CHAPTER

8

Measurement and Geometry

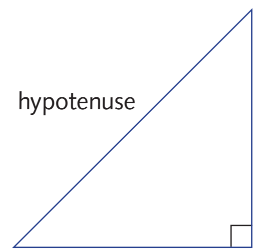
Pythagoras' theorem

Triangles have been a source of fascination for people over many millenniums. You will learn some of the intriguing facts about triangles in this and subsequent chapters.

Pythagoras' theorem is an amazing result that relates the lengths of the sides of a right-angled triangle. It is one of the most widely used results in all of mathematics. In this chapter, we will learn about why it is so useful and important, and how to prove it.

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7 © Brown et al. 2017 Cambridge University Press

Introduction



Among the set of all triangles, there is a special class, known as **right-angled triangles** or **right triangles**. A right-angled triangle has one and only one right angle, and the other two angles are complementary acute angles.

The longest side in a right triangle is called the **hypotenuse**. This name is derived from the Greek word 'teino', which is related to the word 'tension' in English, and has the basic meaning of *to stretch*.

It was well known early in the ancient world that if the two sides adjacent to the right angle have lengths 3 units and 4 units, then the hypotenuse has a length of 5 units.

A number of questions arise:

- Is there anything special about the numbers 3, 4 and 5?
- Are there other sets of three numbers that form the side lengths of a right-angled triangle?
- Is there some relationship between the lengths of the sides of a right-angled triangle?

These were questions that the Greeks examined in some detail. They found and proved a remarkable statement that relates the lengths of the three sides of a right-angled triangle. This work was done by the Pythagorean school of Greek thinkers, and so is known today as **Pythagoras' theorem**.

A **theorem** is a statement of an important mathematical truth.

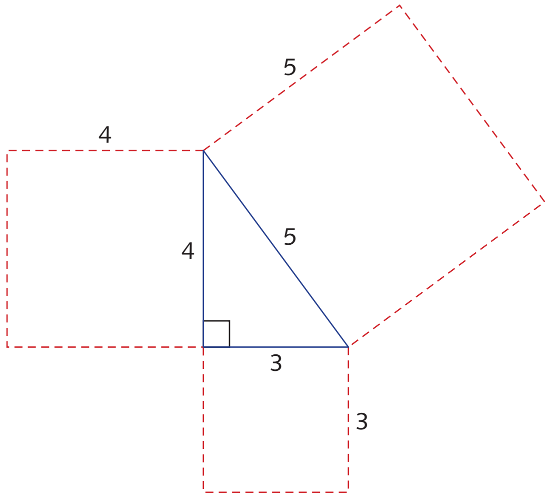
Exercise

Draw a line AB of length 12 cm, leaving yourself some space above the line you have drawn. Using a set square, draw a line AC at right angles to AB, of length 5 cm. Now join C and B to form a right-angled triangle and measure the length of the hypotenuse. You should find that the answer is 13 cm.

Exercise

Repeat the above construction, this time with $AB = 8$ cm and $AC = 6$ cm. Measuring the third side, you should find it is 10 cm. This is because this triangle is an enlargement of the 3, 4, 5 triangle.

To understand what Pythagoras and the Greeks proved, we need to look at the *squares* of the lengths of the sides of a right-angled triangle.



ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

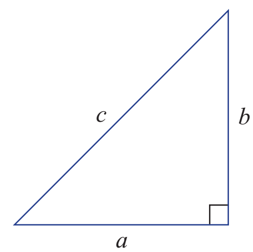
Lengths	3	4	5	**Lengths**	5	12	13	**Lengths**	6	8	10
Squares	9	16	25	**Squares**	25	144	169	**Squares**	36	64	100

Can you see the pattern?

- In the $3, 4, 5$ triangle, $9 + 16 = 25$, that is, $3^2 + 4^2 = 5^2$.
- In the $5, 12, 13$ triangle, $25 + 144 = 169$, that is, $5^2 + 12^2 = 13^2$.
- And in the $6, 8, 10$ triangle, $36 + 64 = 100$, that is, $6^2 + 8^2 = 10^2$.

Pythagoras' theorem

- The square of the length of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the lengths of the other two sides.
- In symbols, if we call the length of the hypotenuse c, and call the lengths of the other two sides a and b, then $a^2 + b^2 = c^2$.

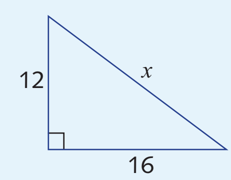


8A Applying Pythagoras' theorem

We can use Pythagoras' theorem to find the length of the hypotenuse of a right-angled triangle, without measuring, if we know the two shorter sides.

Example 1

Find the length of the hypotenuse in the right-angled triangle opposite.



Solution

Let x be the length of the hypotenuse.

Pythagoras' theorem says that:

$$\begin{aligned} x^2 &= 12^2 + 16^2 \\ &= 144 + 256 \\ &= 400 \end{aligned}$$

so $x = 20$

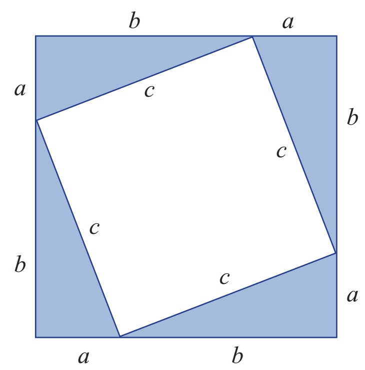
Remember that 400 is the square of the length of the hypotenuse, so we have to take the square root in the final step to find the correct value of x.

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

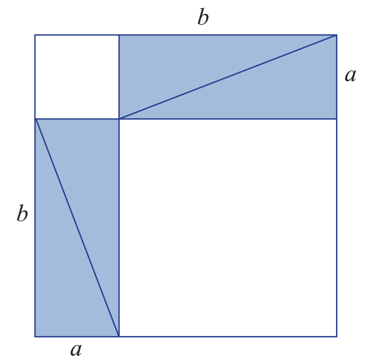
Why the theorem is true

To establish a theorem, we must give a valid proof. There are many ways to prove Pythagoras' theorem, but some are easier than others. Here is a simple geometric demonstration of why the result is true.

The diagram below shows four identical right-angled triangles, each with hypotenuse length c and the other sides of length a and b. They have been arranged within a large square of side length $a+b$, leaving uncovered the white square of side length c.



The triangles and the square, with side length c, can be rearranged within the same large square. The square at the top left has area a^2 and the square at the bottom right has area b^2.

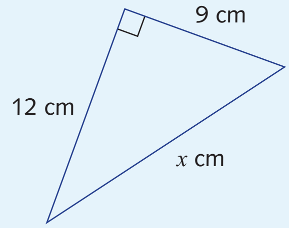


The area of the white square in the first diagram must be equal to the sum of the areas of the two white squares in the second diagram.

From this we can see that $a^2 + b^2 = c^2$.

Example 2

Use Pythagoras' theorem to find the value of x in the triangle below.



ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

Solution

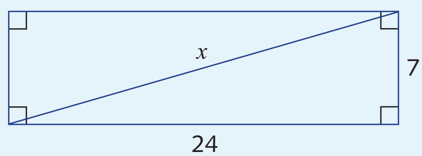
Pythagoras' theorem says that:

$$\begin{aligned} x^2 &= 9^2 + 12^2 \\ &= 81 + 144 \\ &= 225 \end{aligned}$$

so $x = 15$

Example 3

Find the length of the diagonal in the rectangle below.



Solution

The diagonal divides the rectangle into two right-angled triangles. Pythagoras' theorem tells us that:

$$\begin{aligned} x^2 &= 7^2 + 24^2 \\ &= 49 + 576 \\ &= 625 \end{aligned}$$

so $x = 25$

Exercise 8A

1 Write down these squares.

a 9^2 **b** 11^2

c 7^2 **d** 15^2

2 Find the square root of each of these numbers.

a 64 **b** 144

c 196 **d** 900

3 Calculate:

a $7^2 + 24^2$ **b** $9^2 + 12^2$

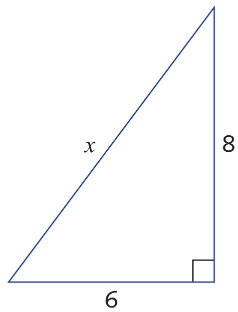
c $20^2 - 16^2$ **d** $17^2 - 15^2$

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

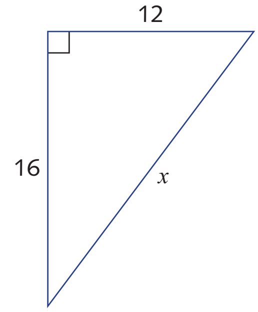
Example 1, 2

4 Use Pythagoras' theorem to find the length of the missing side in each of these triangles.

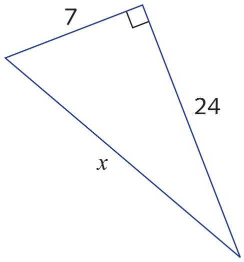
a



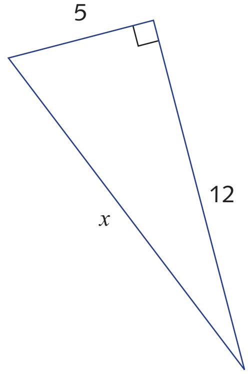
b



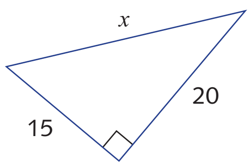
c



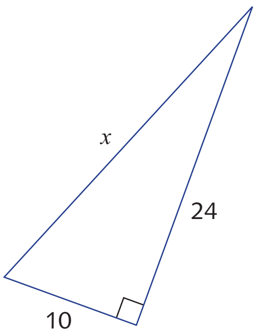
d



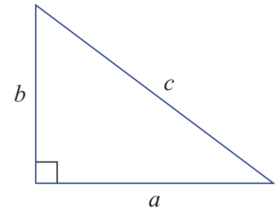
e



f

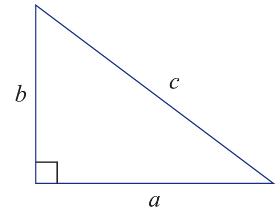


5 The triangle opposite is a right-angled triangle with hypotenuse of length c. The other two sides have lengths a and b. Apply Pythagoras' theorem to complete the table below. The first row is done for you. What pattern do you notice?



Row	a	b	a^2	b^2	$c^2 = a^2 + b^2$	c
1	3	4	9	16	25	5
2	6	8				
3	9	12				
4	12	16				
5	15	20				

6 The triangle opposite is a right-angled triangle with hypotenuse of length c. The other two sides have lengths a and b. Apply Pythagoras' theorem and complete the table below. What pattern do you notice?



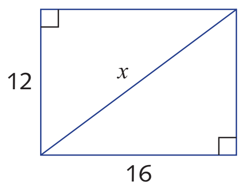
Row	a	b	a^2	b^2	$c^2 = a^2 + b^2$	c
1	3	4				
2	5	12				
3	7	24				
4	9	40				
5	11	60				

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

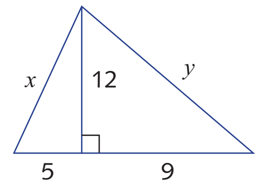
Example 3

7 Find the values of x, y and z in these diagrams.

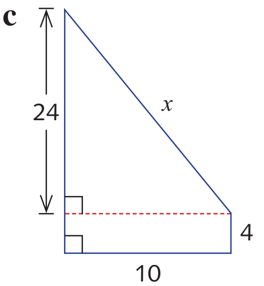
a



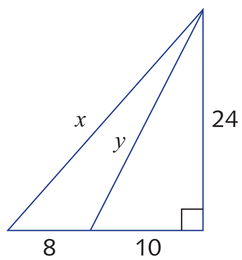
b



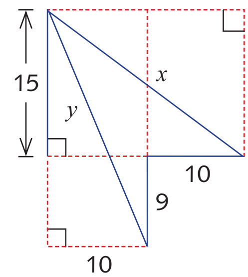
c



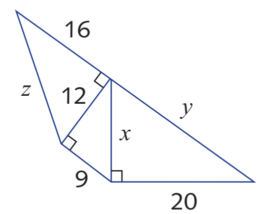
d



e



f

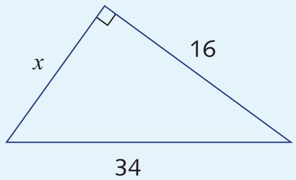


8B Finding a shorter side

We can use Pythagoras' theorem to find the length of one of the shorter sides in a right-angled triangle, given the other two sides.

Example 4

Find the length of the missing side, marked in the diagram as x.



Solution

Applying Pythagoras' theorem

$$x^2 + 16^2 = 34^2$$
$$x^2 + 256 = 1156$$
$$x^2 = 1156 - 256$$
$$= 900$$

so $\quad x = 30$ (taking the square root)

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

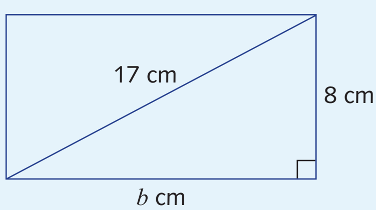
To make such calculations easier, here is a table of squares from 13 to 40. You should know the squares from 1 to 12 already.

Number	Square	Number	Square
13	169	27	729
14	196	28	784
15	225	29	841
16	256	30	900
17	289	31	961
18	324	32	1024
19	361	33	1089
20	400	34	1156
21	441	35	1225
22	484	36	1296
23	529	37	1369
24	576	38	1444
25	625	39	1521
26	676	40	1600

Example 5

A rectangle has width 8 cm and diagonal 17 cm.

What is its length?



Solution

Let b be the length, measured in cm. Then:

$b^2 + 8^2 = 17^2$ (Pythagoras' theorem)

$b^2 + 64 = 289$

$b^2 = 289 - 64$

$= 225$

so $b = 15$

The length of the rectangle is 15 cm.

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

Converse of Pythagoras' theorem

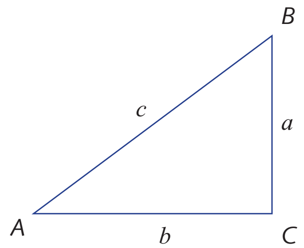
We have seen that in a right-angled triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.

Now suppose that we have a triangle in which the square of the length of the longest side equals the sum of the squares of the lengths of the other two sides. Does it follow that the triangle is right-angled? The answer is yes, and this result is called the **converse** of Pythagoras' theorem.

We will be able to prove this after we have studied congruence in Chapter 12.

Theorem

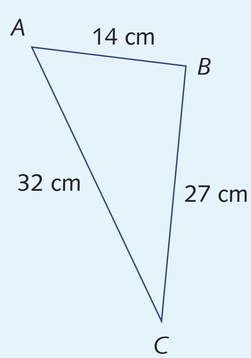
If $c^2 = a^2 + b^2$, then $\angle ACB$ is a right angle.



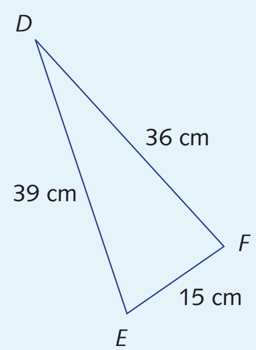
Example 6

Which of the triangles below are right-angled triangles? Name the right angle in each case.

a



b



Solution

In each case, let a and b be the lengths of the two shorter sides of the triangle and let c be the length of the longest side.

If $a^2 = b^2 + c^2$, then the triangle is a right-angled triangle, with the right angle opposite the hypotenuse.

a
$$14^2 + 27^2 = 925$$
$$32^2 = 1024$$
$$14^2 + 27^2 \neq 32^2$$

The triangle is not right-angled.

b
$$15^2 + 36^2 = 1521$$
$$39^2 = 1521$$
$$15^2 + 6^2 = 39^2$$

Triangle DEF is right-angled, with $\angle DFE = 90°$.

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

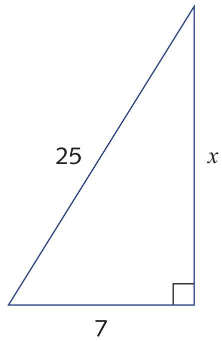
Cambridge University Press

Exercise 8B

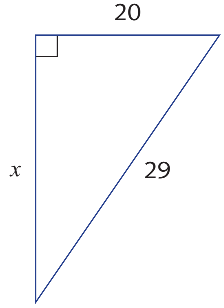
Example 4

1 Use Pythagoras' theorem to find the value of x in each triangle.

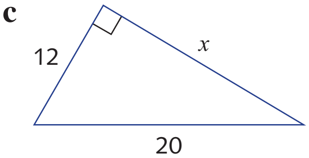
a



b



c



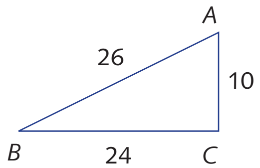
Example 5

2 A rectangle has one side of length 20 cm and its diagonal has length 25 cm. Find the length of the other side.

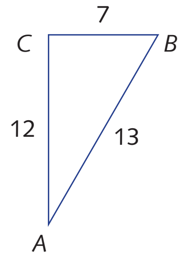
Example 6

3 Decide whether each of the triangles below is right-angled. If it is, name the right angle.

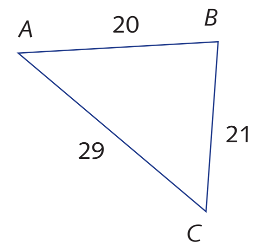
a



b

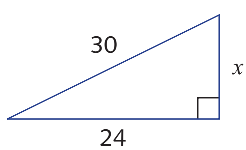


c

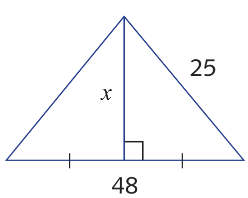


4 Find the values of the pronumerals in these diagrams.

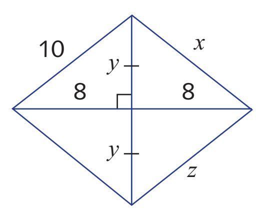
a



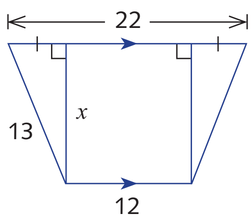
b



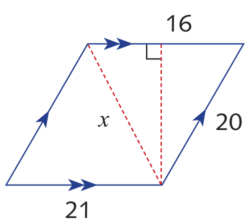
c



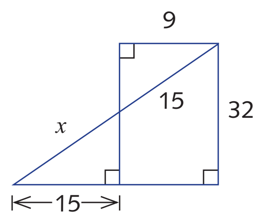
d



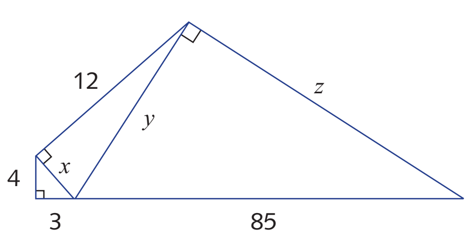
e



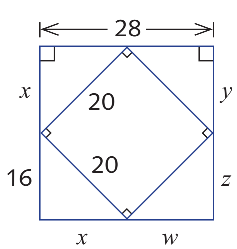
f



g



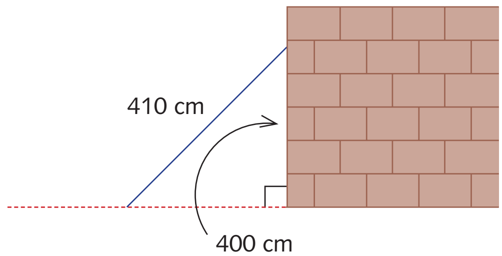
h



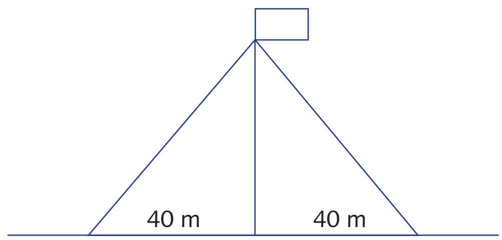
ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

5 The sides of a rectangular table are 150 cm and 360 cm. What is the length of the diagonal?

6 A ladder of length 410 cm is leaning against a wall. It touches the wall 400 cm above the ground. What is the distance between the foot of the ladder and the wall?

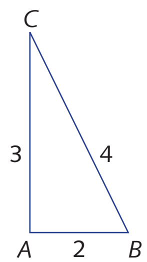


7 A flagpole is supported by two cables of equal length as shown below. The total length of the cables is 100 m. Each cable is attached to a point on level ground 40 m away from the foot of the flagpole. The dimensions of the flag are $1.2 \text{ m} \times 2.4 \text{ m}$. Calculate the height to the top of the flag.

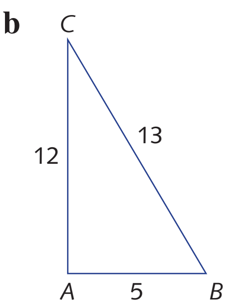


8 Which of the triangles below are right-angled triangles?

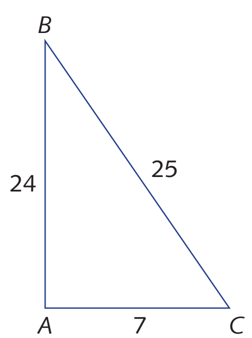
a



b



c



9 A ship sails 32 kilometres north and then 24 kilometres east. How far is it from its starting point?

10 **a** Find the square of the length of the diagonal of a square of side 1 cm.

b Find the square of the length of the diagonal of a square of side length:

i 2 cm **ii** 3 cm **iii** 5 cm **iv** n cm

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

8C Irrational numbers

You may have noticed that in the final step in each example in Section 8B we arrived at a perfect square, such as 25, 169 or 625. We were then able to take the square root and arrive at a whole-number answer. In many cases this will not happen.

The square root of a number that is not a perfect square cannot be written as a fraction, and is an example of what is called an **irrational number**.

The square root of 2 is an example of an irrational number. Since $1^2 = 1$ and $2^2 = 4$, the square root of 2, which is written as $\sqrt{2}$, must be somewhere between 1 and 2.

Although we cannot express the square root of 2 in exact decimal form, we can find an **approximation** to it. In fact, $\sqrt{2} \approx 1.414$, correct to 3 decimal places, and $\sqrt{2} \approx 1.41$ correct to 2 decimal places.

When we use Pythagoras' theorem to solve a problem, we often need to find the square root of a number that is not a perfect square. We can do this approximately by using a calculator, or by looking up a table.

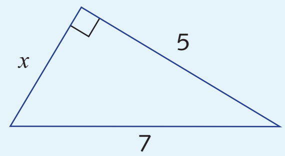
Here is a list of square roots of the numbers from 2 to 29. When the square root is not a whole number, we have given an approximation correct to 2 decimal places.

Number	Square root (2 dec. pl.)	Number	Square root (2 dec. pl.)
2	1.41	16	4
3	1.73	17	4.12
4	2	18	4.24
5	2.24	19	4.36
6	2.45	20	4.47
7	2.65	21	4.58
8	2.83	22	4.69
9	3	23	4.80
10	3.16	24	4.90
11	3.32	25	5
12	3.46	26	5.10
13	3.61	27	5.20
14	3.74	28	5.29
15	3.87	29	5.39

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

Example 7

Find the length, correct to 2 decimal places, of the missing side in the right-angled triangle.



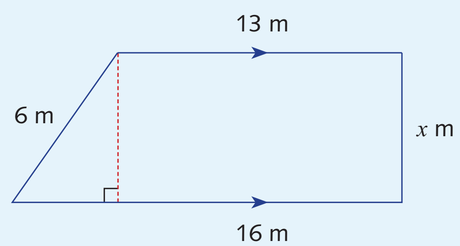
Solution

By Pythagoras' theorem:

$$\begin{aligned} x^2 + 5^2 &= 7^2 \\ x^2 + 25 &= 49 \\ x^2 &= 49 - 25 \\ &= 24 \\ x &= \sqrt{24} \\ &\approx 4.90 \text{ (using the table)} \end{aligned}$$

Example 8

Find the height of the trapezium below.



Solution

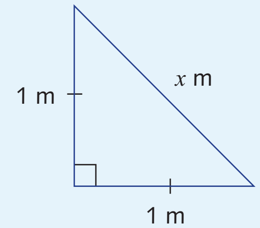
By Pythagoras' theorem:

$$\begin{aligned} x^2 + (16 - 13)^2 &= 6^2 \\ x^2 + 9 &= 36 \\ x^2 &= 36 - 9 \\ &= 27 \\ x &= \sqrt{27} \\ &\approx 5.20 \text{ (using the table)} \end{aligned}$$

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

Example 9

Find the length of the hypotenuse in the triangle to the right, correct to 2 decimal places. What type of triangle is it?



Solution

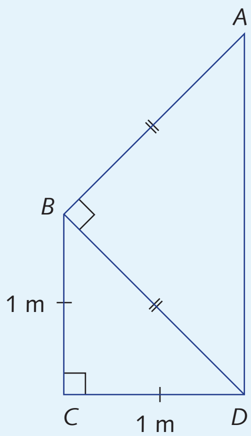
By Pythagoras' theorem:

$$1^2 + 1^2 = x^2$$
$$x^2 = 2$$
$$x = \sqrt{2}$$
$$\approx 1.41 \text{ (to 2 decimal places)}$$

This triangle is an isosceles right-angled triangle. The length of the hypotenuse is 1.41 m, correct to 2 decimal places.

Example 10

Find the length of AD in the diagram to the right.



Solution

By Pythagoras' theorem:

$$1^2 + 1^2 = BD^2$$
$$\text{so} \quad BD^2 = 2$$
$$\text{Also,} \quad BD^2 + AB^2 = AD^2$$
$$\text{so} \quad 2 \times BD^2 = AD^2 \quad (\text{isosceles } \Delta ABD)$$
$$\text{Hence,} \quad 2 \times 2 = AD^2$$
$$AD^2 = 4$$
$$\text{so} \quad AD = 2$$

The length of AD is 2 m.

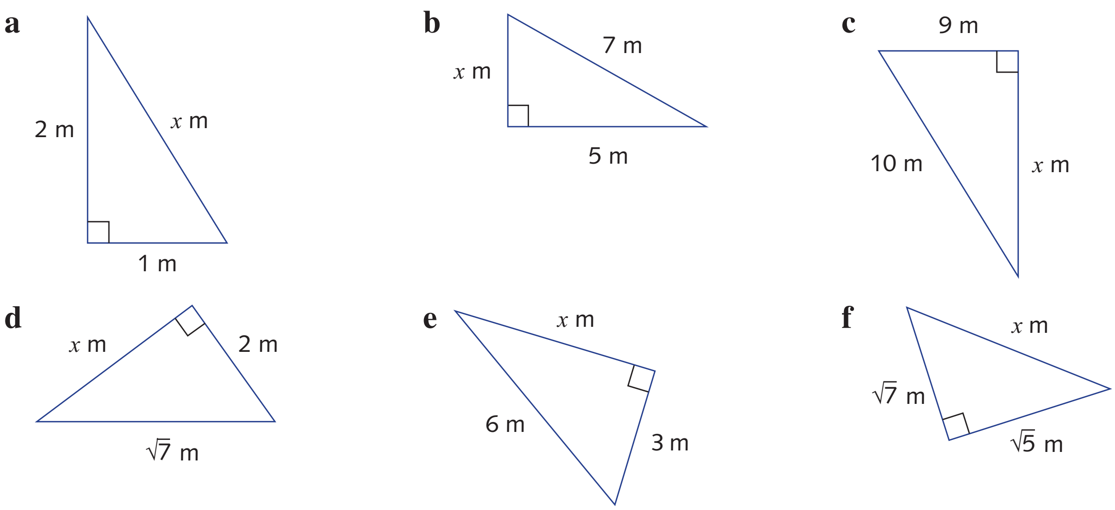
ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7 Cambridge University Press

Exercise 8C

Note: In this exercise, you should use the table on page 206 to work out any square roots you need.

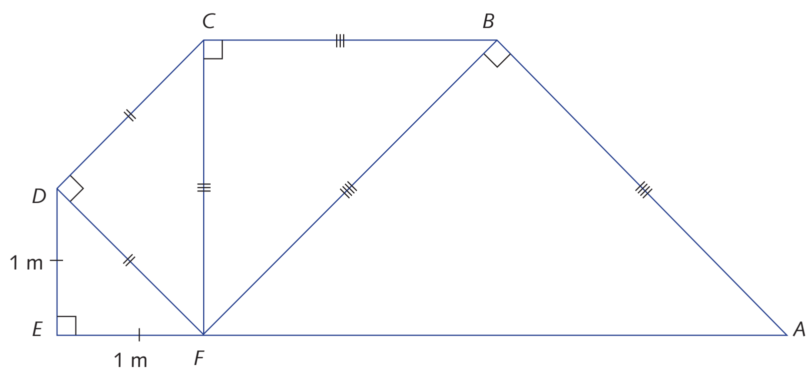
Example 7, 9

1 Use Pythagoras' theorem to find the length of the unknown side in each diagram below. Round your answers correct to 2 decimal places.



Example 10

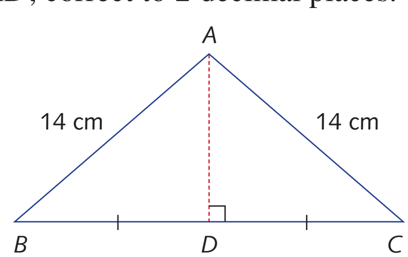
2 Find the length of AF in the diagram below. Only take the square root in the final step.



3 Find the length of the diagonal of a rectangle with side lengths 2 cm and 5 cm. Round off your answer correct to 2 decimal places.

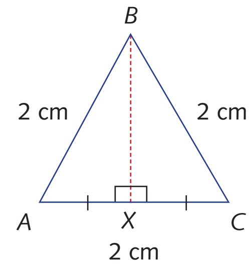
4 Find the perimeter of a square with diagonal length 6 cm. Round your answer to 2 decimal places.

5 An isosceles triangle ABC has equal sides $AB = AC = 14$ cm. The side BC is 26 cm. A line AD is drawn at right angles from A to BC. This divides BC into two intervals of equal length. Find the length of AD, correct to 2 decimal places.



ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

6 Triangle ABC is equilateral, with each side of length 2 cm. Find the length of BX.



7 The equal sides of a right-angled isosceles triangle are 3 cm in length. What is the length of the hypotenuse?

8 A cross-country runner runs 3 km west, then 2 km south and then 8 km east. How far is she from her starting point? Calculate your answer in kilometres, correct to 2 decimal places.

9 A rod 2.5 m long is leaning against a wall. The bottom of the rod is 1.5 m from the wall. How far up the wall does the rod reach?

10 A man starts from a point A and walks 2 km due north to a point B and then 5 km due west to a point C. How far is C from A? Calculate your answer in kilometres, correct to 2 decimal places.

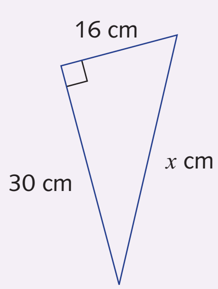
11 Amelia starts from a point A and walks 6 km due south to B. Amelia then walks due east to a point C which is 7 km from A. How far is C from B? Calculate your answer in kilometres, correct to 2 decimal places.

12 A right-angled isosceles triangle has a hypotenuse of length 6 cm. Find the lengths of the other two sides in centimetres, correct to 2 decimal places.

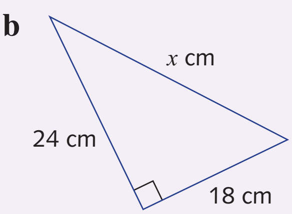
Review exercise

1 Find the value of x in each of these diagrams.

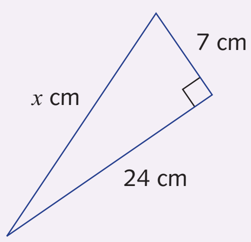
a



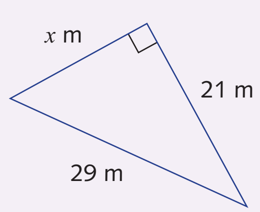
b



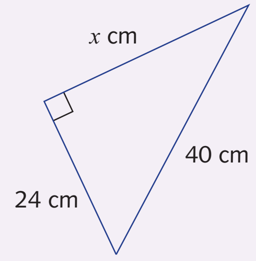
c



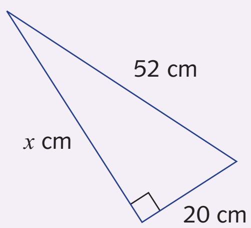
d



e



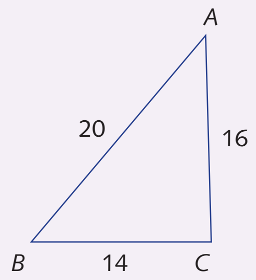
f



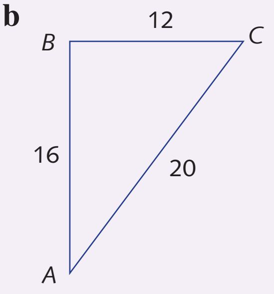
ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

2 Decide whether each of the triangles below is right-angled. If it is, name the right angle.

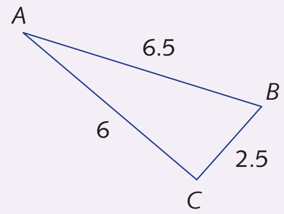
a



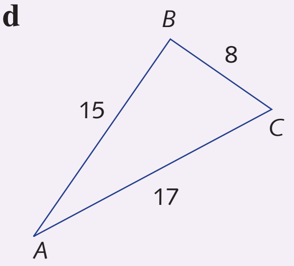
b



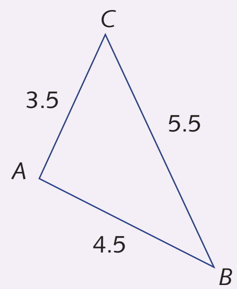
c



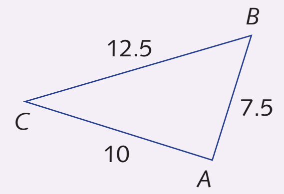
d



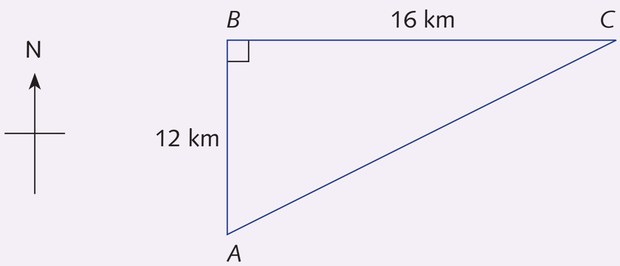
e



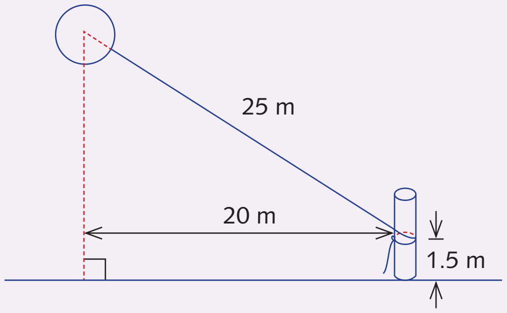
f



3 John starts at a point A and cycles 12 km due north. He turns east at point B, cycles 16 km and reaches point C. How far is he from his starting point?

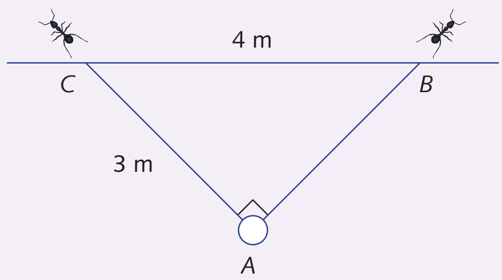


4 How high is the balloon off the ground if it is tied to a wooden pole at a height of 1.5 m by a cord of length 25 m?



ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

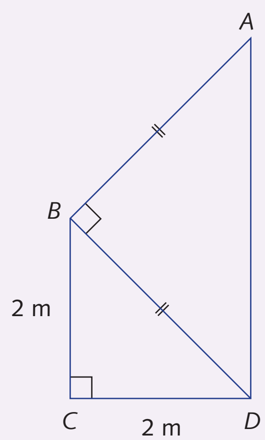
5 Two ants take different paths CA and BA, respectively, while walking back to their nest at point A. How much further does one ant walk than the other? Round your answer to 2 decimal places.



6 **a** A square has side length 3 cm. Find the length of the diagonal correct to 2 decimal places.

b The diagonal length of a square is 4 cm. Find the side length of the square.

7 Find the length of AD in the diagram below.



Challenge exercise

1 Show that the diagonal of a square of side length a is $a\sqrt{2}$.

2 What is the length of the longest diagonal in a cube with edges of length x m?

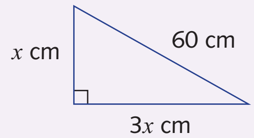
3 What is the area of an equilateral triangle with sides of length x m?

Hint: The height also divides the base into two equal parts.

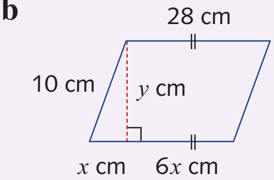
ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

4 Find the exact values of x and y in the following diagrams.

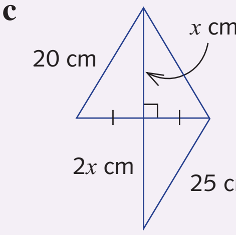
a



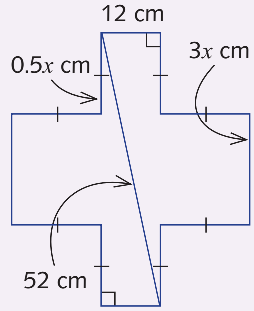
b



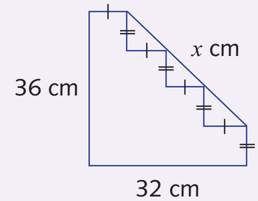
c



d



e

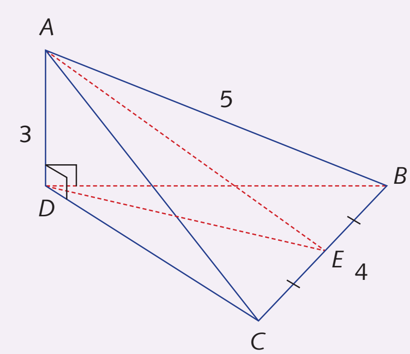


5 A right-angled triangle has hypotenuse of length 41 cm and one of the sides is 9 cm. Find the area of the triangle.

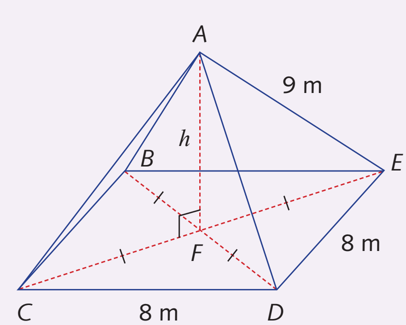
6 Draw an equilateral triangle of side length 4 cm. Find, correct to 2 decimal places, the height of this triangle and its area.

7 In the diagram to the right, ΔABC and ΔDBC are isosceles triangles, with $AB = AC$ and $DB = DC$.

AE and DE are the perpendicular bisectors of ΔABC and ΔDBC, respectively, so $\angle AEB = \angle DEB = 90°$. Find AE, DE and DB, correct to 2 decimal places.



8 A pyramid has a square base $BCDE$ of side length 8 m. The sloping edges of the pyramid are of length 9 m. Find the height h.



ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7 Cambridge University Press