x-axis is an asymptote to the graph.

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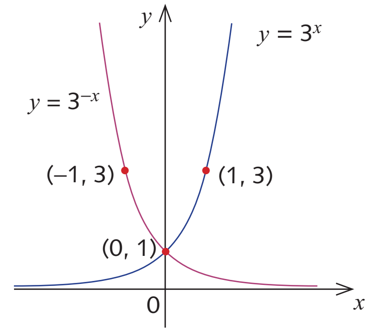
Here are the graphs of $y = 3^x$ and $y = 3^{-x}$ drawn on the same axes.

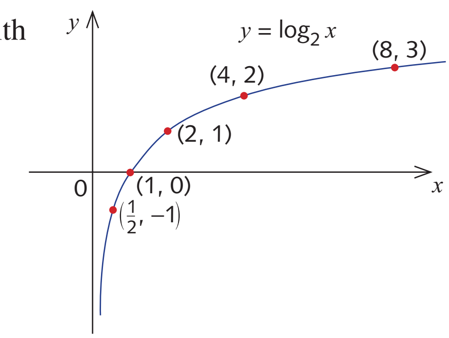
Notice that $y = 3^x$ is the reflection of $y = 3^{-x}$ in the y-axis.



Simple logarithm graphs

We can also draw the graph of $y = \log_2 x$. As usual, we begin with a table of values.

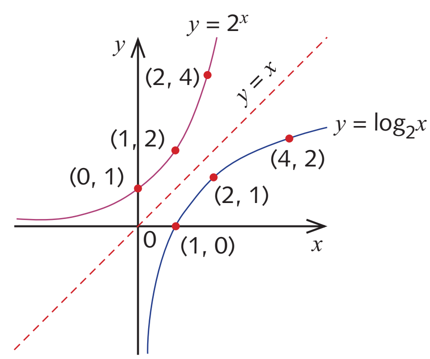
x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$y = \log_2 x$	−4	−3	−2	−1	0	1	2	3	4



How are the graphs of $y = \log_2 x$ and $y = 2^x$ related?

Here is a table of values of $y = 2^x$. The graphs of $y = 2^x$ and $y = \log_2 x$ are shown on the one set of axes.

x	−4	−3	−2	−1	0	1	2	3	4
$y = 2^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



If the point (a, b) lies on $y = 2^x$, then $b = 2^a$.

Hence, we can write $a = \log_2 b$, so (b, a) lies on the graph of $y = \log_2 x$.

Thus, each point on $y = \log_2 x$ can be obtained by taking a point on $y = 2^x$ and interchanging the x and y values.

The midpoint of (a, b) and (b, a) is $\left(\frac{a+b}{2}, \frac{b+a}{2}\right)$, and thus always lies on the line $y = x$.

Graphically this means (a, b) is the reflection of (b, a) in the line $y = x$ and vice versa. This is evident in the above pair of graphs.

From this we can list some of the features of the graph of $y = \log_2 x$.

- The graph is to the right of the y-axis. (This is because the function is only defined for $x > 0$.)
- The y-axis is a vertical asymptote to the graph.
- The x-intercept is $(1, 0)$, corresponding to $\log_2 1 = 0$.
- The graph does not have a y-intercept.
- As x takes very large positive values, $\log_2 x$ becomes large positive.
- As x takes very small positive values, $\log_2 x$ becomes large negative.
- The graph is a reflection of $y = 2^x$ in the line $y = x$.

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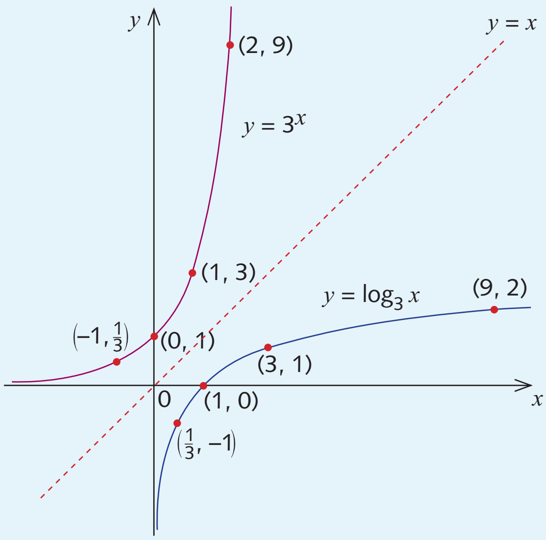
Example 10

Use the graph of $y = 3^x$ to assist in sketching $y = \log_3 x$.

Solution

First draw the graph of $y = 3^x$.



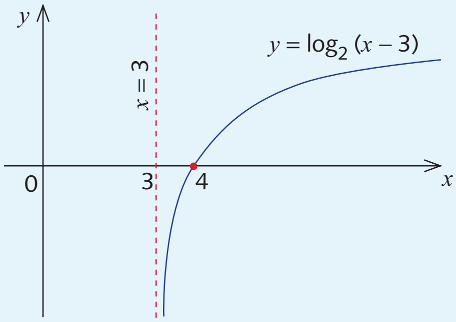
The two graphs are reflections of each other in the line $y = x$.

Example 11

Sketch the graph of $y = \log_2(x - 3)$.

Solution

Translate the graph of $y = \log_2 x$ three units to the right.



Note that the line $x = 3$ is an asymptote to the graph.

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Example 12

Sketch the graphs of $y = \log_3 x$ and $y = \log_5 x$ on the same set of axes.

Solution

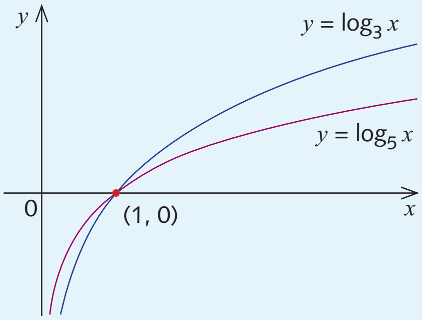
x	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25
$y = \log_3 x$	–2.93	–1.46	0	1.46	2.93
$y = \log_5 x$	–2	–1	0	1	2

$\log_3 5 = \dfrac{\log_{10} 5}{\log_{10} 3} \approx 1.46$

$\log_3 25 = \log_3 5^2 = 2\log_3 5 \approx 2.93$

$\log_3 \dfrac{1}{5} = -\log_3 5 \approx -1.46$

$\log_3 \dfrac{1}{25} = \log_3 5^{-2} = -2\log_3 5 \approx -2.93$



The table of values shows that:

$\log_3 x > \log_5 x$ if $x > 1$ and $\log_5 x > \log_3 x$ if $0 < x < 1$

Exercise 14C

Example 10

1 **a** Use the graph of $y = 4^x$ to draw the graph of $y = \log_4 x$.

b Use the graph of $y = 5^x$ to draw the graph of $y = \log_5 x$.

2 For each of these logarithm functions, produce a table of values for (x, y), using the following y-values: –2, –1, 0, 1, 2. Use the table to draw the graph of the function.

a $y = \log_{10} x$ **b** $y = \log_6 x$

3 Draw each set of graphs on the same axes.

a $y = 3^x$, $y = 3^x + 1$, $y = 3^x - 2$ **b** $y = 5^x$, $y = 2 \times 5^x$, $y = \frac{1}{2} \times 5^x$

c $y = 2^x$, $y = 2^{-x}$ **d** $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{2}\right)^{-x}$

Example 11

4 **a** Sketch the graphs of $y = \log_2 x$ and $y = \log_3 x$ on the same set of axes, for y values between –3 and 3.

b In what ways are the graphs similar?

c How do the graphs differ?

d Without using a table of values, sketch the graph of $y = \log_4 x$ on the same set of axes used in part **a**.

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Example 12

5 Sketch the following graphs.

a $y = \log_3 x,\ x > 0$

b $y = \log_3(x - 1),\ x > 1$

c $y = \log_3(x + 5),\ x > -5$

d $y = 2\log_3 x,\ x > 0$

e $y = \log_3(x) + 2,\ x > 0$

6 Sketch $y = 2^x$, $y = 3^x$, $y = \log_2 x$ and $y = \log_3 x$ on the one set of axes.

14D Applications to science, population growth and finance

In Section 9F of you saw that in a given experiment, the growth in bacteria could be described using an **exponential function**, such as $N = 1000 \times 2^t$.

Here, N is the number of bacteria at time t, measured in hours.

Equations of this type arise in many practical situations in which we know the value of N, but want to solve for t.

Logarithms are needed for such calculations.

Example 13

Initially there are 1000 bacteria in a given culture. The number of bacteria, N, is doubling every hour, so $N = 1000 \times 2^t$, where t is measured in hours.

a How many bacteria are present after 24 hours? Give your answer correct to three significant figures.

b How long is it until there are one million bacteria? Give your answer correct to three significant figures.

Solution

a After 24 hours, $N = 1000 \times 2^{24}$

$$\approx 1.68 \times 10^{10}$$

b If $N = 10^6$, then $10^6 = 1000 \times 2^t$

$$\begin{aligned} 2^t &= 1000 \\ \log_{10} 2^t &= \log_{10} 1000 \\ t \log_{10} 2 &= 3 \\ t &= \frac{3}{\log_{10} 2} \\ &\approx 9.97 \text{ hours} \end{aligned}$$

There are one million bacteria after approximately 9.97 hours.

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The following example illustrates the use of logarithms in estimating the age of fossils.

Example 14

The carbon isotope carbon-14, C^{14}, occurs naturally but decays with time. Measurements of carbon-14 in fossils are used to estimate the age of samples.

If M is the mass of carbon-14 at time t years and M_0 is the mass at time $t = 0$, then $M = M_0 10^{-kt}$ where $k = 5.404\,488\,252 \times 10^{-5}$.

All 10 digits are needed to achieve reasonable accuracy in these calculations.

a Calculate the fraction left after 100 years as a percentage.

b Calculate the fraction left after 10 000 years as a percentage.

c Calculate the half-life of C^{14}. That is, after how long does $M = \frac{1}{2}M_0$?

Solution

a When $t = 100$,

$$\begin{aligned} M &= M_0 10^{-100k} \\ \frac{M}{M_0} &= 10^{-100k} \\ &\approx 0.987\,63 \\ &\approx 98.76\% \end{aligned}$$

That is, the fraction left after 100 years is 98.76%.

b When $t = 10\,000$,

$$\begin{aligned} \frac{M}{M_0} &= 10^{-10\,000k} \\ &\approx 0.288\,11 \\ &\approx 28.81\% \end{aligned}$$

That is, the fraction left after 10 000 years is 28.81%.

c $M = \frac{1}{2}M_0$ when

$$\begin{aligned} \frac{1}{2} &= 10^{-kt} \\ \log_{10}\frac{1}{2} &= -kt \\ kt &= \log_{10} 2 \\ t &= \frac{\log_{10} 2}{k} \\ &\approx 5570.000\,001 \\ &\approx 5570 \text{ years} \end{aligned}$$

That is, the half-life of C^{14} is about 5570 years.

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Compound interest

In Section 1D, we introduced the compound interest formula:

$$A_n = P(1 + R)^n$$

where A_n is the amount that the investment is worth after n units of time, P is the principal and R is the interest rate.

Logarithms can be used to find the value of n in this formula given R, P and A_n.

Example 15

\$50 000 is invested on 1 Jan at 8% per annum. Interest is only paid on 1 Jan of each year. At the end of how many years will the investment be worth

a \$75 000?

b \$100 000?

Solution

a $A_n = P(1 + R)^n$

$A_n = 75\,000,\ P = 50\,000$ and $R = 0.08$, so

$$75\,000 = 50\,000(1.08)^n$$

$$\frac{3}{2} = (1.08)^n$$

$$\log_{10}\frac{3}{2} = n\log_{10}(1.08) \quad \text{(Take logarithms of both sides.)}$$

$$n = \frac{\log_{10}\left(\frac{3}{2}\right)}{\log_{10}(1.08)}$$

$$= 5.268\,44\ldots$$

At the end of the sixth year, the investment will be worth $\$50\,000(1.08)^6 = \$79\,343.72$.

At the end of the fifth year, the investment will be worth $\$50\,000(1.08)^5 = \$73\,466.40$.

The investment will be worth more than \$75 000 at the end of the sixth year.

b $A_n = P(1 + R)^n$

$A_n = 100\,000,\ P = 50\,000$ and $R = 0.08$, so

$$100\,000 = 50\,000(1.08)^n$$

$$2 = (1.08)^n$$

$$\log_{10}(2) = n\log_{10}(1.08) \quad \text{(Take logarithms of both sides.)}$$

$$n = \frac{\log_{10}(2)}{\log_{10}(1.08)}$$

$$= 9.006\,46\ldots$$

At the end of the tenth year, the investment will be worth $\$50\,000(1.08)^{10} = \$107\,946.25$.

At the end of the ninth year, the investment will be worth $\$50\,000(1.08)^9 = \$99\,950.23$.

The investment will be worth more than \$100 000 at the end of the tenth year.

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Exercise 14D

Example 13

1 A culture of bacteria initially has a mass of 3 grams and its mass doubles in size every hour. How long will it take to reach a mass of 60 grams?

2 A culture of bacteria initially weighs 0.72 grams and is multiplying in size by a factor of five every day.

a Write down a formula for M, the weight of bacteria in grams after t days.

b What is the weight after two days?

c How long will the culture take to double its weight?

d The mass of the Earth is about 5.972×10^{24} kg. After how many days will the culture weigh the same as the Earth?

e Discuss your answer to part **d**.

3 The population of the Earth at the beginning of 1976 was four billion. Assume that the rate of growth is 2% per year.

a Write a formula for P, the population of the Earth in year t, $t \geq 1976$.

b What will be the population in 2076?

c When will the population reach 10 billion?

4 The population of the People's Republic of China in 1970 was 750 million. Assume that its rate of growth is 4% per annum.

a Write down a formula for C, the population of China in year t, $t \geq 1970$.

b When would the population of China reach two billion?

c With the assumptions of question **3**, when would the population of China be equal to half the population of the Earth?

d When would everyone in the world be Chinese? (Discuss your answer.)

Example 14

5 The mass M of a radioactive substance is initially 10 g and 20 years later its mass is 9.6 g. If the relationship between M grams and t years is of the from $M = M_o 10^{-kt}$, find:

a M_o and k

b the half-life of the radioactive substance

Example 15

6 An amount of $80 000 is invested on 1 Jan at a compound interest rate of 7% per annum. Interest is only paid on 1 Jan of each year. At the end of how many years will the investment be worth:

a $110 000 ?

b $200 000?

7 A man now owes the bank $47 000, after taking out a loan n years ago with an interest rate of 10% per annum. He borrowed $26 530. Find n.

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8 The formula for the calculation of compound interest is $A_n = P(1 + R)^n$. Find, correct to one decimal place:

a A_n if $P = \$50\ 000$, $R = 8\%$ and $n = 3$

b P if $A_n = \$80\ 000$, $R = 5\%$ and $n = 4$

c n if $A_n = \$60\ 000$, $R = 2\%$ and $P = \$20\ 000$

d n if $A_n = \$90\ 000$, $R = 4\%$ and $P = \$20\ 000$

Review exercise

1 Calculate each logarithm.

a $\log_2 16$ b $\log_5 125$ c $\log_2 512$ d $\log_7 1$

e $\log_3 \frac{1}{27}$ f $\log_2 \frac{1}{64}$ g $\log_{10} 10\ 000$ h $\log_{10} (0.001)$

2 Solve each logarithmic equation.

a $\log_x 16 = 2$ b $\log_x 64 = 6$

c $\log_x 2048 = 11$ d $\log_x 512 = 3$

e $\log_x 25 = 2$ f $\log_x 125 = 3$

3 Write each statement in logarithmic form.

a $1024 = 2^{10}$ b $10^x = a$

c $6^0 = 1$ d $11^1 = 11$

e $3^x = b$ f $5^4 = 625$

4 Write each statement in exponential form.

a $\log_3 81 = 4$ b $\log_2 64 = 6$

c $\log_{10} 0.01 = -2$ d $\log_b c = a$

e $\log_a b = c$

5 Simplify:

a $\log_2 11 + \log_2 5$ b $\log_2 7 + \log_2 5$

c $\log_6 11 + \log_6 7$ d $\log_3 8 - \log_3 32$

e $\log_5 200 - \log_5 40$ f $\log_5 30 - \log_5 6$

6 Simplify:

a $\log_2 5 + \log_2 4 + \log_2 7$ b $\log_5 1000 - \log_5 100 - \log_5 10$

c $\log_7 7 + \log_7 343 - 3\log_7 49$ d $\log_3 25 + 2\log_3 5 - 2\log_3 75$

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7 Use the change of base formula to convert to base 10 and calculate each to four decimal places.

a $\log_7 11$ **b** $\log_5 7$ **c** $\log_3 24$

d $\log_3 35$ **e** $\log_{16} 8$ **f** $\log_3 \frac{1}{4}$

8 Solve for x, correct to four decimal places.

a $2^x = 7$ **b** $3^x = 78$

c $5^x = 28$ **d** $5^x = 132$

e $2^{-x} = 5$ **f** $3^{-x} = 15$

9 Solve for x.

a $\log_2(2x - 3) = 4$ **b** $\log_3 3x = 4$

c $\log_2(3 - x) = 2$ **d** $\log_{10} x = 4$

e $\log_4(5 - 2x) = 3$ **f** $\log_2(x - 6) = 2$

10 Sketch each graph.

a $y = \log_5 x, x > 0$

b $y = \log_3(x - 2), x > 2$

c $y = \log_2(x + 4), x > -4$

d $y = \log_2(x) + 5, x > 0$

11 Express y in terms of x when:

a $\log_{10} y = 1 + \log_{10} x$

b $\log_{10}(y + 1) = 2 + \log_{10} x$

12 Simplify $\log_2\left(\frac{8}{75}\right) - 2\log_2\left(\frac{3}{5}\right) - 4\log_2\left(\frac{3}{2}\right)$.

13 If $\log_{10} x = 0.6$ and $\log_{10} y = 0.2$, evaluate $\log_{10}\left(\frac{x^2}{\sqrt{y}}\right)$.

14 **a** Express $3 + \log_2 5$ as a single logarithm.

b Express $5 - \log_2 5$ as a single logarithm.

15 An amount of \$120 000 is invested on 1 Jan at a compound interest rate of 8% per annum. Interest only paid on 1 Jan of each year. At the end of how many years will the investment be worth:

a \$160 000? **b** \$200 000?

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Challenge exercise

Throughout this exercise, the bases a and b are positive and not equal to 1.

1 Consider a right-angled triangle with side lengths a, b and c, with c the hypotenuse.
Prove that $\log_{10} a = \frac{1}{2}\log_{10}(c+b) + \frac{1}{2}\log_{10}(c-b)$.

2 Simplify $\log_a(a^2 + a) - \log_a(a + 1)$.

3 Show that $3\log_{10} x + 2\log_{10} y - \frac{1}{2}\log_{10} z = \log_{10}\left(\frac{x^3y^2}{\sqrt{z}}\right)$.

4 Solve for x:

a $\log_2(x+1) - \log_2(x-1) = 3$

b $(\log_{10} x)(\log_{10} x^2) + \log_{10} x^3 - 5 = 0$

c $(\log_2 x^2)^2 - \log_2 x^3 - 10 = 0$

d $(\log_3 x)^2 = \log_3 x^5 - 6$

5 Solve each set of simultaneous equations.

a $9^x = 27^{y-3}$, $16^{x+1} = 8^y \times 2$

b $8^x = 32^{y+1}$, $5^{x-1} = 25^y$

c $49^{x+3} = 343^{y-1}$, $2^{x+y} = 8^{x-2y}$

d $8^x = 4^y$, $7^{3x+3} = 343^y$

6 Solve the equation $(\log_a x)(\log_b x) = \log_a b$ for x where a and b are positive numbers different from 1.

7 If $a = \log_8 225$ and $b = \log_2 15$, find a in terms of b.

8 **a** Show that $\log_{10} 3$ cannot be a rational number.

b Show that $\log_{10} n$ cannot be a rational number if n is any positive integer that is not a whole number power of 10.

9 Prove that $\log_a\left(\frac{xy}{z}\right) + \log_a\left(\frac{yz}{x}\right) + \log_a\left(\frac{zx}{y}\right) = \log_a x + \log_a y + \log_a z$.

10 If x and y are distinct positive numbers, $a > 0$ and
$\frac{\log_a x}{y-z} = \frac{\log_a y}{z-x} = \frac{\log_a z}{x-y}$, show $xyz = 1$ and $x^x y^y z^z = 1$.

11 If $2\log_a x = 1 + \log_a(7x - 10a)$, find x in terms of a, where a is a positive constant and x is positive.

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