CHAPTER

9

Review and problem-solving

ICE-EM Mathematics 8 3ed ISBN 978-1-108-40429-7

9A Review

Chapter 1: Whole numbers

1 The population of an ant colony was 14 000 at the start of January. By the end of the month, 256 ants had died, 89 ants had moved into the colony and 123 ants had moved out. The population at the end of January was 14 298. How many ants were born between the start and the end of the month?

2 If each of 37 students in 24 classes ate 5 pies, how many pies were eaten in total?

3 The cost of 20 packets of chips is \$9.50, while each packet costs 65 cents when sold individually. If you want to buy 40 packets of chips, which option is cheaper? How much do you save by taking the cheaper option?

4 The quotient arising from the division of a number by 53 is 29 and the remainder is 23. What is this number?

5 The product of two numbers is 67 267, and the smaller of them is 137. Find the sum of the two numbers.

6 Andrew has 347 marbles and Bashir has 135. Bashir wins 106 marbles from Andrew. How many marbles does each boy have now?

7 What number multiplied by 23 will give the same product as 391 multiplied by 37?

8 How many times must 43 be added to 1649 to give 4186?

9 At an election, there are three candidates, A, B and C. Candidate A obtains 19 878 more votes than C, and B obtains 12 435 more votes than C. Candidates A and B together obtain 137 187 votes. How many voted for C?

Chapter 2: Fractions and decimals

1 Simplify:

a $\dfrac{125}{35}$ **b** $\dfrac{29}{145}$ **c** $\dfrac{289}{136}$ **d** $\dfrac{8}{36}$ **e** $1\frac{15}{9}$

2 Evaluate the following additions and subtractions. Write each answer as a mixed numeral, with the fractional part in simplest form.

a $4\frac{1}{2}+3\frac{2}{5}$ **b** $7\frac{2}{3}-5\frac{1}{4}$ **c** $6\frac{5}{8}+2\frac{5}{6}$

d $9\frac{3}{10}-\dfrac{3}{4}$ **e** $5\frac{7}{12}+3\frac{4}{9}$ **f** $8\frac{2}{9}-6\frac{2}{3}$

3 Evaluate:

a $120\times\dfrac{75}{100}$ **b** $120\div\dfrac{75}{100}$ **c** $5\frac{1}{4}\times\dfrac{2}{3}$ **d** $5\frac{1}{4}\div\dfrac{2}{3}$

e $2\frac{7}{9}\times7\frac{1}{2}$ **f** $2\frac{7}{9}\div7\frac{1}{2}$ **g** $8\div1\frac{5}{7}$ **h** $8\times1\frac{5}{7}$

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4 A rectangular garden uses a straight river bank as its longer side. The other sides are fenced. If the longer side is 25 m, and the shorter side is $\frac{1}{3}$ of the longer side, what is the total length of fencing material?

5 If a year (365 days) is divided into lunar months of 28 days, how many months will there be in a year? Express your answer as a mixed numeral.

6 There were 60 people at a party. If $\frac{1}{5}$ of the people wore red, $\frac{1}{4}$ wore blue and the rest wore white, how many people wore white?

7 Horatia has saved \$1500. If she spends $\frac{2}{5}$ of the money on rent and shopping, and $\frac{2}{9}$ of the remaining money on petrol, how much does she have left?

8 Order each set of fractions from smallest to largest.

a $\frac{2}{3}, \frac{5}{6}, \frac{3}{4}, \frac{1}{2}, \frac{4}{5}$ **b** $\frac{11}{25}, \frac{4}{5}, \frac{13}{20}, \frac{1}{4}, \frac{6}{10}$

9 A balloon shop had 120 balloons in the morning, 24 of which were filled with helium.

a What fraction of the balloons were filled with helium?

b During the day, $\frac{1}{8}$ of the helium balloons were sold and 12 of the ordinary balloons were accidentally popped. What fraction of the balloons left at the end of the day were filled with helium?

10 If Alice can swim 100 m backstroke in 1 minute 20 seconds and she can swim butterfly at $\frac{9}{8}$ times her backstroke speed, and freestyle at $\frac{10}{9}$ times her butterfly speed, how fast can she swim 100 m freestyle?

11 Calculate:

a 1.2×3.8 **b** $1.3 + 2.5 \times 0.6$ **c** $7.1 + 4.5 \div 1.5$

d $13.6 - 7.85$ **e** $12.5 \times 3.1 - 0.8742$ **f** $92.8 \times 87.1 + 0.51$

Chapter 3: Review of factors and indices

1 Find all the factors of:

a 12 **b** 17 **c** 270 **d** 135 **e** 58 **f** 144

2 List all the prime numbers between 100 and 160.

3 Find the lowest common multiple (LCM) of:

a 5 and 35 **b** 18 and 48 **c** 170 and 66 **d** 12, 25 and 75

4 Find the highest common factor (HCF) of:

a 17 and 25 **b** 96 and 39 **c** 150 and 24 **d** 380, 190 and 650

5 Find the result of multiplying the HCF and the LCM of each pair of numbers below. Compare the result in each case to the product of the original pair. What do you notice?

a 24 and 76 **b** 102 and 54

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6 Write in the box the number needed to make each statement true.

a $2^8 \div 2^5 = 2^{\square}$ **b** $5^3 \times 5^2 = 5^{\square}$ **c** $4^3 \times 4^2 = 2^{\square}$

d $6^7 \div 3^7 = \square^7$ **e** $(3^2)^2 = \square$ **f** $\square^5 \div 6^5 = 1$

g $3^3 \times 3^{\square} = 81$ **h** $(5^{\square})^2 = 5^{16}$ **i** $(7^{13} \div 7^8)^{\square} = 7^{15}$

7 Evaluate:

a 8^2 **b** $\sqrt{121}$ **c** $\sqrt{56^2}$ **d** $\sqrt{5^4}$

e $(8^2)^2$ **f** $\sqrt{2209}$ **g** 25^3

8 Copy and fill in the missing digits so that the numbers in each set are:

a divisible by 3: 2567__, 199__4, __56

b divisible by 6: 7365__, 1__08, 222__

c divisible by 3, 4 and 5: 10__0, 12 198__, 59__20

9 Fill in the spaces so that the number __4__20 is divisible by 11.

10 Find the prime factorisations of:

a 210 **b** 5040 **c** 436 **d** 1386

11 **a** Evaluate:

i $2^2 \times 7^3$ **ii** 11×13^2 **iii** $2^3 \times 7 \times 9 \times 15$

b Write the product in part **a iii** as a product of powers of primes.

12 Using prime factorisation, find the HCF and LCM of:

a 12 960 and 3600 **b** 1750 and 1176

13 Calculate mentally:

a 25×16 **b** 125×32 **c** 75×40 **d** 5×874

Chapter 4: Negative numbers

1 Calculate:

a $-15 + 23$ **b** $25 - (-61)$ **c** $37 - 90 + 23$

d $23 - 28 - 6$ **e** $-213 - (-1129) + 81$ **f** $9 - (16 + (-28))$

2 Evaluate:

a $3 \times (-5 + 9) - 6$ **b** $-3 \times -9 - 12$

c $-21 \div 3 + 9 \times -2$ **d** $-12 \times (-4) - (-8 - 2) \times 5$

e $\dfrac{-95 - 6 \times 8}{8 - (-2)}$ **f** $\dfrac{28 \times 8}{5} - \dfrac{8 \times (-12)}{6}$

g $-\dfrac{6}{13} + \dfrac{5}{8}$ **h** $\dfrac{7}{12} \times \left(-\dfrac{9}{7}\right) + \dfrac{5}{3}$

i $2\dfrac{3}{4} - 6\dfrac{2}{5}$ **j** $12.8 - (-10.6) \times \dfrac{2}{3}$

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3 The maximum temperature for Thredbo today was −8°C, while Mount Kosciusko had a top temperature of −24.3°C. How much warmer than Mount Kosciusko was Thredbo today?

4 Plot these points on the Cartesian plane.

$A(-4, 2)$ $\quad B(3, -10)$ $\quad C(9, 8)$ $\quad D(-7, 0)$

5 Points A, B and C have coordinates (1, 0), (3, 2) and (5, 0), respectively. Plot the points and join them with straight lines. What is the area of the triangle ABC?

6 Arrange these numbers in increasing order.

$$-\frac{9}{8}, \frac{12}{8}, 0.2, -0.02, -\frac{7}{9}, \frac{1}{0.4}$$

7 Sam started the week with a debt of \$360.50. He worked hard and earned \$3100, but had \$250 stolen. He won \$80.20 in the lottery and had to pay an electricity bill of \$210. After paying his debt and his electricity bill, how much did Sam have left?

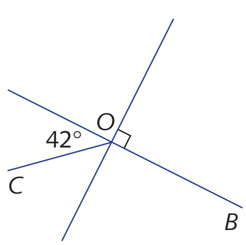
8 Mauna Kea is Hawaii's tallest volcano and the world's highest mountain when measured from its base on the sea floor. The height above sea level is 2405 metres. The base of Mauna Kea on the sea floor is 7286 metres below sea level. What is the height of Mauna Kea from sea floor to summit?

9 In a maths competition, Harry scored 20. Each correct answer was worth 10 marks. For each incorrect answer, 2.5 marks were deducted. If Harry answered 5 questions correctly, how many incorrect answers did he have?

Chapter 5: Review of geometry

1 Find the value of $\angle BOC$ in each diagram below.

a



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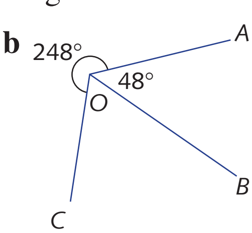

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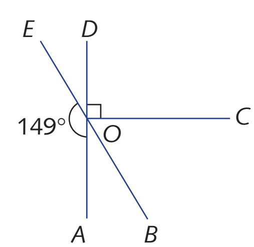
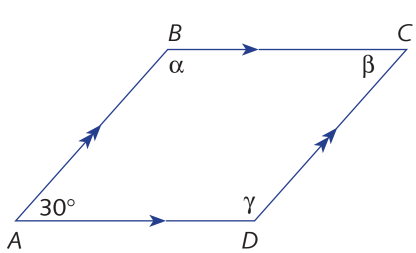


2 Find the values of α, β and γ in these diagrams.

a



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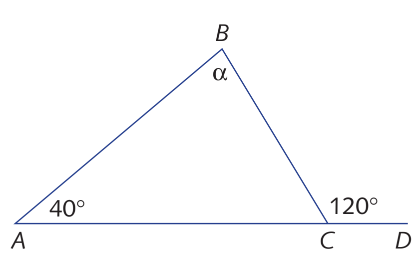

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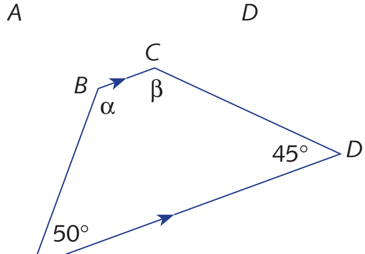
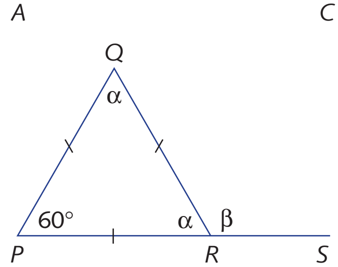


d



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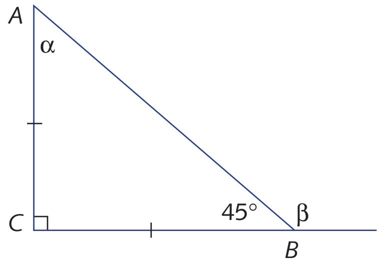
e



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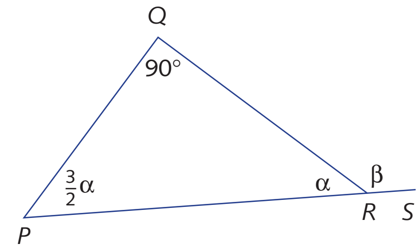
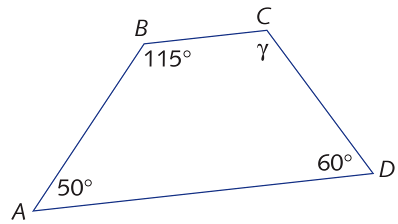


3 For each diagram below, write down an equation involving the pronumerals and then solve the equation.

a



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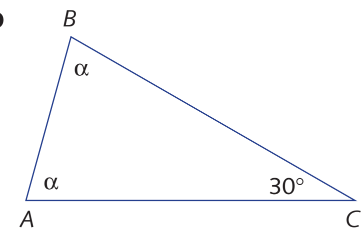

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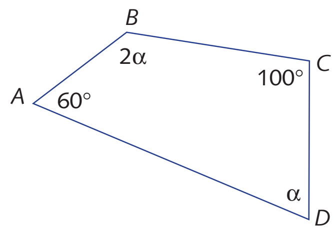

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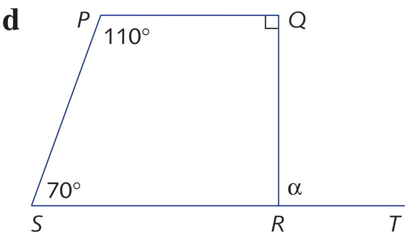

Chapter 6: Algebra – part 1

1 Evaluate each expression for $a = 2$, $b = 1$ and $c = 3$.

a $a + b + c$ **b** abc **c** $\frac{a+b}{c}$ **d** $\frac{a^2 + b^2}{a^2 - b^2}$

2 Substitute $x = -1$ for each expression and evaluate it.

a x^3 **b** x^5 **c** $(2x)^2$ **d** $(2x)^3$

e $-x$ **f** $x^3 + 5$ **g** $x^2 + 5$ **h** $\frac{2-x}{x^2+2}$

3 Solve:

a $m + 3 = 8$ **b** $m - 3 = 11$ **c** $2m = 8$

d $\frac{m}{4} = 6$ **e** $2x - 4 = 11$ **f** $\frac{x}{4} - 8 = 7$

g $5 - x = 11$ **h** $2x - 4 = -6$ **i** $\frac{x}{5} + 8 = -7$

4 Expand brackets.

a $2(x - 3)$ **b** $-3(x + 4)$ **c** $5(x - 2)$

d $a(x + 3)$ **e** $-3(x - 4)$ **f** $-4(3 - x)$

5 Collect like terms to simplify each of these expressions.

a $2x - 3y + 5x + 2y$ **b** $x - y + 2x + 4y$

c $5x - 7x + 6x$ **d** $3x - 11x + 8x$

e $2xy - 8xy + 6xy$ **f** $6x^2y - 2yx^2 + x^2y$

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6 Evaluate each expression for $a = -3$, $b = \frac{2}{5}$ and $c = 10$.

a $a + b - c$ **b** abc **c** $b^2(c + a)$ **d** $(c^2 + a^2) - 2b$

e $\frac{ca^2}{30}$ **f** $\frac{a^2b}{c - b}$ **g** $a^2 - b^2 - c$ **h** $\frac{a}{b}(a - c)$

7 Solve each equation.

a $2x + 32 = 6$ **b** $5b - 12 = 3$ **c** $53 + 4a = 97$

d $\frac{14}{b} + 23 = 10$ **e** $\frac{6x}{5} = 120$ **f** $\frac{7x + 2}{5} = 6$

8 Expand brackets in each of these expressions and collect like terms.

a $23x + 7(x + y)$ **b** $\frac{1}{4}(52y + x) + 2x$

c $3y(4x + 2) + 8x(y + 3)$ **d** $27 - 5x + x(4x + 6)$

e $\frac{x}{3}(9 - x) + 6x(x + 2)$ **f** $32x + 7x(y - 3) + 2x - 5y(x - 1)$

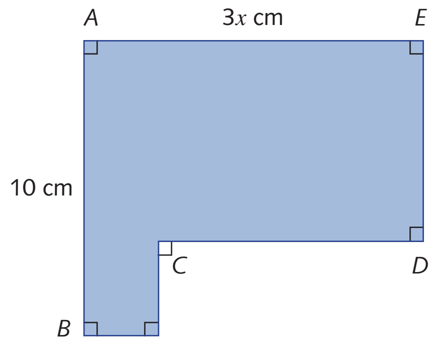
g $a(x - 10 + 3y) - 2a(x + y)$ **h** $4xy(3 + x) - 3yx(6 + 5y)$

9 Select any whole number and add 14. Double the result and then subtract 8 from the number. Divide by 2 and then subtract the original number. Prove that the result is always 10.

10 The entry fee for the Melbourne Show is \$20 per adult and \$8 per child. The cost of each ride is \$5 per person. Stephen's family (2 parents and 3 children) went to the Show and spent \$164 on rides and entrance tickets. If the whole family always went on rides together, how many rides did they go on?

11 Adrian has a habit of reading a book before he sleeps. He reads the same number of pages every night, Monday to Friday. He reads three times more pages on Saturday night than on a weeknight. He does not read any pages on Sunday night. If Adrian took 7 weeks to finish a book that had 280 pages, how many pages does he read on a weeknight, and how many on a Saturday night?

12 The perimeter of the figure shown below is 43 cm. Find the value of x.



Chapter 7: Percentages

1 Express each fraction as a percentage.

a $\frac{2}{5}$ **b** $\frac{5}{6}$ **c** $\frac{7}{8}$ **d** $\frac{7}{9}$ **e** $\frac{5}{7}$ **f** $\frac{3}{40}$

2 Express each percentage as a fraction.

a $8\frac{1}{2}\%$ **b** $16\frac{2}{3}\%$ **c** $15\frac{1}{3}\%$ **d** $8\frac{6}{9}\%$ **e** $4\frac{2}{7}\%$ **f** $5\frac{1}{2}\%$

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3 Express each percentage as a fraction in simplest form.

a 150% **b** 2.5% **c** 136% **d** 204% **e** 0.25% **f** 5.25%

4 Complete each percentage addition and express the result as a fraction.

a 15% + 25% + 14% **b** 45% − 10% + 7%

5 20% of a class of 30 are aged 13 years or under. How many of the class are over 13 years of age?

6 Find:

a 25% of 5000 **b** 12.5% of 16 000 **c** $66\frac{2}{3}$% of 4200

d 6.6% of 9000 **e** 87.5% of 4200 **f** 7% of 320 000

7 Find the new amount after each of the percentage changes.

a $155 000 increased by 20% **b** $60 000 decreased by 5%

c $32 000 increased by 15% **d** 108 000 hectares increased by 24%

8 Calculate the discounted price in each case.

a A discount of 10% on a purchase of $4520

b A discount of 15% on a purchase of $27

c A discount of 5% on a purchase of $86

d A discount of $12\frac{1}{2}$% on $240

9 Calculate:

a 30% of 50% **b** 20% of 30% **c** 50% of 80% **d** 20% of 90%

10 **a** 20% of a number is 1020. Find the number.

b 25% of a number is 600. Find the number.

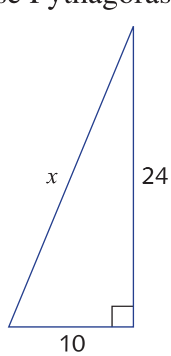
c 15% of a number is 330. Find the number.

11 The price of a book pre-GST is $30. Assume a GST rate of 10%. What is the price of the book including GST?

Chapter 8: Pythagoras' theorem

1 Use Pythagoras' theorem to find the missing side lengths in the triangles below.

a



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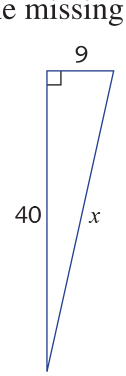

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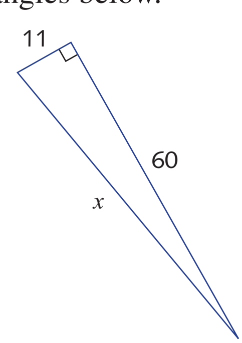

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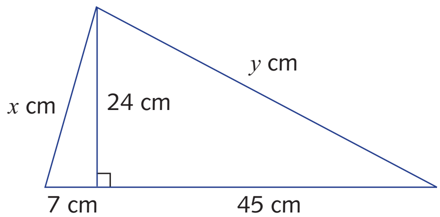
2 Find the values of x and y in the diagrams below.

a



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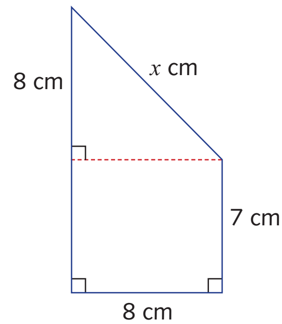
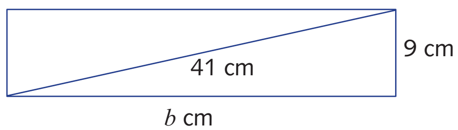
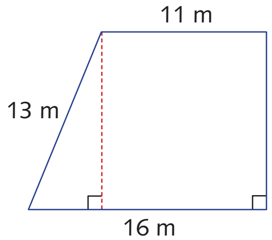


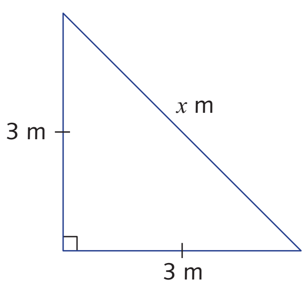
3 A rectangle has width 9 cm and diagonal 41 cm as shown below. What is its length?



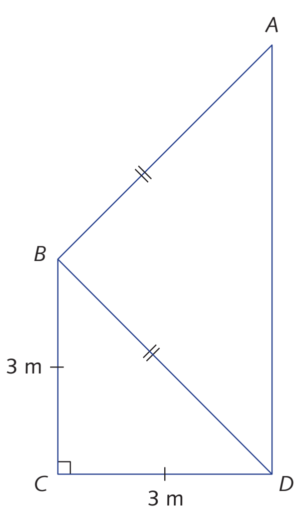
4 Find the height of the trapezium shown below.



5 Find the length of the hypotenuse, correct to 2 decimal places, in the triangle below.



6 Find the length of the side AD in the diagram below.



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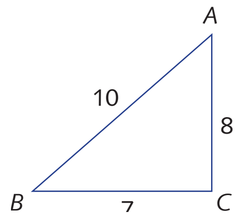
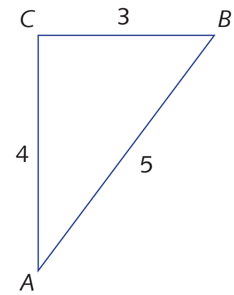
7 For each triangle shown below, decide whether it is a right-angled triangle. If it is, name the right angle.

a



b



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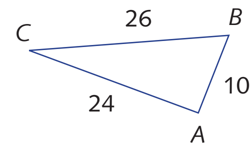

9B Problem-solving

Kaprekar's routine

Choose a set of three digits. Repeats are allowed, but the digits are not *all* allowed to be the same (so 176 and 838 are okay but 444 is not). Now carry out the following steps.

Step 1: Make the largest three-digit number you can out of them.

Step 2: Then reverse this number to get the smallest number possible. This may involve putting a zero at the front of a number.

Step 3: Then subtract the smaller number from the larger one. If the result of the subtraction is two digits, then a 0 is placed to the left of the number.

Now go back to Step 1 with this number and repeat the process until you notice something special happening. For example, if you choose 680, the first step gives 860, step 2 gives 068, step 3 gives $860 - 68 = 792$ and then you start again with $792 \ldots 972 - 279 = \ldots$

1 Describe what eventually happens.

2 What do you notice about the numbers produced on the way?

We call the number at which you finally arrive a **self-producing integer**. With some sets of three digits, you get it straight away. Other sets take a few iterations but never more than six.

If the set you start with contains *four* digits instead of three, you get a different self-producing integer.

3 Find what it is.

4 Find out what happens if you start with a set of *five* digits. (Don't be in too much of a hurry to report on your findings here, because there is a bit more to it than meets the eye. It might be useful to have a partner to help you. Also, if you have been doing all your subtractions without the aid of a calculator up until now, well, now might be a good time to get one out!)

5 *Challenge*: There are three self-producing integers with 10 digits. This is probably a job for a computer program if you feel like having a bash at it.

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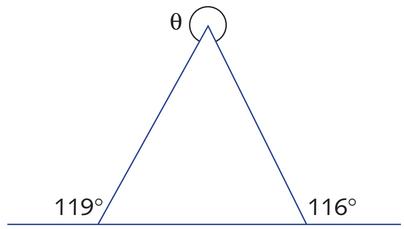
Angle chasing

1 Find θ in each diagram below.

a



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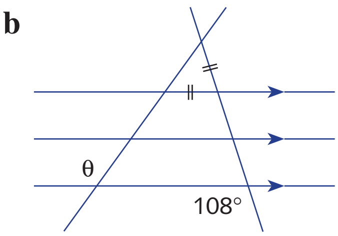

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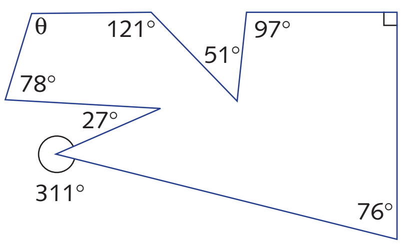

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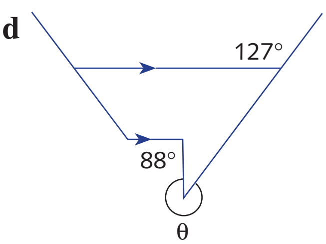
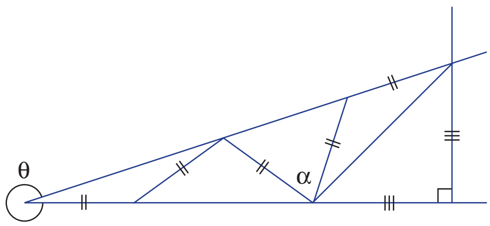


2 What pairs of whole number values can θ and α take in the diagram below? Explain the procedure you followed to obtain your answer.



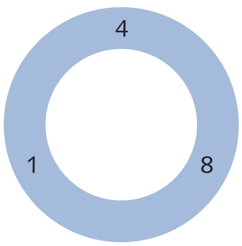
Multiples

1 Take a three-digit multiple of 37:

$13 \times 37 = 481$

Arrange the digits of 481 clockwise in a circle.

Reading around the circle clockwise, we obtain the numbers 481, 814 and 148.



a Show that each of these is a multiple of 37.

b Find out what happens for the three-digit multiples of 37 below:

185, 259, 296, 740, 814 and 925

2 Now look at a multiple of 41 that has five digits:

$1679 \times 41 = 68\,839$

a Show that each of the numbers 68 839, 88 396, 83 968, 39 688 and 96 883 is a multiple of 41.

b Try some more five-digit multiples of 41 to see what happens.

3 **a** Work out each of these multiplications.

i $37 \times 5 \times 3 = \square$ **ii** $37 \times 2 \times 3 = \square$ **iii** $37 \times 7 \times 3 = \square$

What do you notice?

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b Can you predict the value of $37 \times 9 \times 3$? Can you explain what happens?

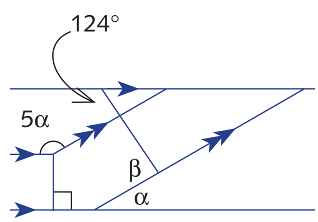
c Now try $37 \times 13 \times 3$. Can you predict the value of $37 \times 17 \times 3$? Can you explain what happens?

d Show that 41 is a factor of 11 111. Can you make up a similar problem to part **c** based on this? (*Hint*: Use the form $41 \times a \times b =$ _____. Replace b with a suitable number and then choose other numbers in turn to replace a.)

Geometry challenge

1 Find the values of the pronumerals in each of these diagrams.

a



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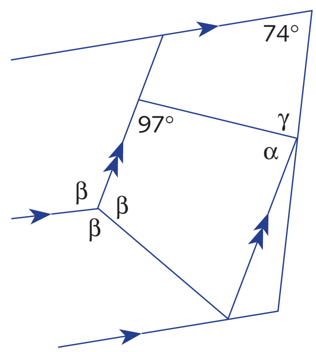

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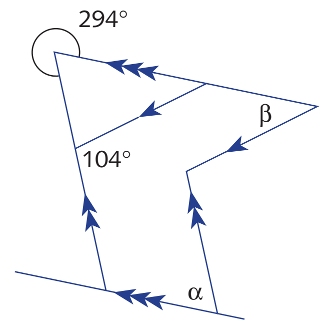

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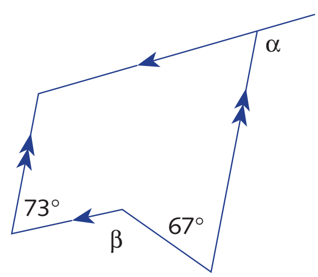

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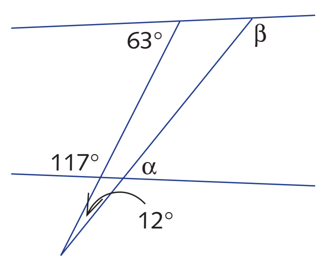

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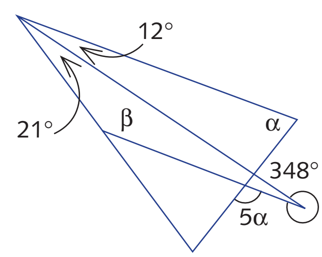
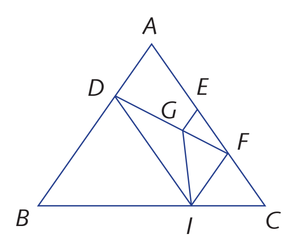


2 The triangle ABC opposite is an isosceles triangle, with $AB = AC$ and $\angle ABC = 55°$. $AD \parallel EG \parallel FI$ and $DI \parallel AF$.



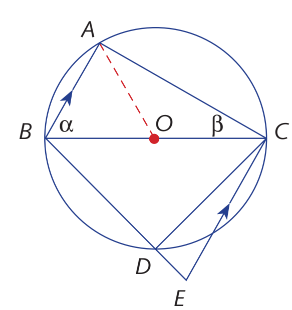
a Find $\angle BAC$ and $\angle ACB$.

b Find $\angle FIC$ and prove that ΔFIC is isosceles.

c Prove that $DB = DI$.

d If $GD = GI$ and $\angle FGI = 34°$, find $\angle GIF$ and $\angle EFG$.

3 O is the centre of the circle shown opposite.



a Let $\angle ABC$ be α. Express $\angle AOC$ in terms of α.

b Let $\angle ACB$ be β. Express $\angle AOB$ in terms of β.

c Prove that ΔABC is a right-angled triangle.
Hence, name two other right-angled triangles in the diagram.

d If $\angle ECD = 25°$, find $\angle ABD$.

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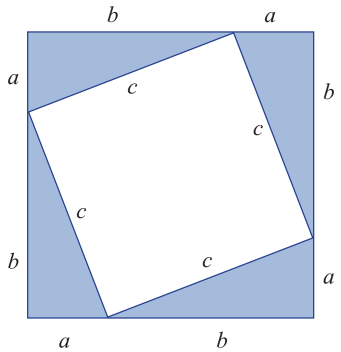
Pythagoras' theorem

In Chapter 8 we gave a proof of Pythagoras' theorem that relied on rearranging triangles and comparing areas in two diagrams.
We now give a proof using just one diagram.

a Show that $(a+b)^2 = a^2 + 2ab + b^2$.

b Show that the total area of the four triangles is $2ab$.

c Hence, show $c^2 = a^2 + b^2$.



Pythagorean triples

In Chapter 8 we encountered a number of right-angled triangles with side lengths that are positive integers. Two such triangles are the ones with side lengths 3, 4 and 5, and 5, 12 and 13.

The numbers 3, 4, 5 are an example of a **Pythagorean triple**. This means that:

$$5^2 = 3^2 + 4^2$$

In a similar way, 5, 12, 13 is a Pythagorean triple because:

$$13^2 = 5^2 + 12^2$$

In general, the set of numbers a, b, c is a Pythagorean triple if a, b and c are positive integers and $a^2 + b^2 = c^2$. The name comes from the fact that a right-angled triangle can be formed with side lengths a, b and c, where c is the length of the hypotenuse.

It is said that Pythagoras invented the following method for finding some of these triples.

Step 1: Take an odd number as the length of one side.

Step 2: Square it and subtract 1.

Step 3: Halve the result (this gives the second side).

Step 4: Add 1 to this result to get the length of the hypotenuse.

For example:

Step 1: Take 11 as the length of the first side.

Step 2: Square it to obtain 121 and subtract 1 to obtain 120.

Step 3: Halve the result to obtain 60 (the length of the second side).

Step 4: Add 1 to get 61 (the length of the hypotenuse).

Check that $11^2 + 60^2 = 61^2$.

Produce some more Pythagorean triples using this method, starting with some other odd numbers.

Not all Pythagorean triples can be produced in this way – an example is the triple 15, 8, 17. Can you see why this triple does not fit the pattern?

Plato, the Greek philosopher and mathematician, recorded the following method for producing Pythagorean triples.

Step 1: Start with a number divisible by 4 as the first side length.

Step 2: Square it, divide by 4 and subtract 1 (this is the second side).

Step 3: Square the original number, divide by 4 and add 1 (this is the hypotenuse).

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For example:

Step 1: Start with 8.

Step 2: Square it to obtain 64, divide by 4 to obtain 16, and subtract 1 to obtain 15.

Step 3: Square 8 to obtain 64, divide by 4 to obtain 16, and add 1 to obtain 17.

Generate some other triples using Plato's method, starting with 12, 16, 20, . . .

Again, not all Pythagorean triples can be produced in this way. For example, 77, 36, 85 is a Pythagorean triple. Can you see why it does not fit Plato's method or Pythagoras' method?

The complete story

Here is a remarkable fact: there is a procedure for producing all possible Pythagorean triples!

To start the explanation, notice that if we have a Pythagorean triple, such as 3, 4, 5, we can get infinitely many others, all related to it, by multiplying each of the numbers by the same factor. For example, 9, 12, 15 (multiplying by 3) is also a Pythagorean triple.

A Pythagorean triple is said to be **primitive** if the three numbers have no common factor other than 1. For example, 3, 4, 5 is primitive but 6, 8, 10 is not. Do the above methods produce primitive Pythagorean triples?

Generating all possible Pythagorean triples

The procedure that produces all of the primitive Pythagorean triples a, b, c uses algebra.

Here it is:

Start with any two positive integers u and v, where $v > u$, u and v have no common factor other than 1, and one of them is odd and the other is even. Now calculate:

$$a = v^2 - u^2$$
$$b = 2uv$$
$$c = u^2 + v^2$$

Then a, b, c is a primitive Pythagorean triple, and every primitive Pythagorean triple can be produced in this way.

Use this method to reproduce all of the primitive Pythagorean triples you created using Pythagoras' method and Plato's method.

Sports percentages

It is suitable to use a calculator or spreadsheet for this exercise.

A common use of percentages is to decide how sporting teams are placed on a ladder or league table. When you look at a ladder, you expect to see the teams with the most wins at the top and those with fewest wins near the bottom. If matches are drawn, half the points for a win are given to both teams involved. The points for wins and draws are called **premiership points**. A win is often worth four points and a draw is worth two.

A problem arises when two or more teams have the same number of premiership points. They cannot both occupy the same rung on the ladder, so a way of deciding which team is to be placed higher is needed. This is where percentages come in.

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The total number of points (or goals) scored by a team in the season to date is expressed as a percentage of the total number of points its opponents have scored against it. The formula is:

$$\text{percentage} = \frac{\text{points for}}{\text{points against}} \times 100\%$$

When a team has scored more points than its opponents have scored against it to date, its percentage at that stage will be more than 100%. If it has been outscored by its opponents overall, its percentage will be less than 100%.

This method does not favour teams that are high-scoring, but that allow their opponents to score highly against them, over low-scoring teams with good defensive set-ups. It is commonly used in some football codes, netball, basketball, ten-pin bowling and volleyball. For other sports, this method is less appropriate due to the scoring systems involved. For this reason, it is not usually used for hockey, soccer, cricket or softball.

The table below is a netball ladder after five games of a season have been played. Only two of the percentages have been calculated.

	TEAM	Wins	Losses	Draws	Goals For	Goals Against	Percentage	Prem. Points
1	Pelicans	5	–	–	115	80	143.8%	20
2	Cassowaries	4	1	–	126	90		16
3	Jabirus	3	2	–	104	96		12
4	Hawks	2	3	–	130	104		8
5	Brolgas	2	3	–	105	112		8
6	Kookaburras	2	3	–	90	120		8
7	Emus	1	4	–	77	105	73.3%	4
8	Eagles	1	4	–	85	125		4

Activity 1

Your first task is to calculate the percentages of the other six teams, and confirm that the teams are in the right order. Remember, if a team has more premiership points, it will be placed above a team with fewer premiership points, even though its percentage may not be as high.

The teams then play the sixth round of games. The results are:

- Brolgas (27) beat Jabirus (16)
- Emus (20) beat Kookaburras (15)
- Hawks (30) beat Eagles (28)
- Cassowaries (31) beat Pelicans (27).

These results change the teams' numbers of wins, losses, goals for and goals against for the season, and their premiership points. It may also change their places on the ladder.

Activity 2

Update the ladder. This means you will need to work out all the new figures and percentages for each team. For example, the Brolgas' Goals For will increase by 27 and their Goals Against will increase by 16. The Jabirus' Goals For will go up by 16 and their Goals Against will go up by 27. A calculator may help you work out the percentages.

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Activity 3

1. Which team's percentage did not change? Can you explain why?
2. How many of the teams that won had their percentages go down? Which ones were they?
3. Is it possible to lose and have your percentage increase? Give an example of a team that had this happen.
4. Add up the Goals For column. Add up the Goals Against column. Explain your results.

9C Modular arithmetic and congruence

Arithmetic modulo 7 means that you divide a number by 7 and just look at the remainder. The quotient gets discarded. For example, $15 = 7 \times 2 + 1$, so we write:

$15 \equiv 1 \pmod{7}$

Read this as '15 is congruent to 1, modulo 7'. Such an equation is called a **congruence.**

Thinking in modulo 7 can relate to the days of the week.

Today is Monday. What day of the week will it be in 15 days time?

In 14 days time it will be Monday again, so in 15 days time it will be Tuesday.

Today is Monday 4 May. What day of the week will it be on 4 June?

There are 31 days in May, and $31 = 7 \times 4 + 3$. In 28 days time it will be Monday again, so in 31 days time it will be Thursday.

There are seven possible remainders 0, 1, 2, 3, 4, 5 and 6 after division by 7, and there are seven days in the week.

Arithmetic modulo 5 means that you divide the number by 5 and just look at the remainder. Someone counted on the fingers of one hand the people coming into the Wombat Heights Football Ground, but forgot to count how many times he'd gone around his hand:

1, 2, 3, 4, 0, 1, 2, 3, 4, 0, 1, 2, 3, … 2, 3

'I don't know how many people went into the Football Ground today, but I do know that the number ended in a 3 or an 8.'

Arithmetic modulo 12 and arithmetic modulo 24 are related to counting the hours around a 12-hour clock and around a 24-hour clock.

Dividing 90 by 12 and by 24 gives

$90 \equiv 6 \pmod{12}$ and $90 \equiv 18 \pmod{24}$

'It is now 8:30 p.m. What will my 12-hour watch and my 24-hour clock say in 90 hours time?'

'My 12-hour watch will say 2:30, and my 24-hour clock will say 14:30.'

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Arithmetic modulo *n* where n is any whole number greater than one is possible.

There are three possible remainders 0, 1, 2 after division by 3. Similarly, there are ten possible remainders 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 after division by 10.

In each case every whole number is congruent to one of the remainders. The plural of modulus is moduli.

Here are some more congruences, using various moduli:

$20 \equiv 2 \pmod{3}$ $\quad 67 \equiv 7 \pmod{10}$

$42 \equiv 0 \pmod{7}$ $\quad 35 \equiv 1 \pmod{2}$

$100 \equiv 1 \pmod{11}$ $\quad 80 \equiv 8 \pmod{9}$

We can use any whole number greater than 1 as the **modulus**.

Congruence with negative integers

Here is how to divide −10 by 7 and get a whole number remainder less than 7:

$-10 = 7 \times -2 + 4$, so $-10 \equiv 4 \pmod{7}$

Here are two everyday interpretations of this idea:

Arithmetic modulo 7 can also be used with negative integers.

Today is Monday. What day of the week was it 10 days ago?

Fourteen days ago it was also Monday, so 10 days ago it was Friday.

Today is Monday 4 May. What day of the week was it on 4 April?

There are 30 days in April, and $-30 = 7 \times (-5) + 5$. Thirty-five days ago it was also Monday, so 30 days ago it was Saturday.

Arithmetic modulo 12 and **arithmetic modulo 24** can be used with negative integers.

Here is an example with 12-hour and 24-hour clocks.

'It is now 8:30 p.m. What did my 12-hour watch and my 24-hour clock say 31 hours ago?'

Dividing −31 by 12 and by 24 gives:

$-31 = 12 \times -3 + 5$ and $-31 = 24 \times -2 + 17$

Writing this in congruences gives:

$-31 \equiv 5 \pmod{12}$ and $-31 \equiv 17 \pmod{24}$

'My 12-hour watch said 1:30, and my 24-hour clock said 13:30.'

Here are some more congruences involving negative integers and various moduli:

$-20 \equiv 1 \pmod{3}$ $\quad -67 \equiv 3 \pmod{10}$

$-42 \equiv 0 \pmod{7}$ $\quad -35 \equiv 1 \pmod{2}$

$-100 \equiv 10 \pmod{11}$ $\quad -80 \equiv 1 \pmod{9}$

Congruence modulo a whole number

The integers 9 and 23 are both congruent to 2 modulo 7, so we say that they are congruent to each other:

$9 \equiv 23 \pmod{7}$

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We could have checked this directly simply by subtracting 9 from 23:

$23-9=14$, which is a multiple of 7

Working with days of the week this becomes:

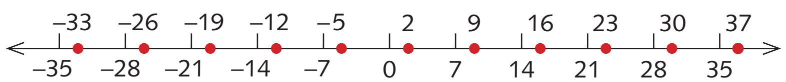
'It will be the same day of the week in 9 days time and in 23 days time.'

This provides us with the standard definition of congruence modulo 7. Two integers are called **congruent modulo 7** if their difference is a multiple of 7. By the division algorithm, every integer is congruent modulo 7 to one, and only one, of the seven remainders 0, 1, 2, 3, 4, 5 and 6.

The integers 9 and −33 are also congruent modulo 7, because

$9-(-33)=9+33=42$, which is a multiple of 7.
'It will be the same day of the week in 9 days time as it was 33 days ago.'

On the number line below, we have marked both the multiples of 7 and all the integers congruent to 2 modulo 7.



Each integer congruent to 2 modulo 7 is 2 more than a multiple of 7, no matter whether the integer is positive or negative.

Here are some more congruences. Check them by subtracting the integers:

$10\equiv 40 \pmod 3$	$-27\equiv -67 \pmod{10}$
$-42\equiv 21 \pmod 7$	$30\equiv 514 \pmod 2$
$-100\equiv -540 \pmod{11}$	$30\equiv -51 \pmod 9$

Modular arithmetic

- Two integers are called **congruent modulo 7** if their difference is a multiple of 7.
- Every integer is congruent modulo 7 to exactly one whole number less than 7; that is, to 0, 1, 2, 3, 4, 5 or 6.
- The **modulus** can be any whole number greater than 1.

Exercise 9C

1 Copy and complete:

a $27=7\times\_\_\_+\_\_\_$, hence $27\equiv\_\_\_ \pmod 7$

b $359\,412\,294\equiv 10\times\_\_\_+\_\_\_$, hence $359\,412\,294\equiv\_\_\_ \pmod{10}$

c $53=12\times\_\_\_+\_\_\_$, hence $53\equiv\_\_\_ \pmod{12}$

d $3=8\times\_\_\_+\_\_\_$, hence $3\equiv\_\_\_ \pmod 8$

2 Copy and complete:

a $36\equiv\_\_\_ \pmod 2$

b $100\equiv\_\_\_ \pmod{12}$

c $156\equiv\_\_\_ \pmod{25}$

d $94\equiv\_\_\_ \pmod 9$

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3 Find the first number over 50 that is congruent to 1:

a modulo 2 **b** modulo 8 **c** modulo 10 **d** modulo 3

e modulo 19 **f** modulo 15 **g** modulo 30 **h** modulo 66

4 Roger says 'She loves me, she loves me not, she loves me, she loves me not. . .' as he counts the petals of a flower. The flower has 47 petals. Is happiness assured for Roger or not?

5 Six children – Adam, Bessie, Charles, Doreen, Edwin and Florence – choose chocolates in turn from a box of 80 chocolates. Adam starts, and they keep going in alphabetical order. Who gets the last chocolate?

6 It is 4 p.m. now. What will the time be in 110 hours?

7 If the first of November is a Sunday, what dates of November are Wednesdays? What day of the week is the 27 November?

8 In what years between 1890 and 1920 could one have truthfully said, 'Next year is a leap year'?

9 **a** What is 52 348 181 modulo 4? Is there an easier way than dividing?

b What is 52 348 181 modulo 100? Is there an easier way than dividing?

c What is 52 348 181 modulo 50? Is there an easier way than dividing?

d What is 52 348 181 modulo 25? Is there an easier way than dividing?

10 **a** What is the time 27 hours after 11 a.m.?

b What day of the week is 176 days after a Wednesday?

c A man is facing north. He turns through 90° in a clockwise direction 57 times. In which direction is he facing now?

11 **a** Write -13 as a multiple of 5 plus remainder, where the remainder is 0, 1, 2, 3 or 4.

b Hence, copy and complete, '$-13 \equiv$ ___ (mod 5).'

c Mark the multiples of 5 between -26 and 14 on a number line. Mark also -13 and all integers between -26 and 14 congruent to -13 modulo 5.

12 **a** Mark on the number line all integers between -10 and 5 congruent to 2 modulo 5.

b Mark on the number line all integers between -10 and 5 congruent to 3 modulo 5.

13 Copy and complete:

a $-20 = 7 \times$ ___ + ___, hence $-20 \equiv$ ___ (mod 7)

b $-30 = 8 \times$ ___ + ___, hence $-30 \equiv$ ___ (mod 8)

c $-65 \equiv$ ___ (mod 11) **d** $-70 \equiv$ ___ (mod 9)

e $-100 \equiv$ ___ (mod 11) **f** $-72 \equiv$ ___ (mod 6)

14 **a** Divide the numbers $-16, -11, -1, 0, 4, 15$ and 18 by 4. Hence, decide which of the numbers are congruent mod 4.

b Divide the numbers $-14, -30, 5, 18, 45, 53$ and 73 by 12. Hence, decide which of the numbers are congruent mod 12.

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9D Addition, multiplication and powers in modular arithmetic

Addition

The wonderful thing about adding in modular arithmetic is that you can simply add the remainders. For example:

$$31 = 7 \times 4 + 3 \text{ and } 30 = 7 \times 4 + 2, \text{ so } 31 + 30 = 7 \times 8 + 5$$

and writing this in modulo 7 arithmetic:

$$30 + 31 \equiv 3 + 2 \equiv 5 \pmod 7$$

Check this by adding the integers first: $30 + 31 = 61 \equiv 5 \pmod 7$.

Today is Monday 4 May. What day of the week will it be on 4 July? May has 31 days and June has 30 days, and $31 + 30 \equiv 5 \pmod 7$, so it will be Saturday.

When the remainders add to more than 7, we reduce the remainder modulo 7:

$$31 + 30 + 31 \equiv 3 + 2 + 3 \equiv 8 \equiv 1 \pmod 7$$

This is because $31 + 30 + 31 = 7 \times 12 + 8 = 7 \times 13 + 1$.

Today is Monday 4 May. What day of the week will it be on 4 August?
Since $31 + 30 + 31 \equiv 1 \pmod 7$, it will be Tuesday.

Here are some more additions with various moduli.

$10 + 25 \equiv 1 + 1 \equiv 2 \pmod 3$ $\qquad$ $26 + 39 \equiv 6 + 9 \equiv 5 \pmod{10}$

$9 + 75 \equiv 2 + 5 \equiv 0 \pmod 7$ $\qquad$ $-11 + 21 \equiv 1 + 1 \equiv 0 \pmod 2$

$-100 + 12 \equiv 10 + 1 \equiv 0 \pmod{11}$ $\qquad$ $30 + 50 \equiv 3 + 5 \equiv 8 \pmod 9$

Check them by adding the integers first. For example, $10 + 25 \equiv 35 \equiv 2 \pmod 3$.

Multiplication

Multiplication of integers is repeated addition, so we can use the same method for multiplying in modular arithmetic. For example:

$$10 \times 12 \equiv 3 \times 5 \equiv 1 \pmod 7 \text{ and } 65 \times 12 \equiv 2 \times 5 \equiv 3 \pmod 7$$

Check the first calculation by multiplying the integers first: $10 \times 12 = 120 \equiv 1 \pmod 7$.
You can check the second if you feel the need to do so.

A mine organises its shifts in 12-day cycles, with the first cycle beginning on a Monday. On what days of the week will the 11th cycle and the 66th cycle begin?

Since $10 \times 12 \equiv 1 \pmod 7$, the 11th cycle will begin on a Tuesday.

Since $65 \times 12 \equiv 3 \pmod 7$, the 66th cycle will begin on a Thursday.

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Here are some more multiplications with various moduli.

$$10 \times 25 \equiv 1 \times 1 \equiv 1 \pmod{3}$$
$$25 \times 38 \equiv 5 \times 8 \equiv 0 \pmod{10}$$
$$9 \times 75 \equiv 2 \times 5 \equiv 3 \pmod{7}$$
$$-10 \times 25 \equiv 0 \times 1 \equiv 0 \pmod{2}$$
$$-100 \times 12 \equiv 10 \times 1 \equiv 10 \pmod{11}$$
$$30 \times 50 \equiv 3 \times 5 \equiv 6 \pmod{9}$$

Check at least some of them by multiplying the integers first.

Powers

Taking powers is repeated multiplication, so once again we can use the same method for taking powers in modular arithmetic. For example:

$$10^3 \equiv 36^3 \equiv 27 \equiv 6 \pmod{7} \text{ and } 30^6 \equiv 2^6 \equiv 64 \equiv 1 \pmod{7}$$

Here are some similar calculations:

$14^4 \equiv 2^4 \equiv 16 \equiv 1 \pmod{3}$ $\quad$ $25^3 \equiv 5^3 \equiv 125 \equiv 5 \pmod{10}$

$4^{10} \equiv (4^2)^5 \equiv 16^5 \equiv 0^5 \equiv 0 \pmod{8}$ $\quad$ $(-11)^6 \equiv 1^6 \equiv 1 \pmod{2}$

$(-3)^8 \equiv 9^4 \equiv 4^2 \equiv 5 \pmod{11}$ $\quad$ $(-2)^9 \equiv (-8)^3 \equiv 1^3 \equiv 1 \pmod{9}$

Fibonacci sequences in modular arithmetic

Fibonacci was a mediaeval Italian mathematician who studied, among other things, a curious sequence of numbers which is now named after him. The Fibonacci sequence starts with 1 and 1, then every term is the sum of the two previous terms:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

The numbers soon get very big and take a bit of calculating. But now try writing out the sequence in arithmetic modulo 8. The rules are the same – the first two terms are 1 and 1, then every term is the sum modulo 8 of the two previous terms:

$$1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, \ldots$$

We get a zero every 6 terms, and the whole thing repeats itself after 12 terms. The final questions in the following exercises continue this experimentation.

Exercise 9D

1 Perform these additions in modulo arithmetic. It is best to reduce the numbers *before* you add them, but do the ones marked ✓ both ways for comparison.

✓ **a** $16 + 2 \pmod{5}$ ✓ **b** $902 + 30 \pmod{9}$

✓ **c** $7 + 7 \pmod{2}$ ✓ **d** $6 + 8 + 3 + 15 \pmod{3}$

e $4 + 4 + 8 + 7 + 6 \pmod{5}$ **f** $1 + 2 + 3 + 4 + 5 \pmod{4}$

✓ **g** $1 + 3 + 5 + 7 + 9 \pmod{5}$ **h** $79 + 79 + 79 + 79 \pmod{80}$

i $243 + 527 + 384 + 22 + 3 \pmod{5}$

j $3\,628\,598 + 21\,242\,522 + 376\,554\,004 \pmod{100}$

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2 Reduce the numbers first before you multiply, but do those marked ✓ both ways.

✓**a** $7 \times 8 \pmod 5$

✓**b** $11 \times 12 \pmod 8$

✓**c** $4 \times 4 \times 4 \pmod 3$

d $9 \times 8 \pmod{12}$

e $247 \times 3482 \pmod{10}$

f $552 \times 661 \times 776 \pmod 5$

3 If you have a string of numbers to multiply, reduce them all first, but also reduce the product at each step as you go along.

a $26 \times 13 \times 17 \times 35 \times 62 \pmod 3$

b $10 \times 23 \times 59 \times 44 \times 33 \pmod 6$

c $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \pmod 8$

d $1 \times 3 \times 5 \times 7 \times 9 \pmod 8$

4 Reducing as you go along is really important when taking powers. You can also group the factors in pairs, or in threes, if that will make the calculations easier.

a $5 \times 5 \times 5 \times 5 \times 5 \times 5 \pmod 7$

b $4 \times 4 \times 4 \times 4 \pmod 7$

c $16 \times 16 \times 16 \times 16 \times 16 \pmod 7$

d $23^8 \pmod 5$

e $5^6 \pmod{11}$

f $4^7 \pmod{12}$

g $2^{12} \pmod{12}$

h $9^{26} \pmod{10}$

5 These questions involve both sums and products. Reduce the numbers to start with, then keep reducing the numbers after each step so they do not get too big.

a $14 \times 5 + 21 \times 4 \pmod 3$

b $15 \times 4 + 25 \times 23 \pmod 6$

c $10^5 + 2^5 \pmod{11}$

d $3 + 3^2 + 3^3 + 3^4 \pmod 5$

6 Perform these additions, then explain what general result they illustrate.

a $2 + 2 + 2 \pmod 3$

b $1 + 1 + 1 + 1 \pmod 4$

c $3 + 3 + 3 + 3 + 3 \pmod 5$

d $5 + 5 + 5 + 5 + 5 + 5 \pmod 6$

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7 My watch has 6 functions, which the silver button goes through in order when I push it. If I push the button 74 times, how many more times must I push the button to get it back to the start? Use modulo arithmetic.

8 A satellite goes around the Earth once every 7 hours. If it is 9 a.m. now, what time will it be in 247 revolutions? (Don't calculate 7×247, use modulo arithmetic.)

9 It is Wednesday today. Christmas is 280 days away. Then 25 days holiday at the beach. Then 211 days at work. Then overseas for 143 days. What day of the week is the last day overseas? Use modulo arithmetic.

10 Eric has a packet of 37 Smarties. He counts them out on his five fingers, over and over again, 19 times in fact. What finger does he end on? (Don't calculate 37×19, use modulo arithmetic.)

11 A dancer turns 60° nine times, then 40° twelve times, then 90° thirteen times. What angle does he make with his original position? Use modulo arithmetic.

12 **a** Prove that every power of 10 is equal to 1 modulo 9.

b Hence, prove that every whole number is congruent modulo 9 to the sum of its digits.

c Hence, prove that every whole number is congruent modulo 3 to the sum of its digits.

13 **a** Prove that every odd power of 10 is congruent to -1 modulo 11, and that every even power of 10 is congruent to 1 modulo 11.

b Hence, prove that every whole number such as 123 456 is congruent modulo 11 to the alternating sum of its digits, where alternating sum means $6-5+4-3+2-1$ with the units taken as positive.

14 Write out the Fibonacci sequences using the following moduli. Continue each sequence until it repeats itself. In each case, say how often the zeros occur, and how long it takes for the whole thing to repeat itself.

a modulo 3 **b** modulo 2 **c** modulo 4 **d** modulo 5

e modulo 6 **f** modulo 10 **g** modulo 9 **h** modulo 11

i modulo 10 (this one takes quite a while – don't give up)

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