CHAPTER

15

Statistics and Probability

Probability

In this chapter, we continue our study of probability. In particular, we introduce the important ideas of sampling with and without replacement. The other important new ideas in this chapter are the concepts of conditional probability and independence.

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15A Review of probability

We first review the basic ideas of probability that we introduced in Chapter 12 of *ICE-EM Mathematics Year 9*.

Sample spaces with equally likely outcomes

In *ICE-EM Mathematics Year 9*, we looked at the experiment of throwing two dice and recording the values on the uppermost faces. The results can be displayed in an array, as shown here.

Die 1 \ Die 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The **sample space**, ξ, for this experiment is the set of ordered pairs displayed in the array. That is, $\xi = \{(1, 1), (1, 2), ..., (6, 6)\}$. The 36 **outcomes** of this experiment are equally likely and each outcome has probability $\frac{1}{36}$.

Example 1

Two dice are thrown and the value on each die is recorded. Find the probability that:

a the sum of the two values is 5

b the sum of the two values is less than or equal to 3

Solution

The sample space ξ is as described as above. The size of ξ is 36.

a Let A be the event that the sum is 5.

$A = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$P(A) = \frac{4}{36} = \frac{1}{9}$

b Let B be the event that the sum is less than or equal to 3.

$B = \{(1, 1), (1, 2), (2, 1)\}$

$P(B) = \frac{3}{36} = \frac{1}{12}$

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Sample spaces with non-equally likely outcomes

We can change the experiment to: Two dice are thrown and the *sum* of the values on the uppermost faces is recorded.

This leads to a different sample space:

$$\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

The outcomes are no longer equally likely, since, for example, we can only obtain a total of 2 by throwing a 1 and a 1, but there are 5 ways to obtain a sum of 6.

We can determine the probability of each of these outcomes from the array on the previous page.

The probabilities are listed in the table below.

Outcome	2	3	4	5	6	7	8	9	10	11	12
P(outcome)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The sum of the probabilities of the outcomes is 1.

Events

An **event** is a subset of the sample space. For example, in the experiment of throwing two dice and recording the sum of the uppermost faces, an event is a subset of:

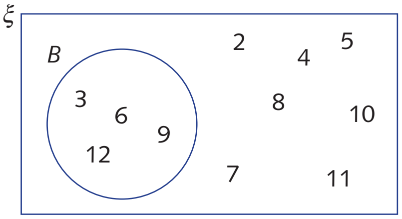
$$\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

For example, the event B {outcomes whose sum is divisible by 3} is the subset:

$$B = \{3, 6, 9, 12\}$$

We will often use a more colloquial description of such events. For example, we will say B is the event 'the sum is divisible by 3'.

An outcome is **favourable** to an event if it is a member of that event. For example, $6 \in B$ and $5 \notin B$. The event B can be illustrated with a Venn diagram.



Probability of an event

The probability p of an outcome is a number between 0 and 1 inclusive.

Probabilities are assigned to outcomes in such a way that the sum of the probabilities of all the outcomes in the sample space ξ is 1.

The probability of the event A is written as $P(A)$. Thus, $P(A)$ is the sum of the probabilities of the outcomes that are favourable to the event A.

Hence, $0 \le P(A) \le 1$, for each event A. That is, the probability of an event is a number between 0 and 1 inclusive. In particular, $P(\xi) = 1$.

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For the event B {outcomes whose sum is divisible by 3} in the previous example:

$$\begin{aligned} P(B) &= P(3) + P(6) + P(9) + P(12) \\ &= \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} \\ &= \frac{1}{3} \end{aligned}$$

For an experiment in which all of the outcomes are equally likely:

$$\text{Probability of an event} = \frac{\text{number of outcomes favourable to that event}}{\text{total number of outcomes}}$$

This is *not* the case for the experiment of throwing two dice and recording the sum, as we learned that such an event had non-equally likely outcomes.

Example 2

If a die is rolled, what is the probability that a number greater than 4 is obtained?

Solution

When a die is rolled once, there are six equally likely outcomes
$\xi = \{1, 2, 3, 4, 5, 6\}$

Let A be the event 'a number greater than four is obtained'.
Then $A = \{5, 6\}$

Hence, $P(A) = \frac{2}{6} = \frac{1}{3}$

Example 3

A standard pack of playing cards consists of four suits: Hearts, Diamonds, Clubs and Spades. Each suit has 13 cards consisting of an Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King.

The pack is shuffled and a card is drawn at random.

For this experiment the size of ξ is 52.

a What is the probability that it is a King?

b What is the probability that it is a Heart?

Solution

a Let K be the event 'drawing a King'.

There are four Kings in the pack of 52 cards.

$P(K) = \frac{4}{52} = \frac{1}{13}$

b Let H be the event 'drawing a Heart'.

There are 13 Hearts in the pack of 52 cards.

$P(H) = \frac{13}{52} = \frac{1}{4}$

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Example 4

One box contains 4 discs labelled as shown.

1 3 6 8

A second box contains 5 discs labelled as as shown.

2 4 5 7 9

A disc is taken from each of the boxes and the larger of the two numbers is recorded.

a What is a sample space for the experiment?

b Find the probability of each outcome.

c Find the probability that the number obtained is less than 5.

Solution

There are 20 different pairs that can be drawn from the two boxes. Each of these pairs is equally likely to occur. The larger of the two numbers is recorded in the array.

Box 1 \ Box 2	2	4	5	7	9
1	2	4	5	7	9
3	3	4	5	7	9
6	6	6	6	7	9
8	8	8	8	8	9

a The sample space is $\xi = \{2, 3, 4, 5, 6, 7, 8, 9\}$

b From the array:

$$P(2) = \frac{1}{20},\ P(3) = \frac{1}{20},\ P(4) = \frac{2}{20},\ P(5) = \frac{2}{20},$$

$$P(6) = \frac{3}{20},\ P(7) = \frac{3}{20},\ P(8) = \frac{4}{20},\ P(9) = \frac{4}{20}$$

c
$$\begin{aligned} P(\{2, 3, 4\}) &= P(2) + P(3) + P(4) \\ &= \frac{1}{20} + \frac{1}{20} + \frac{2}{20} \\ &= \frac{1}{5} \end{aligned}$$

Review of probability

- A **sample space**, ξ, consists of all possible outcomes of an experiment.
- Each outcome has a probability p between 0 and 1. That is, $0 \le p \le 1$.
- The sum of the probabilities of all outcomes is 1.
- An **event**, A, is a subset of ξ. A member of A is called an outcome **favourable** to A.
- $P(A)$ is the sum of the probabilities of all outcomes favourable to A.
- For an experiment in which all the outcomes are equally likely:

$$\text{Probability of an event} = \frac{\text{number of outcomes favourable to that event}}{\text{total number of outcomes}}$$

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Exercise 15A

Example 1, 2

1 David has 13 marbles. Five of them are pink, three are blue, three are green and two are black. If he chooses a marble at random, what is the probability that it is green?

Example 3

2 A debating team consists of five boys and seven girls. If one of the team is chosen at random to be the leader, what is the probability that the leader is a girl?

3 A basketball team consists of five players: Adams, Brown, Cattogio, O'Leary and Nguyen. If a player is chosen at random, what is the probability that his name starts with a consonant?

4 Slips of paper numbered $1, 2, 3, \ldots, 10$ are placed in a hat and one is drawn at random. What is the probability that the number on the slip of paper is not a multiple of four?

5 A bag contains 11 balls. Three of these are black and eight are blue. A ball is taken from the bag at random. What is the probability that it is blue?

Example 4

6 One box contains 4 discs labelled as shown.

(2) (3) (7) (8)

A second box contains 5 discs labelled as shown.

(5) (6) (7) (8) (9)

A disc is taken from each of the boxes and the larger of the two numbers is recorded.

a List the sample space for the experiment.

b Find the probability of each outcome.

c Find the probability that the number obtained is greater than 6.

7 A box contains 3 discs labelled as shown.

(1) (2) (3)

A second box contains 3 discs labelled as shown.

(2) (4) (7)

A disc is taken randomly from each box and the result is recorded as an ordered pair, for example, $(1, 7)$.

a List the sample space for the experiment.

b Find the probability of each outcome.

c Find the probability that there is an even number on both of the selected discs.

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8 A box contains 4 discs labelled as shown.

(1) (2) (3) (4)

A second box contains 3 discs labelled as shown.

(1) (2) (3)

A disc is taken randomly from each box and the sum of the numbers on the two discs is recorded.

a List the sample space for the experiment.

b Find the probability of each outcome.

c Find the probability that the sum is less than 5.

9 Two dice are thrown and the values on the uppermost faces recorded. What is the probability of:

a obtaining an even number on both dice?

b obtaining exactly one 6?

c obtaining a 3 on one die and an even number on the other?

10 Two dice are thrown and the difference of the values on the uppermost faces is recorded: outcome = value on die 1 – value on die 2

a List the sample space for this experiment.

b What is the probability of obtaining a negative number?

c What is the probability of obtaining a difference of 0?

d What is the probability of obtaining a difference of -1?

e What is the probability of obtaining a difference that is exactly divisible by 3?

11 A bag contains six balls: three red balls numbered 1 to 3, two white balls numbered 1 and 2, and one yellow ball. Two balls are selected one after the other, at random, and the first is replaced before the second is withdrawn.

a List the sample space.

b Find the probability that:

i both balls are the same colour

ii the two balls selected are different colours

12 The surnames of 800 male students on a school roll vary in length from 3 letters to 11 letters as follows:

Number of letters	3	4	5	6	7	8	9	10	11
Number of boys	16	100	171	206	144	97	51	13	2

If a boy is selected at random from those in this school, what is the probability that his surname contains:

a four letters? **b** more than eight letters? **c** less than five letters?

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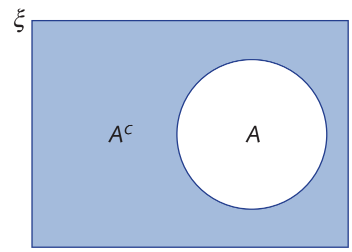
15B The complement, union and intersection

The complement of *A*

In some problems, the outcomes in the event A can be difficult to count; whereas the event 'not A' may be easier to deal with.

The event 'not A' consists of every possible outcome in the sample space ξ that it is not in A. The set 'not A' is called the **complement** of A and is denoted by A^c.



Every outcome in the sample space ξ is contained in exactly one of A or A^c.

Therefore:

$$P(A) + P(A^c) = 1 \text{ and so } P(A^c) = 1 - P(A)$$

This can be illustrated with a Venn diagram.

Example 5

A card is drawn from a standard pack. What is the probability that it is not the King of Hearts?

Solution

Let A be the event 'the King of Hearts is drawn'.

Then A^c is the event 'the King of Hearts is not drawn'.

$$P(A) = \frac{1}{52}$$

$$\begin{aligned} P(A^c) &= 1 - P(A) \\ &= 1 - \frac{1}{52} \\ &= \frac{51}{52} \end{aligned}$$

The probability that the card drawn is not the King of Hearts is $\frac{51}{52}$.

Union and intersection

Sometimes, rather than just considering a single event, we want to look at two or more events.

We return to our example of throwing two dice and taking the sum of the numbers on the uppermost faces.

Recall that $\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Let A be the event 'a number divisible by 3 is obtained'.

Let B be the event 'a number greater than 5 is obtained'.

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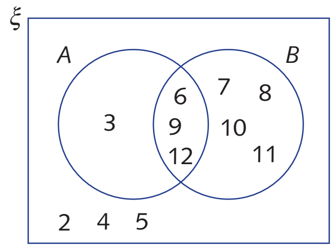
The events A and B are:

$A = \{3, 6, 9, 12\}$

$B = \{6, 7, 8, 9, 10, 11, 12\}$

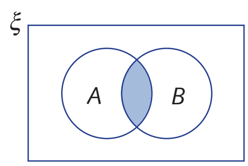
and $A \cap B = \{6, 9, 12\}$

Here is the Venn diagram illustrating these events.

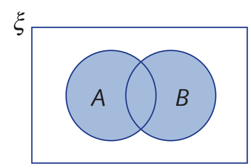


The outcomes favourable to the event 'the number is divisible by 3 *and* greater than 5' is the **intersection** of the sets A and B; that is, $A \cap B$. The event $A \cap B$ is often called 'A *and* B'.

The outcomes favourable to the event 'the number is divisible by 3 *or* greater than 5' is the **union** of the sets A and B; that is, $A \cup B$. The event $A \cup B$ is often called 'A *or* B'.



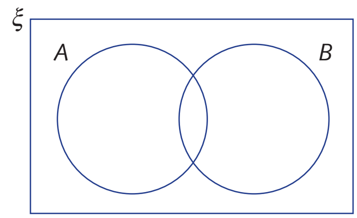
$A \cap B$ is shaded



$A \cup B$ is shaded

For an outcome to be in the event $A \cup B$, it must be in *either* the set of outcomes for A or the set of outcomes for B. Of course, it could be in both sets.

For an outcome to be in the event $A \cap B$, it must be in *both* the set of outcomes for A and the set of outcomes for B.



We recall the **addition rule** for probability.

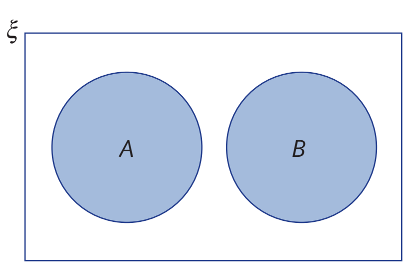
For any two events, A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is clear from the Venn diagram.

Two events are **mutually exclusive** if they have no outcomes in common. That is:

$A \cap B = \varnothing$, where $\varnothing$ is the empty set



In this case, when A and B are mutually exclusive, the addition rule becomes:

$$P(A \cup B) = P(A) + P(B)$$

Here are some examples using these ideas.

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Example 6

Two dice are thrown and the sum of the numbers on the uppermost faces is recorded. What is the probability that the sum is:

a even?

b greater than 7?

c less than 5?

d greater than 7 or less than 5?

e even and greater than 7?

f even or greater than 7?

Solution

Recall that $\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Outcome	2	3	4	5	6	7	8	9	10	11	12
P(outcome)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Let A be the event 'the sum is even'

B be the event 'the sum is greater than 7'

C be the event 'the sum is less than 5'

Then $A = \{2, 4, 6, 8, 10, 12\}$

$B = \{8, 9, 10, 11, 12\}$

$C = \{2, 3, 4\}$

a Using the table:

$$\begin{aligned} P(A) &= P(2) + P(4) + P(6) + P(8) + P(10) + P(12) \\ &= \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} \\ &= \frac{1}{2} \end{aligned}$$

That is, the probability that the sum is even is $\frac{1}{2}$.

b
$$\begin{aligned} P(B) &= P(8) + P(9) + P(10) + P(11) + P(12) \\ &= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} \\ &= \frac{5}{12} \end{aligned}$$

That is, the probability that the sum is greater than 7 is $\frac{5}{12}$.

c
$$\begin{aligned} P(C) &= P(2) + P(3) + P(4) \\ &= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} \\ &= \frac{1}{6} \end{aligned}$$

That is, the probability that the sum is less than 5 is $\frac{1}{6}$.

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d $P(\text{the sum is greater than 7 or less than 5}) = P(B \cup C)$

Now $B \cap C = \varnothing$, so B and C are mutually exclusive events

$$\begin{aligned} P(B \cup C) &= P(B) + P(C) \\ &= \frac{7}{12} \end{aligned}$$

e $P(\text{the sum is even and greater than 7}) = P(A \cap B)$

$$\begin{aligned} P(A \cap B) &= P(8) + P(10) + P(12) \\ &= \frac{5}{36} + \frac{3}{36} + \frac{1}{36} \\ &= \frac{1}{4} \end{aligned}$$

f $P(\text{the sum is even or greater than 7}) = P(A \cup B)$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{2} + \frac{5}{12} - \frac{1}{4} \\ &= \frac{2}{3} \end{aligned}$$

Example 7

The eye colour and gender of 150 people were recorded. The results are shown in the table below.

Gender \ Eye colour	Blue	Brown	Green	Grey
Male	20	25	5	10
Female	40	35	5	10

What is the probability that a person chosen at random from the sample:

a has blue eyes?

b is male?

c is male and has green eyes?

d is female and does not have blue eyes?

e has blue eyes or is female?

f is male or does not have green eyes?

Solution

Let A be the event 'has blue eyes'

B be the event 'has brown eyes'

G be the event 'has green eyes'

M be the event 'is male'

F be the event 'is female'

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a $P(A) = \frac{60}{150} = \frac{2}{5}$

b $P(M) = \frac{60}{150} = \frac{2}{5}$

c $P(M \cap G) = \frac{5}{150} = \frac{1}{30}$

d $P(F \cup A^c) = \frac{35 + 5 + 10}{150}$
$= \frac{50}{150}$
$= \frac{1}{3}$

e $P(A \cup F) = \frac{20 + 40 + 35 + 5 + 10}{150}$
$= \frac{110}{150}$
$= \frac{11}{15}$

[Alternatively, using the addition rule,
$P(A \cup F) = P(A) + P(F) - P(A \cap F)$
$= \frac{2}{5} + \frac{3}{5} - \frac{40}{150}$ *Note*: $P(F) = 1 - P(M)$
$= \frac{11}{15}$]

f $P(M \cup G^c) = \frac{20 + 25 + 5 + 10 + 40 + 35 + 10}{150}$
$= \frac{145}{150}$
$= \frac{29}{30}$

Note: This can also be calculated using the addition rule or by noting that this is the complement of the event 'The person has green eyes and is female',
$1 - P(F \cap G) = 1 - \frac{5}{150} = \frac{29}{30}$.

Complement, or, and

- The event 'not A' includes every outcome of the sample space ξ that is not in A. The event 'not A' is called the complement of A and is denoted by A^c.

  $P(A^c) = 1 - P(A)$
- An outcome in the event $A \cup B$, is either in A or B, or both.
- An outcome in the event $A \cap B$, is in both A and B.
- For any two events A and B:

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Two events A and B are mutually exclusive if $A \cap B = \varnothing$ and in this case $P(A \cup B) = P(A) + P(B)$.

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Exercise 15B

Example 5

1 A number is chosen at random from the first 15 positive whole numbers. What is the probability that it is not a prime number?

2 A card is drawn at random from an ordinary pack of 52 playing cards. What is the probability that it is not a King?

3 A number is chosen at random from the first 30 positive whole numbers. What is the probability that it is not divisible by 7?

4 In a raffle, 1000 tickets are sold. If you buy 50 tickets, what is the probability that you will not win first prize?

5 A letter is chosen at random from the 10 letters of the word COMMISSION. What is the probability that the letter is:

a N? **b** S? **c** a vowel? **d** not S?

Example 6

6 A card is drawn at random from a pack of playing cards. Find the probability that the card chosen:

a is a Club

b is a court card (i.e. an Ace, King, Queen or Jack)

c is a Club and a court card **d** is a Club or a court card

e has a face value between 2 and 5 inclusive and is a court card

f has a face value between 2 and 5 inclusive or is a court card

7 A standard die is thrown and the uppermost number is noted. Find the probability that the number is:

a even and a six **b** even or a six

c less than or equal to four and a six **d** less than or equal to three or a six

e even and less than or equal to four **f** odd or less than or equal to three

Example 7

8 A survey of 200 people was carried out to determine hair and eye colour. The results are shown in the table below.

Eye colour \ Hair colour	Fair	Brown	Red	Black
Blue	25	9	6	18
Brown	16	16	18	22
Green	15	17	22	16

What is the probability that a person chosen at random from this group has:

a blue eyes? **b** red hair?

c fair or brown hair? **d** blue or brown eyes?

e red hair and green eyes? **f** eyes that are not green?

g hair that is not red? **h** fair hair and blue eyes?

i eyes that are not blue or hair that is not fair?

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In the following questions, use an appropriate Venn diagram.

9 In a group of 100 students, 60 study mathematics, 70 study physics and 30 study both mathematics and physics.

a Represent this information on a Venn diagram.

b One student is selected at random from the group. What is the probability that the student studies:

i mathematics but not physics?

ii physics but not mathematics?

iii neither physics nor mathematics?

10 In a group of 40 students, 26 play tennis and 19 play soccer. Assuming that each of the 40 students plays at least one of these sports, find the probability that a student chosen at random from this group:

a plays both tennis and soccer

b plays only tennis

c plays only one sport

d plays only soccer

11 In a group of 65 students, 30 students study geography, 42 study history and 20 study both history and geography. If a student is chosen at random from the group of 65 students, find the probability that the student studies:

a history or geography

b neither history nor geography

c history but not geography

d exactly one of history or geography

12 A number is selected at random from the integers 1 to 1000 inclusive. Find the probability the number is:

a divisible by 5

b divisible by 9

c divisible by 11

d divisible by 5 and 9

e divisible by 5 and 11

f divisible by 9 and 11

g divisible by $5,\ 9$ and 11

13 In a group of 85 people, 33 own a microwave, 28 own a DVD player and 38 own a computer. In addition, 6 people own both a microwave and a DVD player, 9 own both a DVD player and a computer, 7 own both a computer and a microwave and 2 people own all three items. Draw a Venn diagram representing this information. If a person is chosen at random from the group, what is the probability that the person:

a does not own a microwave, a computer or a DVD player?

b owns exactly one of the three items?

c owns exactly two of the three items?

14 If a card is drawn at random from a pack of 52 playing cards, what is the probability that it will be:

a a Heart or the Ace of Clubs?

b a Heart or an Ace?

c a Heart or a Diamond?

15 From a set of 15 cards whose faces are numbered 1 to 15, one card is drawn at random. What is the probability that it is a multiple of 3 or 5?

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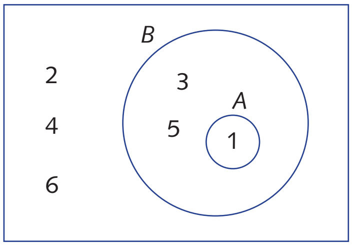
15C Conditional probability

The probability of an event, A, occurring when it is known that some event, B, has occurred is called the probability of ***A* given *B*** and is written $P(A \mid B)$. This is the idea of **conditional probability**.



Suppose we roll a fair die and define event A as 'rolling a one' and event B as 'rolling an odd number'.

The events A and B are shown on the Venn diagram to the right.

What is the probability that a one was rolled given the information that an odd number was rolled?

We are being asked to find $P(A \mid B)$.

The knowledge that event B has occurred restricts the sample space for this calculation to $B = \{1, 3, 5\}$. Since the outcomes of a fair die are equally likely to occur, we can calculate $P(A \mid B)$:

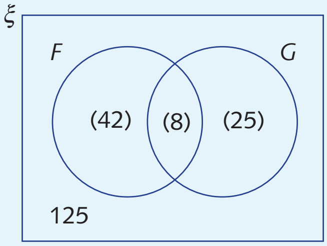
$$P(A \mid B) = \frac{1}{3}$$

The understanding that an event has occurred requires us to adjust our probability calculations in the light of this information.

Example 8

In a group of 200 students, 42 study French only, 25 study German only and 8 study both. Find the probability that a student studies French given that they study German.

Solution



The sample space ξ is the set of 200 students.

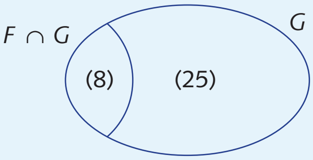
Let F be the event 'a student studies French'.

Let G be the event 'a student studies German'.

This information can be represented in a Venn diagram.

P (a student studies French given that they study German) is written as $P(F \mid G)$.

To find this, we consider G as a new sample space. The corresponding Venn diagram is as shown.



$|G| = 33$ and $|F \cap G| = 8$. Hence, $P(F \mid G) = \dfrac{8}{33}$.

In this problem, we are regarding G as a sample space in its own right and calculating the probablity of $F \cap G$ as an event in the sample space G. Thus, we have:

$$P(F \mid G) = \frac{|F \cap G|}{|G|}$$

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Example 9

A bowl contains blue and black marbles. Some of the marbles have A marked on them and others have B marked on them. The number of each type is given in the table below.

	Black marble	Blue marble
Marked A	50	27
Marked B	22	13

A marble is randomly taken out of the bowl. Find the probability that:

a it is a marble marked A

b it is a marble marked A given that it is blue

c it is a blue marble

d it is a blue marble given that it is marked B

Solution

There are 112 marbles.

a $P(\text{a marble marked } A) = \frac{77}{112} = \frac{11}{16}$

b $P(\text{a marble marked } A \mid \text{it is blue}) = \frac{27}{40}$

c $P(\text{a blue marble}) = \frac{40}{112} = \frac{5}{14}$

d $P(\text{a blue marble} \mid \text{it is marked } B) = \frac{13}{35}$

Note that in Example 9, $P(\text{Is blue} \cap \text{Marked } B) = \frac{13}{112}$ and $P(\text{Marked } B) = \frac{35}{112}$.

Hence $Pr(\text{Is blue} \mid \text{Marked } B) = \frac{P(\text{Is blue} \cap \text{Marked } B)}{P(\text{Marked } B)} = \frac{13}{112} \div \frac{35}{112} = \frac{13}{35}$.

In general:

Conditional probability

Suppose that A and B are two subsets of a sample space ξ. Then for the events A and B

$$P(A \text{ given } B) = P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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Example 10

Given that for two events, A and B, $P(A) = 0.6$, $P(B) = 0.4$ and $P(A \cup B) = 0.8$, find:

a $P(A \mid B)$ **b** $P(B \mid A)$

Solution

a $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

We know $P(B)$, but $P(A \cap B)$ is required.

The addition rule states, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$$\begin{aligned}\text{Therefore, } P(A \cap B) &= P(A) + P(B) - P(A \cup B)\\ &= 0.6 + 0.4 - 0.8\\ &= 0.2\end{aligned}$$

$$\begin{aligned}\text{Hence, } P(A \mid B) &= \frac{P(A \cap B)}{P(B)}\\ &= \frac{0.2}{0.4} = \frac{1}{2}\end{aligned}$$

b $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$ $\quad (B \cap A = A \cap B)$

$$= \frac{0.2}{0.6} = \frac{1}{3}$$

Exercise 15C

Example 8, 9

1 A bowl contains green and red normal jelly beans and green and red double-flavoured jelly beans. The number of each type is given in the following table.

	Green	Red
Normal jelly bean	13	18
Double-flavoured jelly bean	9	8

A jelly bean is randomly taken out of the bowl. Find the probability that:

a it is a double-flavoured jelly bean

b it is a green jelly bean

c it is a green normal jelly bean

d it is a green jelly bean given that it is a normal jelly bean

e it is a double-flavoured jelly bean given that it is a red jelly bean

f it is a double-flavoured jelly bean given that it is a green jelly bean

2 A die is tossed. What is the probability that an outcome greater than 4 is obtained, given that:

a an even number is obtained?

b a number greater than 2 is obtained?

3 Two coins are tossed. What is the probability of obtaining two heads given that at least one head is obtained?

4 A card is drawn from a standard pack of cards. What is the probability that:

a a court card is drawn given that it is known that the card is a Heart?

b the 8 or 9 of Clubs is drawn given that it is known that a black card is drawn?

5 In a traffic survey during a 30-minute period, the number of people in each passing car was noted, and the results tabulated as follows.

Number of people in a car	1	2	3	4	5
Number of cars	60	50	40	10	5

Total: 165

a What is the probability that there was 1 person in a car during this period?

b What is the probability that there was more than 1 person in a car during this period?

c What is the probability that there were less than 2 people in a car during this period given that there were less than 4 people in the car?

d What is the probability that there were 5 people in a car during this period given that there were more than 3 people in the car?

6 A group of 2000 people, eligible to vote, were asked their age and candidate preference in an upcoming election, with the following results.

	18–25 years	26–40 years	Over 40 years	Total
Candidate A	400	200	170	770
Candidate B	500	460	100	1060
No preference	100	40	30	170
Total	1000	700	300	2000

What is the probability that a person chosen at random from this group:

a is from the 26–40 age group?

b prefers Candidate B?

c is from 26–40 age group given that they prefer Candidate A?

d prefers Candidate B given that they are in the 18–25 years age group?

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7 A prize is going to be awarded at the end of a concert. It is announced that the winner will be chosen randomly.

The number of people at the concert in different age groups is given in the following table.

Age group	0–5	6–11	12–18	19–29	30–40	Older than 40
Number of people In age group	10	150	350	420	125	85

What is the probability that the prize winner is:

a 40 or less?

b between 12 and 29?

c older than 11?

d older than 18 given they are older than 11?

e 29 or less given that they are 40 or less?

8 A game is devised by two friends, Aalia and Rachael. They roll two dice and take the smaller number from the larger, or they write 0 if the numbers are the same. Aalia wins if the difference is less than 3.

a Complete the table of differences.

Die 1 \ Die 2	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2					
4	3					
5	4					
6	5					

b Draw up a table giving the outcomes of the experiment and their probabilities.

c Find the probability that Aalia wins.

d Find the probability that Rachael wins.

e Find the probability that Aalia wins given that the difference is less than 4.

f Find the probability of Rachael winning given that the difference is less than 4.

9 An urn contains 25 marbles numbered from 1 to 25 . A marble is drawn from the urn. What is the probability that:

a the marble numbered 3 is drawn given that it is odd?

b a marble with a number less than 10 is drawn given that it is less than 20?

c a marble with a number greater than 10 is drawn given that it is greater than 5?

d a marble with a number greater than 10 is drawn given that it is less than 20?

e a marble with a number divisible by 10 is drawn given that it is divisible by 5?

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10 In a group of 85 people, 33 own a microwave, 28 own a DVD player and 38 own a computer. In addition, 6 people own both a microwave and a DVD player, 9 own both a DVD player and a computer, 7 own both a computer and a microwave and 2 people own all three items. If a person is chosen at random from the group, what is the probability that the person:

a owns a microwave given that they own a DVD player?

b owns a computer given that they own a DVD player?

c owns a computer given that they own a DVD player and a microwave?

Example 10

11 Given that for two events A and B, $P(A) = 0.2$, $P(B) = 0.6$ and $P(A \cup B) = 0.7$, Find:

a $P(A \mid B)$

b $P(B \mid A)$

12 Given that for two events A and B, $P(B) = 0.5$, $P(A \mid B) = 0.2$ and $P(A \cup B) = 0.7$. Find $P(A)$.

15D Independent events

Consider the situation where a coin is tossed twice.

Toss 1 \ Toss 2	H	T
H	(H, H)	(H, T)
T	(T, H)	(T, T)

If we define A as the event 'the second toss is a head' and B as the event 'the first toss is a head', then $A = \{(T,H),(H, H)\}$ and $B = \{(H, T),(H, H)\}$. What is $P(A \mid B)$?

$$\text{By definition, } P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Hence, $P(A \mid B) = P(A)$. This is unsurprising since there are two separate coin tosses driving events A and B and one does not impact upon the other. That is, the probability of event A occurring is unaffected by event B having occurred. This is an example of independent events.

Two events A and B are **independent** if the occurrence of one event does not affect the probability of the occurrence of the other. That is, if:

$P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

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Thus when two events A and B are independent:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = P(A), \quad \text{if } P(B) \neq 0$$

Therefore:

$$P(A \cap B) = P(A) \times P(B)$$

This equation provides a convenient alternative to testing whether two events A and B are independent.

In the special case that one event or the other is impossible to occur, i.e. $P(A) = 0$ or $P(B) = 0$, this rule is also satisfied since both sides of the equation are zero. In this special case, we say that A and B are also independent.

Independent events

Events A and B are independent if and only if:

$$P(A \cap B) = P(A) \times P(B)$$

Example 11

200 teenagers and young adults less than 23 years old were interviewed about their use of the social medium, Snapchatter. The results, as well as their ages are given in the following table.

Use Snapchatter	Age (years), A		Total
	$13 \le A < 18$	$18 \le A < 23$	
Yes	77	63	140
No	28	32	60
Total	105	95	200

Is the use of Snapchatter independent of age among teenagers and young adults?

Solution

From the table:

$$P(13 \le A < 18 \cap Yes) = \frac{77}{200} = 0.335$$

$$P(13 \le A < 18) \times P(Yes) = \frac{105}{200} \times \frac{140}{200} = \frac{147}{400} = 0.3675$$

Hence,

$$P(13 \le A < 18 \cap Yes) \neq P(13 \le A < 18) \times P(Yes)$$

Therefore, these events are not independent.

That is, Snapchatter use is not independent of age among teenagers and young adults.

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Example 12

In a certain rural town, the probability that a randomly selected person has more than one sibling (S) is 0.6 and the probability that they live 'in the east of town' (between the north east and south east of the centre) (E) is 0.3. If these events are independent, then find the following probabilities.

a A person from the east of town has more than one sibling.

b A person has no more than one sibling and does not live in the east of town.

Solution

a A person from the east of town who has more than one sibling is represented by $E \cap S$,

$$P(E \cap S) = P(E) \times P(S) \qquad (E \text{ and } S \text{ are independent})$$
$$= 0.6 \times 0.3 = 0.18$$

b A person not from the east of town who has no more than one sibling is represented by $E^c \cap S^c$,

$$P(E^c \cap S^c) = P(E^c) \times P(S^c) \quad (E \text{ and } S \text{ are independent, therefore } E^c \text{ and } S^c \text{ are independent.})$$
$$= 0.4 \times 0.7 = 0.28$$

Confusion often arises between independent and mutually exclusive events. As discussed previously, two events A and B are mutually exclusive means that $A \cap B = \varnothing$ and hence that $P(A \cap B) = 0$. Therefore, two events will only be mutually exclusive **and** independent if the probability of at least one of them is zero.

Example 13

Consider rolling a die. Define event A as 'rolling a number divisible by 3' and event B as 'rolling an even number'.

a Are events A and B independent?

b Are events A and B mutually exclusive?

Solution

a Favourable outcomes for the these events are $A = \{3, 6\}$, $B = \{2, 4, 6\}$.

Therefore, $A \cap B = \{6\}$.

Since all outcomes $\{1, 2, 3, 4, 5, 6\}$ are equally likely,

$$P(A) = \frac{2}{6} = \frac{1}{3}, \quad P(B) = \frac{3}{6} = \frac{1}{2} \quad \text{and } P(A \cap B) = \frac{1}{6}.$$

This means, $P(A \cap \mathrm{B}) = P(A) \times P(B) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

Hence, events A and B are independent.

b $P(A \cap B) \neq 0$. Therefore, events A and B are not mutually exclusive.

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Exercise 15D

Example 11

1 100 people were surveyed about their attitudes to the use of headgear in professional boxing and classified according to sex. The results are shown in the table below.

Should the use of headgear be mandatory in professional boxing?

	Male	Female	Total
Yes	25	30	55
No	35	10	45
Total	60	40	100

Is attitude to the use of headgear in professional boxing independent of sex?

2 80 adults were surveyed about whether they play computer games more than once a week and their age category was recorded ('30 years or older' and 'less than 30'). The results are shown below.

Play computer games >1 per week	Age (years)		Total
	< 30	≥ 30	
Yes	35	20	55
No	10	15	25
Total	45	35	80

Is playing computer games multiple times a week independent of age category?

3 200 road accidents recorded by the Traffic Authority were studied in terms of vehicle speed at the time of collision, relative to the local speed limit, and the accident severity. The results are shown below.

Accident severity	Speed of vehicle at collision			Total
	Not speeding	≤ 10 km/h over limit	> 10 km/h over limit	
Minor	41	88	17	146
Major	2	31	21	54
Total	43	109	38	200

a Find the probability that a randomly selected accident in the study is classed as 'major'.

b Find the probability that an accident was classed as major given that it collided at a speed greater than 10 km/h over the local speed limit.

c Hence, explain why accidents in this study classed as 'major' are not independent of the event that they were travelling at a speed greater than 10 km/h over the limit.

d By focussing on the events, 'Minor' accident and 'Not speeding', conduct an alternative test to that used in part **c** to show that they are also not independent events.

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Example 12

4 The probability that a person does their grocery shopping at Colesworth is $\frac{3}{5}$, and the probability that a person is left-handed is $\frac{1}{7}$. If these events are independent, find the following probabilities.

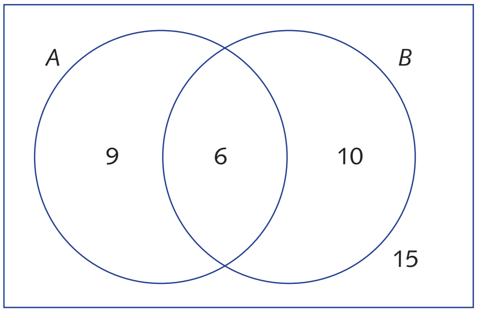
a A person does their grocery shopping at Colesworth and is left-handed.

b A person is not left-handed but does their grocery shopping at Colesworth.

c A person is not left-handed and does not do their grocery shopping at Colesworth.

d A person does their grocery shopping at Colesworth or is left-handed.

5 Events A and B are shown in the Venn diagram. Show that A and B are independent.



Example 13

6 Consider rolling a die on two separate occasions. Define event A as 'rolling a 4 on the first throw' and event B as 'rolling at least 10 as the sum of the two numbers shown'.

a Create a table (array) showing the sample space of a die rolled twice.

b Determine whether events A and B are independent.

c Determine whether events A and B are mutually exclusive.

15E Sampling with replacement and without replacement

A random experiment is any repeatable procedure with clear but unpredictable outcomes like, for example, tossing a coin or rolling a die. **Sampling** is a type of experiment that concerns making a number of random selections from a set of things. This set from which random selections are drawn is known as the **population**. We will be considering sampling with replacement and without replacement.

As we know, the conditional probability of an event A given that event B has already occurred is given by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

The formula can be re-arranged to give the **multiplication rule of probability**:

$$P(A \cap B) = P(A \mid B) \times P(B)$$

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This rule can be used when calculating probabilities in multi-stage experiments. Sampling with replacement involves making a selection, observing the outcome and then returning the item to the population before another selection is made. Since replacement occurs, the outcome at one stage is not affected by the outcome at any other stage.

Multi-stage sampling with replacement

A bag contains three red balls, R_1, R_2 and R_3, and two black balls, B_1 and B_2. A ball is drawn at random and its colour recorded. It is then put back in the bag, the balls are mixed thoroughly and a second ball is drawn. Its colour is also noted. The sample space is shown in the array below.

First ball \ Second ball	R_1	R_2	R_3	B_1	B_2
R_1	(R_1, R_1)	(R_1, R_2)	(R_1, R_3)	(R_1, B_1)	(R_1, B_2)
R_2	(R_2, R_1)	(R_2, R_2)	(R_2, R_3)	(R_2, B_1)	(R_2, B_2)
R_3	(R_3, R_1)	(R_3, R_2)	(R_3, R_3)	(R_3, B_1)	(R_3, B_2)
B_1	(B_1, R_1)	(B_1, R_2)	(B_1, R_3)	(B_1, B_1)	(B_1, B_2)
B_2	(B_2, R_1)	(B_2, R_2)	(B_2, R_3)	(B_2, B_1)	(B_2, B_2)

The sample space ξ contains the 25 pairs listed above. They are equally likely and each outcome has probability $\frac{1}{25}$ of occurring.

Let A be the event 'both balls are red', B be the event 'the first ball is red' and C be the event 'the second ball is red'.

From the array:

$P(A) = \frac{9}{25}$, $P(B) = \frac{15}{25} = \frac{3}{5}$ and $P(C) = \frac{15}{25} = \frac{3}{5}$

The event $B \cap C$ is 'the first and second balls are red', which is the same as event A. That is, $A = B \cap C$.

We note that $P(A) = \frac{9}{25}$ and $P(B) \times P(C) = \frac{9}{25}$.

$P(A) = P(B \cap C) = P(B) \times P(C)$ is true.

Multi-stage sampling without replacement

We start with the same bag of coloured balls as previously described. A ball is drawn at random and its colour recorded. The ball is *not* put back in the bag. A second ball is drawn at random from the remaining balls and its colour recorded.

The sample space ξ is listed in the array on the next page. There are the $5 \times 4 = 20$ outcomes in the sample space ξ.

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Second ball / First ball	R_1	R_2	R_3	B_1	B_2
R_1	–	(R_1, R_2)	(R_1, R_3)	(R_1, B_1)	(R_1, B_2)
R_2	(R_2, R_1)	–	(R_2, R_3)	(R_2, B_1)	(R_2, B_2)
R_3	(R_3, R_1)	(R_3, R_2)	–	(R_3, B_1)	(R_3, B_2)
B_1	(B_1, R_1)	(B_1, R_2)	(B_1, R_3)	–	(B_1, B_2)
B_2	(B_2, R_1)	(B_2, R_2)	(B_2, R_3)	(B_2, B_1)	–

The – indicates that the pair cannot occur.

Again, let A be the event 'both balls are red', B be the event' 'the first ball is red' and C be the event 'the second ball is red'. It is also the case that event $B \cap C$ is the same as event A. That is, $A = B \cap C$.

From the array, $P(A) = \frac{6}{20} = \frac{3}{10}$, $P(B) = \frac{12}{20} = \frac{3}{5}$ and $P(C) = \frac{12}{20} = \frac{3}{5}$.

We note that $P(A) = \frac{3}{10}$ and $P(B) \times P(C) = \frac{9}{25}$.

So in this case $P(B \cap C) \neq P(B) \times P(C)$.

The events B and C are not independent. The result of the second draw is not independent of the result of the first. This should not be a surprise since, if the first ball drawn is red, there are two reds and two blacks left. On the other hand, if the first ball drawn is black, there are three reds and one black left.

To apply the multiplication principle in this case, we need to calculate $P(C \mid B)$. We note that if B has occurred then there are four balls left; two of them are red and two black.

Therefore, $P(C \mid B) = \frac{2}{4} = \frac{1}{2}$

$$\begin{aligned} P(B \cap C) &= P(B) \times P(C \mid B) \\ &= \frac{3}{5} \times \frac{1}{2} \\ &= \frac{3}{10} \\ &= P(A), \text{ as expected} \end{aligned}$$

Tree diagrams and probability

Drawing an array containing all possibilities is only sensible for small cases such as those dealt with earlier in this section. For example, if one draws two cards from a pack of cards without replacement then there are 52×51 possibilities.

Another useful method for calculating probabilities is a **tree diagram**.

Tree diagrams were introduced in *ICE-EM Mathematics Year 9* as a means of methodically listing the sample space of a multi-stage experiment involving equally likely outcomes. However, they can also be used to visually support the multiplication rule of probability in any multi-stage experiment via branches of the tree.

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Consider again the experiment of drawing two balls from a bag containing 3 red and 2 black balls without replacement.

On the first draw, the probability of choosing a red is $\frac{3}{5}$ and the probability of choosing a black $\frac{2}{5}$.

If a red ball is chosen first, then there are 2 red and 2 black balls to choose from on the second draw. This means we can determine the following conditional probabilities:

$$P(\textit{red second} \mid \textit{red first}) = \frac{2}{4} = \frac{1}{2}$$

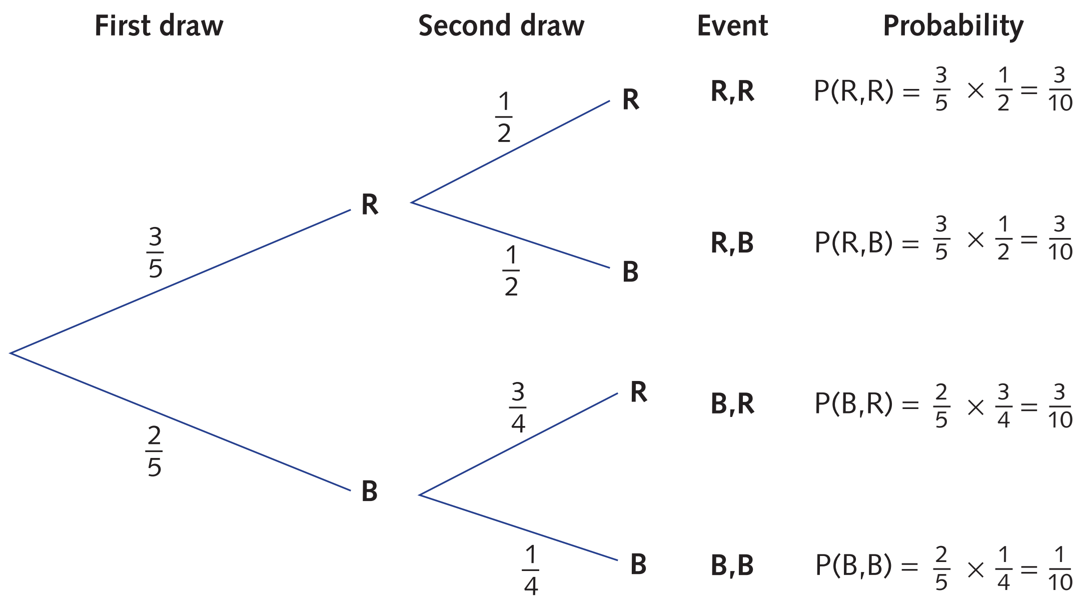
$$P(\textit{black second} \mid \textit{red first}) = \frac{2}{4} = \frac{1}{2}$$

If a black ball is chosen first, then there are 3 red and 1 black ball to choose from on the second draw. Therefore:

$$P(\textit{red second} \mid \textit{black first}) = \frac{3}{4}$$

$$P(\textit{black second} \mid \textit{black first}) = \frac{1}{4}$$

Overall, there are four possible events involving colour in this two-stage event; red first – red second (R, R), red first – black second (R, B), black first – red second (B, R) and black first – black second (B, B). These events are represented in the four branch tree diagram below. Respective probabilities are placed along each arm, with conditional probabilities placed along the second arms, as shown. Event probability is then determined by multiplying probabilities along the respective branch. This is justified by the multiplication rule of probability.



A 20 branch tree diagram (5×4) could have been used to model the situation in a similar manner to the array on page 464, in terms of equally likely outcomes $(R_1, R_2, R_3, B_1, B_2)$. This however, would have been far less efficient.

Consider the following example that does *not* concern sampling.

Example 14

A coin is tossed three times and the uppermost face is recorded each time.

a List the sample space.

b Find the probability of obtaining two heads.

Solution

a $\xi = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

b **Method 1**

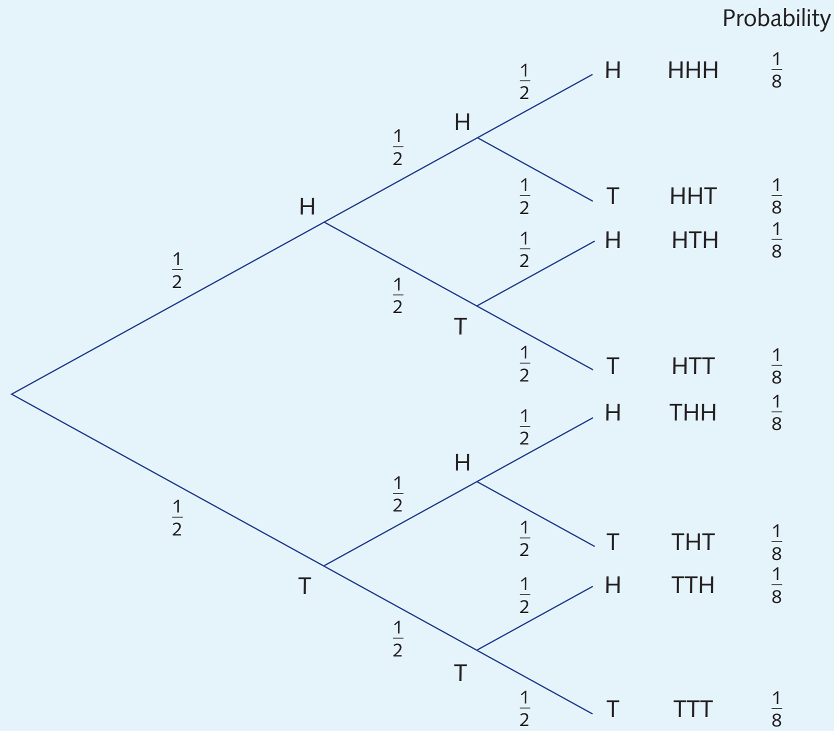
Let A be the event two heads are obtained.

$A = \{HHT, HTH, THH\}$

$$P(A) = \frac{3}{8}$$

Method 2

We draw a tree diagram. It is clear in this case that each throw is independent of each of the others, so probabilities along successive branches remain fixed.



The probability of a head or a tail at each stage is $\frac{1}{2}$.

The three required arms of the tree are HHT, HTH and THH with two heads.

$$P\text{ (two heads)} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{3}{8}$$

Note: The above example involves equally likely outcomes, so a tree diagram may be only useful for methodically listing the sample space.

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Cards

A deck of cards consists of 52 cards –13 Hearts, 13 Diamonds, 13 Spades and 13 Clubs. Each suit consists of a 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K and A.

If one card is drawn, then the probabilities are easy to calculate. For example:

$$P(\text{King of Hearts is drawn}) = \frac{1}{52},\ P(\text{a Heart is drawn}) = \frac{13}{52} = \frac{1}{4}$$

$$P(\text{a King is drawn}) = \frac{4}{52} = \frac{1}{13}$$

If two cards are drawn *with replacement* then, for example:

$$P(\text{two Hearts are drawn}) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

On the other hand, if two cards are drawn *without replacement*, then the conditional probabilities vary depending on the first drawn card. For example, if the King of Hearts is drawn, then on the second draw:

$$P(\text{Heart}) = \frac{12}{51},\ P(\text{Club}) = \frac{13}{51},\ P(\text{Spade}) = \frac{13}{51} \text{ and } P(\text{Diamond}) = \frac{13}{51}$$

In this type of situation, a tree diagram is useful for assisting with the application of the multiplication principle.

Example 15

A card is taken at random from a pack and not replaced. A second card is then taken from the pack and the result noted.

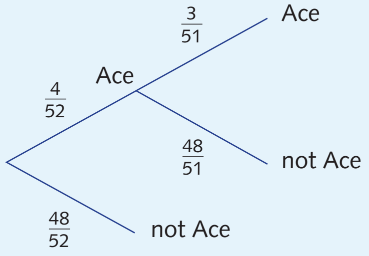
a What is the probability that the two cards are Aces?

b What is the probability that the two cards are Hearts?

c What is the probability of obtaining one Heart and one Club?

Solution

a



$$P(\text{two Aces}) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

(continued over page)

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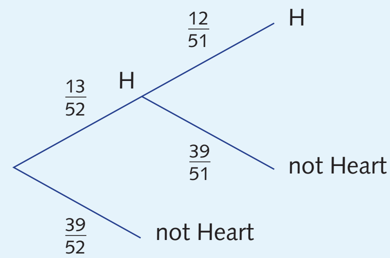
b $P(\text{two Hearts}) = \frac{13}{52} \times \frac{12}{51} = \frac{1}{17}$



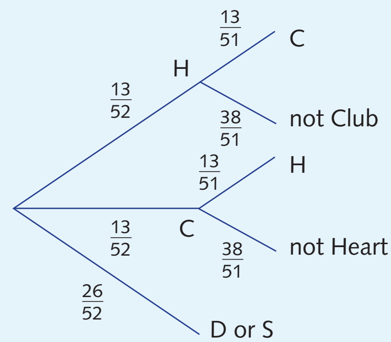
c $P(\text{one Heart and one Club})$

$= P(\text{the first card is a Heart and the second card a Club})$

$+ P(\text{the first card is a Club and the second card a Heart})$

$= \frac{13}{52} \times \frac{13}{51} + \frac{13}{52} \times \frac{13}{51}$

$= \frac{13}{102}$



Exercise 15E

Example 14

1 A card is drawn at random from a pack of 52 playing cards. It is replaced and the pack is shuffled. A second card is then drawn. What is the probability of the event:

a both cards are Diamonds?

b neither card is a Diamond?

c only one of the cards is a Diamond?

d only the first card is a Diamond?

e only the second card is a Diamond?

f at least one of the cards is a Diamond?

Example 15

2 One card is drawn at random from a pack of 52 playing cards. It is not replaced. A second card is then drawn. What is the probability of the event:

a both cards are Diamonds?

b neither card is a Diamond?

c only one of the cards is a Diamond?

d only the first card is a Diamond?

e only the second card is a Diamond?

f at least one of the cards is a Diamond?

3 A bag contains 8 red balls and 5 black balls. A ball is taken and its colour noted. It is not replaced. A second ball is taken and its colour noted. Find the probability of obtaining:

a a red ball followed by a black ball

b a red and a black ball

c two red balls

d two black balls

4 Giorgia has 5 red ribbons, 3 blue ribbons and 6 green ribbons in a drawer. Giorgia randomly takes one ribbon out and then a second (no replacement). What is the probability that she obtains:

a 2 red ribbons?

b a red and a blue ribbon?

c a green and a red ribbon?

d 2 blue ribbons?

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5 Cube A has 5 red faces and 1 white face, cube B has 3 red faces and 3 white faces and cube C has 2 red faces and 4 white faces. The 3 cubes are tossed. What is the probability of:

a 3 red faces uppermost?

b 3 white faces uppermost?

c red with A and B and white with C?

d red with A and white with B and C?

e at least 1 red face uppermost?

6 A bag of confectionary has 27 chocolates and 35 toffees in it.

Sanjesh takes out one item from the bag and then a second without replacing the first. What is the probability of obtaining:

a two chocolates?

b two toffees?

c a chocolate and a toffee?

7 A bag contains 15 blue balls and 10 green balls. A ball is taken out and its colour noted. It is replaced. A second ball is taken out and its colour noted. Find the probability of obtaining:

a a green ball followed by a blue ball

b a green and a blue ball

c two green balls

d two balls of the same colour

8 A coin is tossed four times. What is the probability of:

a four heads?

b four tails?

c head, tail, head, tail, in that order?

d heads in the first three tosses but not in the fourth?

e a head in at least one of the four tosses?

9 A die is tossed three times. What is the probability of obtaining:

a three 6s?

b no 6s?

c three odd numbers?

d three even numbers?

e a 6 in the first two tosses only?

f a 6, not a 6, and a 6 in that order?

10 A box contains chocolates and toffees with green and red wrapping. The number of each type of confectionary and its wrapping colour is given below. One item is removed from the box.

	Green wrapping	Red wrapping
Chocolate	48	60
Toffee	20	25

Find the probability of obtaining an item with green wrapping and the probability of obtaining an item with green wrapping given that it is a chocolate.

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Review exercise

1 **a** A bag contains 2 red marbles and 3 black marbles. Two marbles are drawn from the bag. Each marble is replaced after it is drawn and the bag is shaken. Find the probability of selecting:

i two black marbles

ii two red marbles

iii a red and a black marble

iv at least one red marble

b From the same bag of marbles as in part **a**, two marbles are selected without replacing the first marble. Find the probability that the selection contains:

i two red marbles

ii a black and a red marble

iii two marbles of the same colour

2 Discs with the digits 0 to 9 are placed in a box. A disc is drawn at random, its digit is recorded, then it is replaced in the box. A second disc is then drawn and its digit is recorded. Find the probability:

a that the two digits are the same

b of drawing an even digit and an odd digit

c that the first digit is a 6 and the second digit is odd

3 A number is chosen by throwing a die in the shape of a regular tetrahedron with the numbers 2, 4, 6, 8 on the faces, and noting the number that is face down. A second number is obtained by throwing a fair six-sided die and noting the number on its uppermost face. These two numbers are then added together.

a Complete the table, showing all possible outcomes.

		Roll of the six-sided die					
		1	2	3	4	5	6
Roll of the four-sided die	2	3					
	4						
	6			9			
	8					13	

b Find the probability of:

i A: the event in which the total score exceeds 8

ii B: the event in which the total score is 10

c If C is the event in which the total score is less than 13, find $A \cap C$ and $P(A \cap C)$.

d Are the events A and C independent? Justify your answer.

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4 In a group of 100 students, 60 study mathematics, 50 study physics and 20 study both mathematics and physics. One of the mathematics students is selected at random. What is the probability that he also studies physics?

5 An odd digit is selected at random and then a second odd digit is chosen at random (they may be equal). What is the probability that the sum of the two digits is greater than 10?

6 An urn contains 8 red marbles, 7 white marbles and 5 black marbles. One marble is drawn at random from the urn. What is the probability that it is:

a red or black?

b not white?

c neither black nor white?

7 A cube has 4 red faces and 2 white faces; another has 3 red and 3 white; another 2 red and 4 white. The 3 cubes are tossed. What is the probability that there are at least 2 red faces uppermost?

8 A number is selected at random from the integers 1 to 100 inclusive. What is the probability it is:

a divisible by 3?

b divisible by 7?

c divisible by both 3 and 7?

d divisible by 3 but not by 7?

e divisible by 7 but not by 3?

Challenge exercise

1 A tennis team consists of 4 players who must be chosen from a group of 6 boys and 5 girls.

a Find the number of ways the team can be picked:

i without restriction

ii with 2 boys and 2 girls in the team

iii if at least 2 girls must be in the team

iv if no more than 2 boys are to be in the team

b If the team consists of 2 girls (Joanne and Freda) and two boys (Peter and Stuart) from which two pairs of mixed doubles must be selected, how many ways can the mixed doubles pairs be selected?

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c During a particular tournament, the probability of the first mixed doubles pair winning each match it plays is 0.4 and the probability of the second pair winning each match it plays is 0.7.

i Find the probability that both pairs win their first match.

ii Find the probability that the first pair wins 2 and loses 1 of their first 3 matches.

iii Find the probability that the second pair wins their second and third match, given that they won their first match.

2 A box contains 35 apples, of which 25 are red and 10 are green. Of the red apples, five contain an insect and of the green apples, one contains an insect. Two apples are chosen at random from the box. Find the probability that:

a both apples are red and at least one contains an insect

b at least one apple contains an insect given that both apples are red

c both apples are red given that at least one is red

3 Four-digit numbers are to be formed from the digits 4, 5, 6, 7, 8, 9.

a For each of the cases below, find how many four-digit numbers can be formed if:

i any digit may appear up to four times in the number

ii no digit may appear more than once in the number

iii there is at least one repeated digit, but no digit appears more than twice in a number

b Find the probability that a four-digit number chosen at random from the set of numbers in part **a i** contains at least one six.

4 Each of three boxes has two drawers. One box contains a diamond in each drawer, another contains a pearl in each drawer, and the third contains a diamond in one drawer and a pearl in the other. A box is chosen, a drawer is opened and found to contain a diamond. What is the probability that there is a diamond in the other drawer of that box?

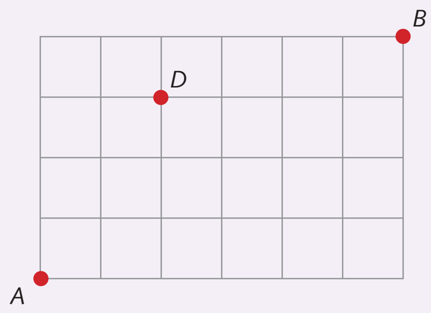
5 If you hold two tickets in a lottery for which n tickets were sold and 5 prizes are to be given, what is the probability that you will win at least one prize?

6 If A and B are mutually exclusive events, show that

$$P(A \mid A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

7 **a** In the diagram shown, in how many different ways can you get from A to B if you are only allowed to move to the right and upwards?

b What is the probability that a random journey from A to B passes through the point D? (Only moves to the right and up are allowed.)



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