CHAPTER

12

Measurement and Geometry

Congruent triangles

Figures are copied everywhere – a photocopier can produce a copy of a figure, craft workers use copies of figures to create designs or patterns, copies of figures can be obtained by tracing, and so on.

The idea of a 'copy' also arises in geometry and is called '**congruence**'.

You have previously met the three kinds of geometrical transformation – translation, reflection and rotation. Two figures are called **congruent** if one is the image of the other when you apply one or more of these transformations.

We will see that two triangles are congruent when the sides and angles of one triangle are equal to the sides and angles of the other.

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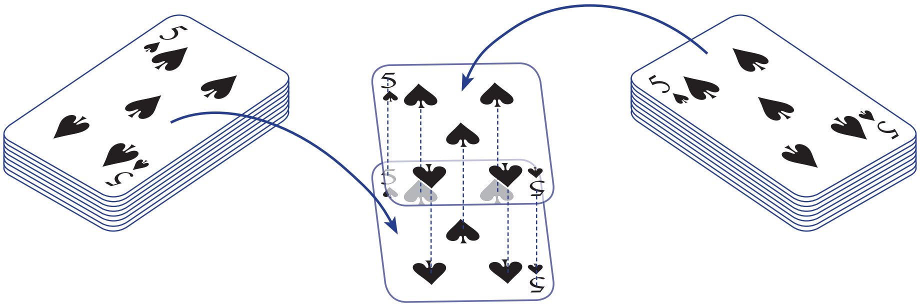
12A Congruence of figures in the plane

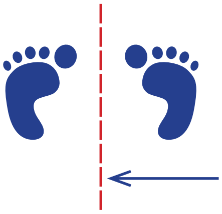
If we take the five of spades from each of two identical decks of cards, they look exactly the same. We can move one card and place it on top of the other one so that the pictures on the two cards coincide exactly, as shown below.



Two objects like this are called **congruent**. The word 'congruent' comes from Latin and means 'in agreement' or 'in harmony'. Here is a more precise definition:

> Two plane figures are called **congruent** if one figure can be moved on top of the other, by a sequence of translations, rotations and reflections, so that they coincide exactly. Thus congruent figures have the same size and the same shape.

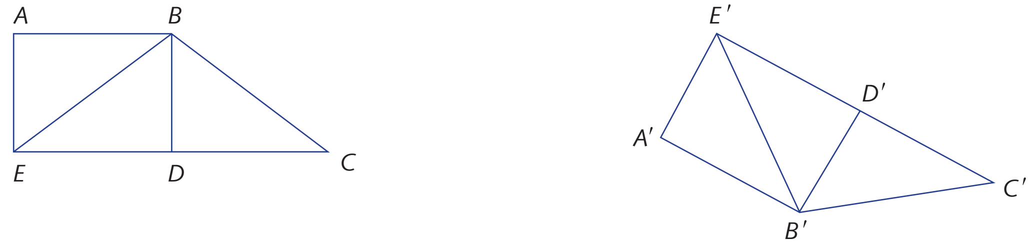
For example, the two footprints opposite are congruent because the footprint on the right is a reflection of the footprint on the left.



Pairing the parts of congruent figures

When two figures are congruent, we can pair every part of one figure with the corresponding part of the other by finding a sequence of translations, rotations and reflections which moves one figure exactly on top of the other. For example, the two figures below are congruent. Some of the pairings of points with points, intervals with intervals, and angles with angles are shown in the list below.

Can you see how to transform one figure to the other? There are several ways of doing it. Once you have done this you will see the following pairings. We use the notation P' for the image of the point P under transformation, as introduced in Chapter 18 of *ICE-EM Mathematics Year 7*.



$A \leftrightarrow A'$	$D \leftrightarrow D'$	$BC \leftrightarrow B'C'$	$\angle EAB \leftrightarrow \angle E'A'B'$
$B \leftrightarrow B'$	$E \leftrightarrow E'$	$BD \leftrightarrow B'D'$	$\angle BCD \leftrightarrow \angle B'C'D'$
$C \leftrightarrow C'$	$AB \leftrightarrow A'B'$	$BE \leftrightarrow B'E'$	$\angle DBE \leftrightarrow \angle D'B'E'$

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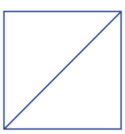
If two figures are congruent, then paired intervals have the same length, paired angles have the same size, and paired regions have the same area.

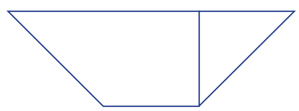
Congruent figures

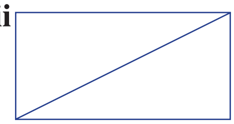
- Two plane figures are called **congruent** if one figure can be moved on top of the other, by a sequence of translations, rotations and reflections, so that they coincide exactly.
- Congruent figures have the same shape and the same size.
- When two figures are congruent, we can find a transformation that pairs every part of one figure with the corresponding part of the other, so that:
  - paired angles have the same size
  - paired intervals have the same length
  - paired regions have the same area.

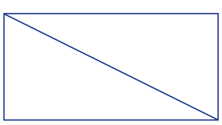
Exercise 12A

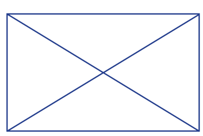
1 List the figures in the collection below that are congruent to each of the figures **i**, **ii** and **iii**.

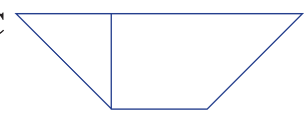
i


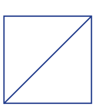
ii


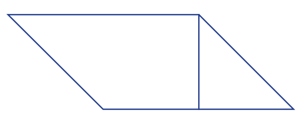
iii


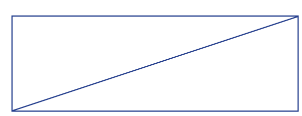
A


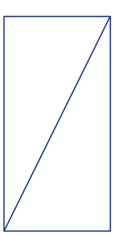
B


C


D


E


F


G


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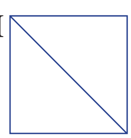
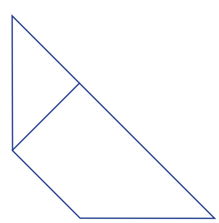
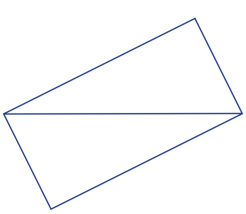
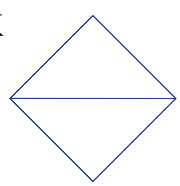
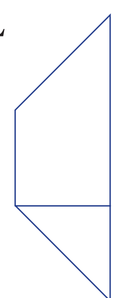


I


J


K


L


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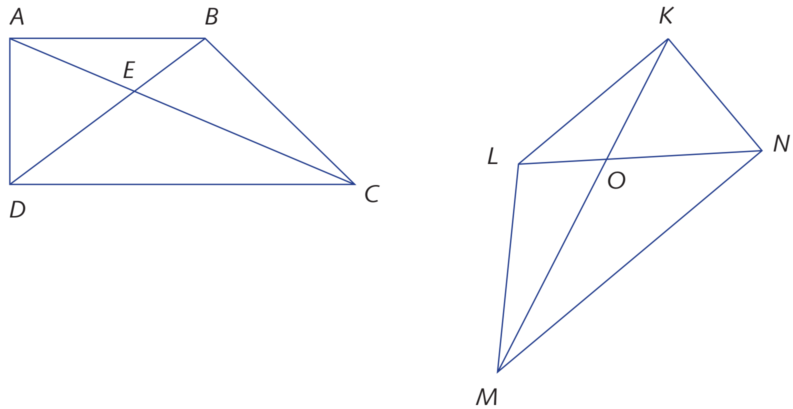
2 Complete the pairings of vertices, sides and angles of these two congruent figures.



a $A \leftrightarrow$ **b** $B \leftrightarrow$ **c** $C \leftrightarrow$

d $D \leftrightarrow$ **e** $E \leftrightarrow$ **f** $AB \leftrightarrow$

g $AC \leftrightarrow$ **h** $BD \leftrightarrow$ **i** $ED \leftrightarrow$

j $\angle ABC \leftrightarrow$ **k** $\angle EAB \leftrightarrow$ **l** $\angle DAB \leftrightarrow$

3 **Discussion questions**

a A page of writing is reflected in a mirror. Is the image congruent to the real page?

b Think of two large jacaranda trees that you have seen. Are they congruent?

c Is a model of the Sydney Harbour Bridge congruent to the real bridge?

d Two identical twins stand at attention in army uniform. Are they congruent?

e Think of two cumulus clouds of about the same size. Are they congruent?

f Are two copies of the same photograph congruent?

g What lower-case letter, apart from b itself, is congruent to b?

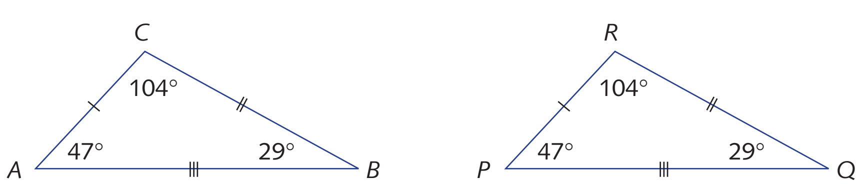
h What digit, apart from 9 itself, is congruent to 9?

12B Congruent triangles

Most geometrical reasoning about congruence that we are going to do involves only congruent triangles.

Translations

Here are two congruent triangles. The sides AB and PQ are on the same line.



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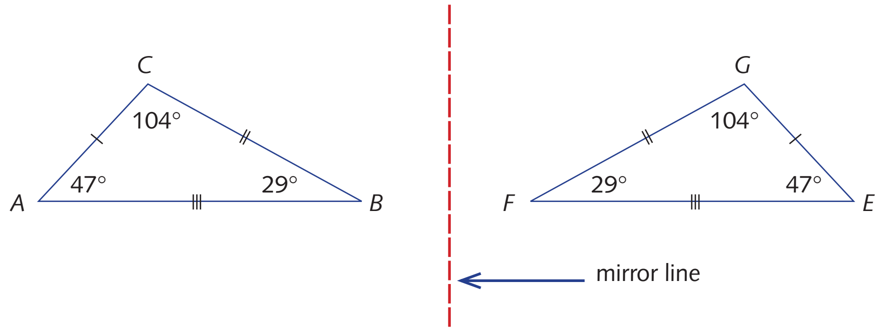
The translation that takes P to A also takes Q to B and R to C. Thus the two triangles are congruent because ΔABC is the image of ΔPQR under a translation.

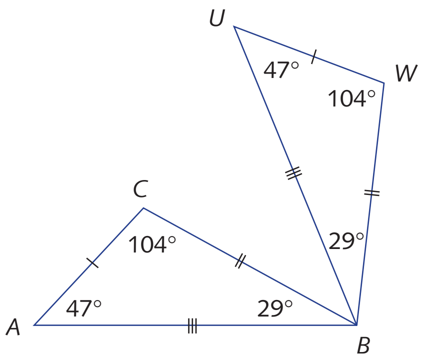
Reflections



The triangle FGE is the reflection of the triangle BCA. Thus the two triangles are congruent.

Rotations

Triangle UBW is a rotation of triangle ABC about B.
Thus the two triangles are congruent.



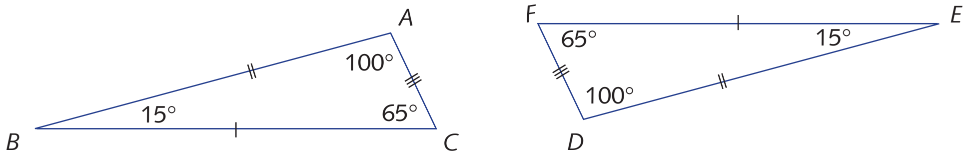
When are two triangles congruent?

We can see from the examples above that when two triangles are congruent, paired sides have the same length and paired angles have the same size.

The rest of this chapter will develop four tests for two triangles to be congruent. At this stage, however, we simply note that:

> If the vertices of two triangles can be paired up so that paired sides have the same length and paired angles have the same size, then they are congruent.

To demonstrate this, cut out one triangle and move it so that it lies exactly on top of the other triangle. This process involves only translations and rotations, and a reflection if you have to turn the triangle over.



Thus the two triangles above are congruent. You should be able to image how you would transform one triangle so that it lies on top of the other.

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Writing a congruent statement with vertices in matching order

When writing symbolic congruence statements, we use the symbol ≡ for ‘is congruent to’.

For example, the triangles in the diagram on the previous page are congruent. This is written as:

$\triangle ABC \equiv \triangle DEF$ (Read this as ‘triangle *ABC* is congruent to triangle *DEF*.’)

It is vitally important *to write the vertices of the two triangles in matching order*. In the statement above, we wrote the two triangles as $\triangle ABC$ and $\triangle DEF$ because the paired vertices are:

$A \leftrightarrow D$ and $B \leftrightarrow E$ and $A \leftrightarrow F$

This attention to detail is useful when it comes to paired sides. We can read off the paired sides from the congruence statement $\triangle ABC \equiv \triangle DEF$ in the natural way:

$AB \leftrightarrow DE$ and $BC \leftrightarrow EF$ and $CA \leftrightarrow FD$

Congruent triangles

- If the vertices of two triangles can be paired up so that paired angles have equal size and paired sides have equal length, then they are congruent.
- When writing a **congruence statement**, always write the vertices of the two congruent triangles in matching order. We can then read off paired angles and paired sides so that the pairings can be read off in the natural way. For example, the statement $\triangle ABC \equiv \triangle DEF$ means that:

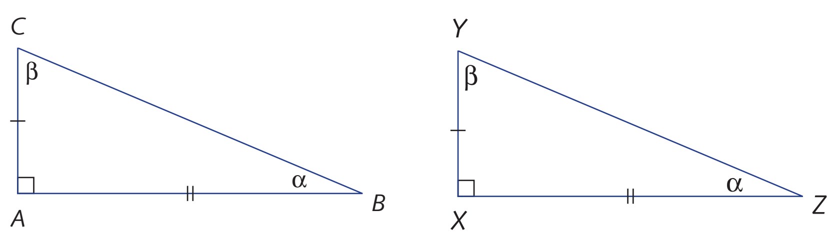
  $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$,

  $AB = DE$, $BC = EF$, $CA = FD$.

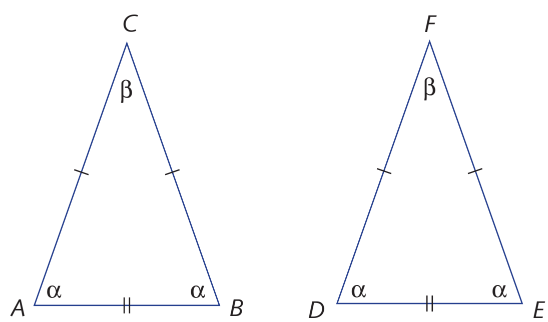
Exercise 12B

1 Write a congruence statement for each pair of congruent triangles.

a



b



$\triangle ABC$ is isosceles with $CA = CB$.

$\triangle DEF$ is isosceles with $FD = FE$

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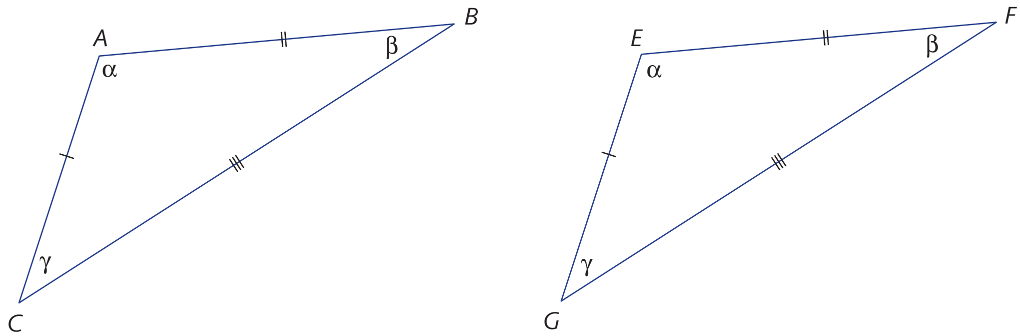
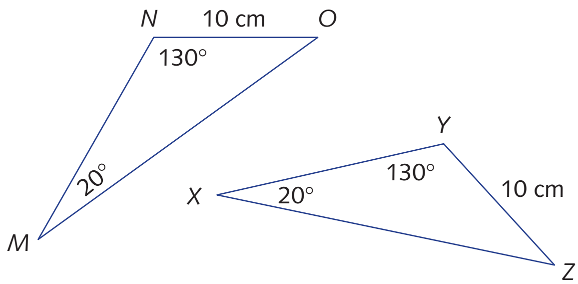
c



d

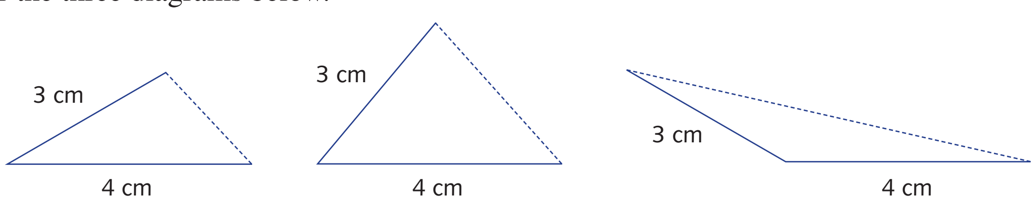


12C Congruent triangles: The SSS and AAS tests

The SSS congruence test

To prove that two triangles are congruent, it is not necessary to prove that they have three pairs of equal sides and three pairs of equal angles. In this section and the next, we will investigate the minimum amount of information needed to establish that two triangles are congruent.

Consider the three diagrams below.



The three triangles shown are clearly not congruent because the 3 cm and 4 cm can flap about. This shows that just knowing that two pairs of sides are equal is not enough information to establish congruence. Knowing that *three* pairs of sides are equal, however, is enough to establish congruence, as the discussion below will demonstrate.

This test for congruence is called the **SSS congruence test**. The initials **SSS** stand for **S**ide, **S**ide, **S**ide.

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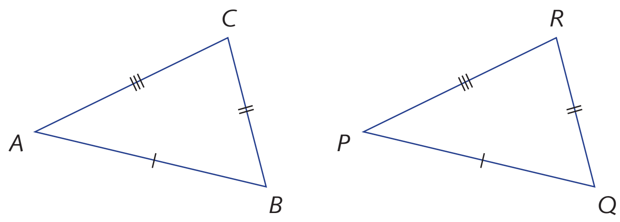
SSS congruence test

If the three sides of one triangle are equal to the three sides of another, then the two triangles are congruent.

For example, in the diagram below, $\Delta ABC \equiv \Delta PQR$ (SSS).



Thus the SSS congruence test tells us that the angles of any triangle are determined by the three sides.

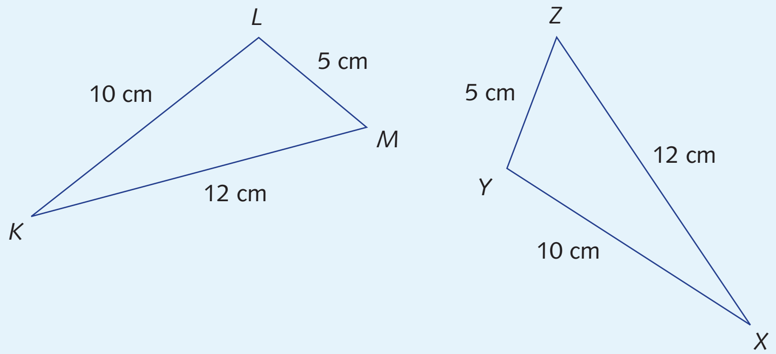
We can now conclude from the congruence that:

$$\angle A = \angle P$$
$$\angle C = \angle R$$
$$\angle B = \angle Q$$

You should always write the initials of the test used to demonstrate the congruence after the congruence statement.

Example 1

Write down a statement that the two triangles below are congruent, giving the appropriate congruence test as a reason.



Solution

$\Delta KLM \equiv \Delta XYZ$ (SSS)

We review the construction of triangles with given side lengths.

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Constructing a triangle given three sides

We will construct a triangle with three given side lengths. Here is an example using the side lengths 12 cm, 10 cm and 5 cm.

Step 1: Draw an interval AB of length 12 cm.

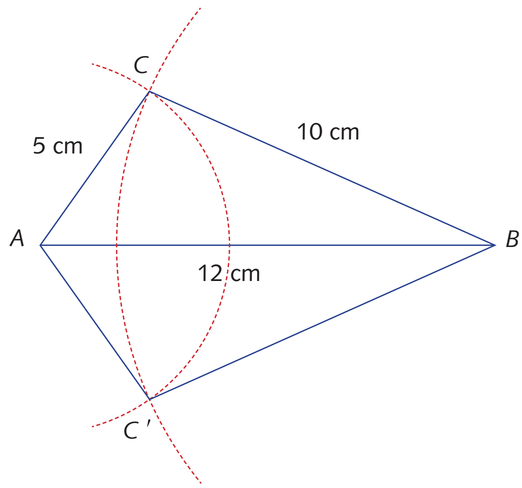
Step 2: Draw a circle with centre A and radius 5 cm.

Step 3: Draw a circle with centre B and radius 10 cm crossing the first circle at C and C'.

Step 4: Join up the triangle ABC and the triangle ABC'.

The two triangles ABC and ABC' are reflections of each other in AB and so are congruent.

Note: The three angles of any triangle are determined by the lengths of the three sides.



The AAS congruence test

We first consider two important ideas about triangles.

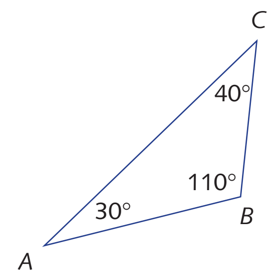
Sides opposite angles

Here is a triangle with the angles marked.

We say that the side AB is **opposite** the angle 40° at C.

The side BC is opposite the angle 30° at A.

The side AC is opposite the angle 110° at B.

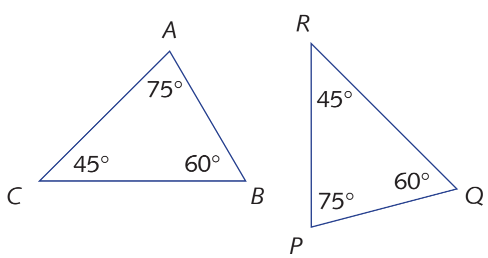


Matching sides and angles

Here are two triangles with the same three angles in both. We see that A matches P, B matches Q and C matches R.

We also say that AB and PQ are matching sides because:

- AB is opposite the angle 45° in ΔABC, and
- PQ is opposite the angle 45° in ΔPQR.



Similarly, BC and QR are matching sides because they are both opposite 75°.

Also CA and RP are matching sides because they are both opposite 60°.

Matching sides

When two triangles each have the same three angles, we say that a side of one triangle **matches** a side in the second if the two sides are opposite the same angle.

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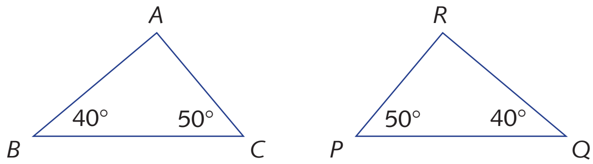
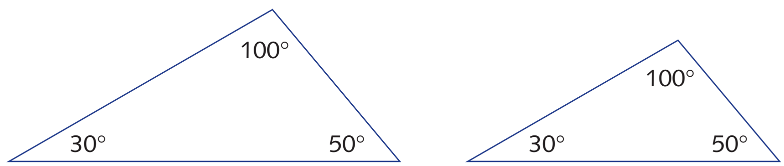
Notice that if two angles of one triangle are equal to two angles of another, then the third pair of angles are also equal. This is because the angle sum of a triangle is 180°. For example, the third angle in each triangle below is 90°.



Thus *BC* and *PQ* are matching sides in this situation also. (They are matching because they are opposite equal angles.)

Now consider the diagrams below.



The two triangles above both have angles of 30°, 50° and 100°, but they are not congruent, because they have different sizes. This shows that just knowing that three pairs of angles are equal is not enough to establish congruence.

If, however, we know also that a pair of matching sides are equal, then the two triangles are congruent. This test for congruence is called the **AAS congruence test**. The initials **AAS** stand for **A**ngle, **A**ngle, **S**ide.

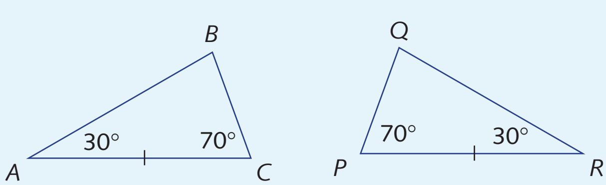
AAS congruence test

If two angles and one side of one triangle are equal to two angles and the matching side of another triangle, then the two triangles are congruent.

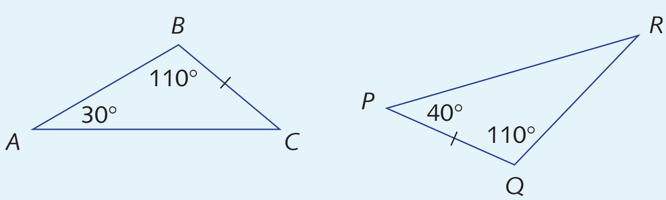
Example 2

In each part, write down a statement that the two triangles are congruent, giving the appropriate congruence test as a reason.

a



b



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Solution

a In the triangles ABC and PQR:

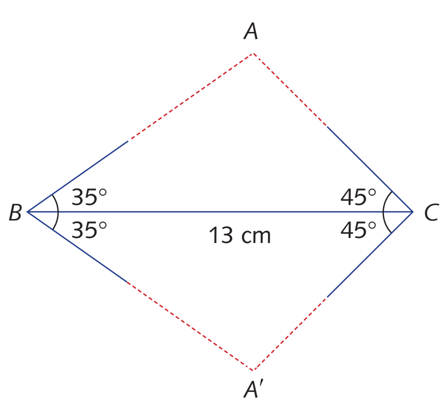
$$\angle A = \angle R = 30^\circ$$
$$\angle C = \angle P = 70^\circ$$
$$AC = PR \text{ (matching side)}$$
$$\text{so } \Delta ABC \equiv \Delta RQP \text{ (AAS)}$$

(Note that the second triangle is named in matching order. A matches R, B matches Q and C matches P.)

b In the triangles ABC and PQR:

$$\angle B = \angle Q = 110^\circ \text{ (given)}$$
$$\angle C = 40^\circ \text{ (angle sum of } \Delta ABC)$$
$$\text{so } \angle C = \angle P$$
$$BC = PQ \text{ (given)}$$
$$\text{so } \Delta ABC \equiv \Delta RQP \text{ (AAS)}$$

To see why the AAS test is valid, we look at a particular case of a triangle ABC with base $BC = 13$ cm and base angles equal to $B = 35^\circ$ and $C = 45^\circ$. How many such triangles are there? Are they all congruent to one another? We can construct two such triangles by drawing the rays shown and extending them to meet at the points A and A'.



Note: $\Delta A'BC$ is the reflection of ΔABC in the line BC because reflection preserves angles. So these two triangles are congruent.

Using congruence to find lengths and angles

Once the congruence of two triangles has been established, we can draw conclusions about the remaining pairs of matching angles or matching sides. This allows us to draw conclusions about lengths and angles.

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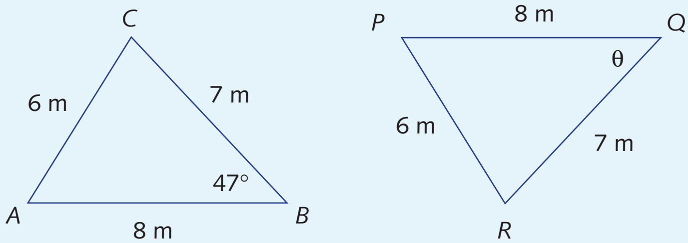
Example 3

Use congruence to find the value of θ.



Solution

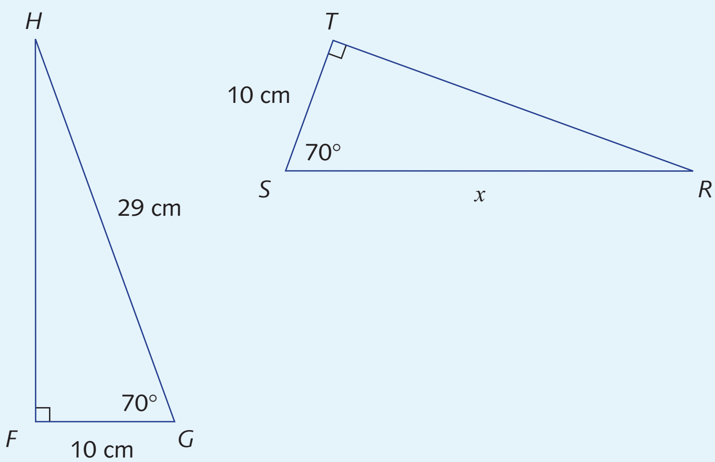
From the diagram, $\Delta ABC \equiv \Delta PQR$ (SSS)

so $\theta = \angle B$ (matching angles of congruent triangles)

$= 47°$

Example 4

Use congruence to find the value of x.



Solution

From the diagram, $\Delta FGH \equiv \Delta TSR$ (AAS)

so $x = HG$ (matching sides of congruent triangles)

$= 29$ cm

You may have noticed that the left-hand diagrams in Examples 3 and 4 have too much information. In both triangles, at least one measurement must be only approximate. All of this can safely be ignored at this stage.

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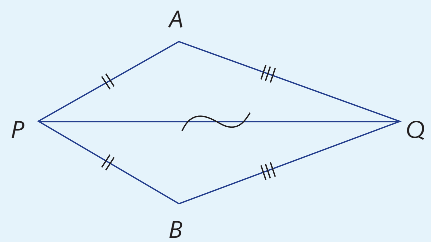
Using congruence to show that two lengths or two angles are equal

Congruence can be used to show that two matching lengths or two matching angles are equal.

Example 5

Use congruence to show that $\angle PAQ = \angle PBQ$.



Solution

From the diagram, $\Delta PAQ \equiv \Delta PBQ$ (SSS)

so $\Delta PAQ = \Delta PBQ$ (matching angles of congruent triangles)

Note: The side PQ is called **common** to both triangles, and is marked on the diagram with the wavy symbol ~. In the above example it provides the third pair of equal sides.

Exercise 12C

Question **1** is for class discussion.

1 **a** Three metal rods, of lengths 2 m, 3 m and 4 m, are bolted together with hinges at the corners. What congruence test explains why the structure is rigid?

b A rope is attached to the top of a vertical pole. The rope is then tied down to the ground so that it makes an angle of 20° to the vertical. Another identical pole nearby is secured in the same way. Do the two ropes necessarily have the same length? Why or why not?

2 **a** Using a ruler and compasses only, construct a triangle with side lengths of 6 cm, 8 cm and 5 cm.

b **i** What happens when you try to construct a triangle with side lengths of 4 cm, 5 cm and 12 cm?

ii Copy and complete: 'The longest side of a triangle …'

c Jack says that in the Great Outback there are three towns A, B and C. Town B is 34 km from A and 68 km from C. Town A is 110 km from C. (All distances are as the crow flies.) Comment on Jack's claim.

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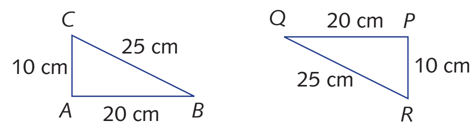
3 **a** Using a ruler and compasses only, construct a triangle ABC in which $AB = 12$ cm, $\angle A = 45°$ and $\angle B = 60°$. Measure the length of AC. Which congruence test tells you that your neighbour gets the same length?

b Ajun claims to have constructed a triangle in which one side is 10 cm and the angles at each end of this side are 105° and 80°. Comment on Ajun's claim.

Example 1, 2

4 In each part below, say whether the two triangles are congruent. If they are congruent, write a congruence statement, including the appropriate congruence test.

a



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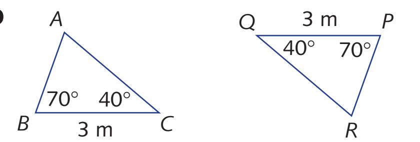

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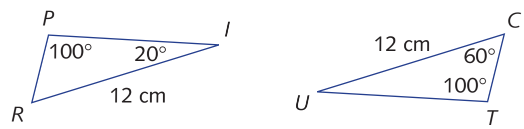

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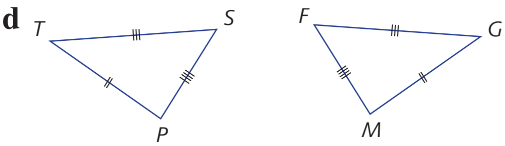

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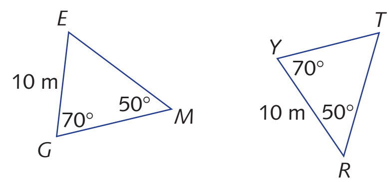

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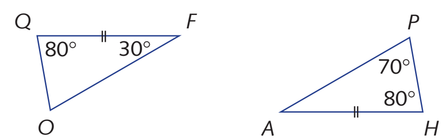
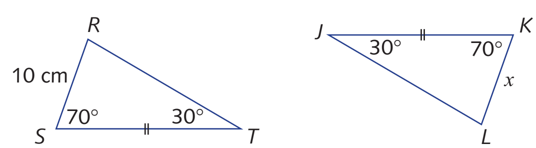


Example 3, 4

5 If possible, write down a congruence statement (with the appropriate congruence test) in each part. Then find the value of x or θ, giving reasons. Keep the vertices written in matching order.

a



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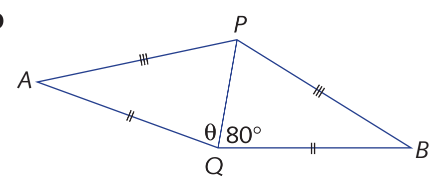

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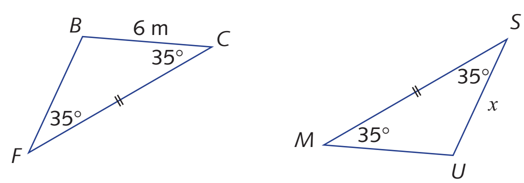
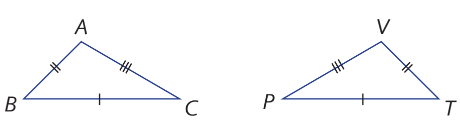


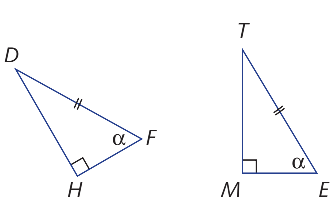
Example 5

6 In each part, first write a congruence statement. Then prove the required result, giving all reasons.

a Prove that $\angle ACB = \angle VPT$.



b Prove that $FH = EM$.



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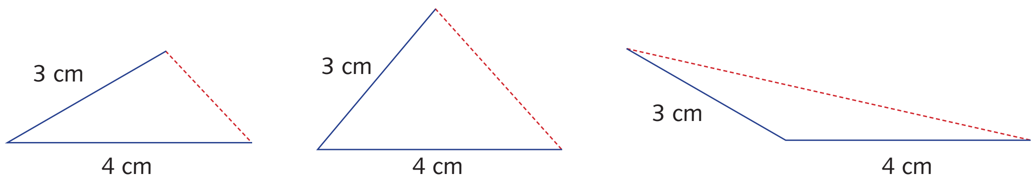
12D Congruent triangles: The SAS and RHS tests

We saw in the preceding section that having two pairs of equal sides is not enough to establish that two triangles are congruent, because the two sides can flap about freely, like the blades of a pair of scissors.



The two tests discussed in this section stop the two sides flapping about by specifying one angle of the triangle as well.

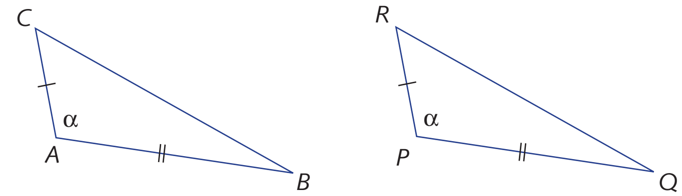
The SAS congruence test

The easiest way to stop the two sides flapping is to specify the angle between them. This angle is called the **included angle**. ('Included' means 'shut in'.) The resulting test is called the **SAS congruence test**. The letters SAS stand for **S**ide, **A**ngle, **S**ide.

SAS congruence test

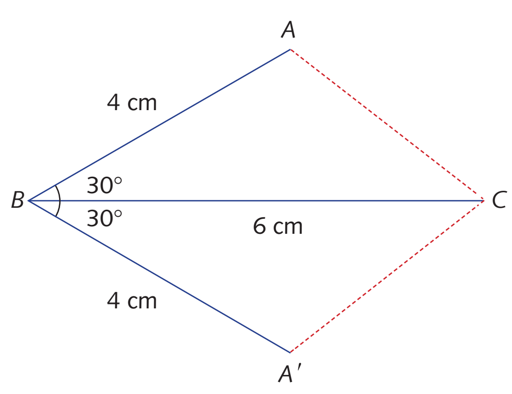
If two sides of one triangle are respectively equal to two sides of another triangle, and the angles included between the sides in each pair are equal, then the triangles are congruent.

Be careful! For the test to work, the angles *must* be the ones included between the sides in each pair.



In the diagram above, $\Delta ABC \equiv \Delta PQR$ (SAS).

To see why this is true, take a triangle ABC with side lengths $AB = 4$ cm, $BC = 6$ cm and included angle, $\angle B = 30°$. Its reflection in the line BC is also shown. Triangle ABC is congruent to triangle $A'BC$ because reflection in BC moves A to A'.



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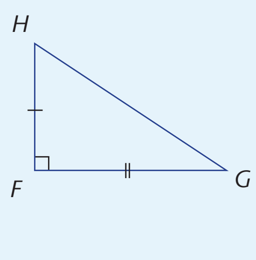
Check for yourself that any other triangle in the plane with the given side lengths and included angle is one of the triangles we have drawn with BC in a different position and orientation.

Example 6

Write down a statement that the two triangles below are congruent, giving the appropriate congruence test as a reason.



Solution

$\Delta HFG \equiv \Delta RTS$ (SAS)

The RHS congruence test

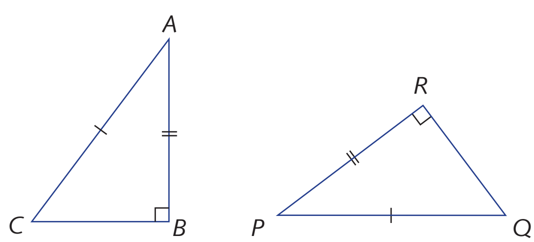
A useful test that can be applied to right-angled triangles, and is easy to apply, is the **RHS congruence test**. (The letters RHS stand for **R**ight angle, **H**ypotenuse, **S**ide.)

RHS congruence test

If two right-angled triangles have equal hypotenuses, and another pair of equal sides, then the triangles are congruent.

Proof of the RHS congruence test

$$\begin{aligned} BC^2 &= AC^2 - AB^2 \text{ (Pythagoras' theorem in } \Delta ABC) \\ &= PQ^2 - PR^2 \text{ (equal hypotenuse and equal side)} \\ &= QR^2 \text{ (right-angled } \Delta PQR) \end{aligned}$$



So $BC = QR$ and the two triangles have equal sides. Hence, they are congruent by the SSS congruence test.

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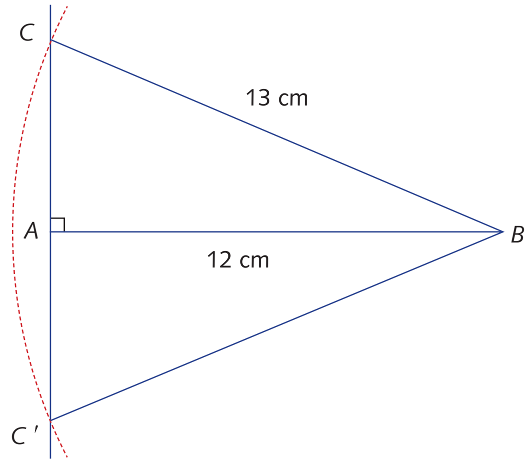
Constructing a right-angled triangle with a given hypotenuse and side

Construct a right-angled triangle ABC with $\angle A = 90°$, $BC = 13$ cm and $AB = 12$ cm.



The side BC is the hypotenuse of the right-angled triangle. By Pythagoras' theorem, it must be the longest side.

Step 1: In the middle of your page, draw an interval AB of length 12 cm.

Step 2: Draw a line through A at right angles to AB.

Step 3: With centre B and radius 13 cm, draw a circle crossing the vertical line at two points C and C'.

Step 4: Join up the triangle ABC and the triangle ABC'.

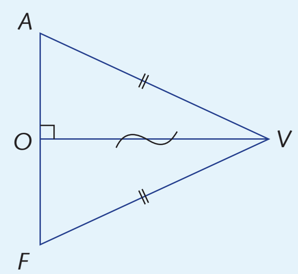
The two triangles you have drawn are congruent, being reflections of each other in AB.

Example 7

Write down a statement that the two triangles below are congruent, giving the appropriate congruence test as a reason.

Solution

$\Delta AOV \equiv \Delta FOV$ (RHS)



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The four standard congruence tests for triangles

Two triangles are **congruent** if:

SSS: the three sides of one triangle are respectively equal to the three sides of the other triangle, **or**

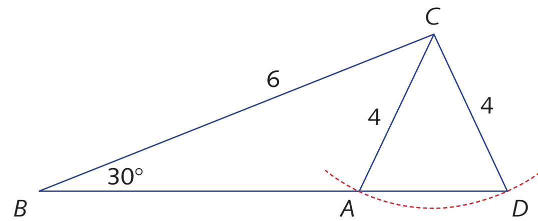
AAS: two angles and one side of one triangle are respectively equal to two angles and the matching side of the other triangle, **or**

SAS: two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle, **or**

RHS: the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of the other right-angled triangle.

The case of the non-included angle

In the diagram, triangles BCA and BCD have a common side BC, a common angle $\angle B$, and $AC = DC$. As you can see, the triangles are not congruent.



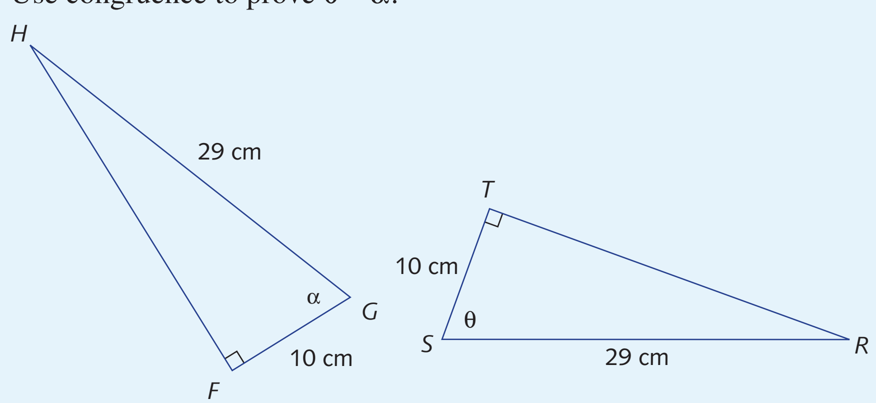
Therefore two sides and a non-included angle are not sufficient to determine congruency. Further discussion of the case of the non-included angle is to be found in Question **6** of Exercise 12D and Question **8** of the Challenge exercise.

Using congruence to find sides or angles

When two triangles are congruent, and we are given side lengths and/or angles of one of them, we can find the matching sides and/or angles of the other.

Example 8

Use congruence to prove $\theta = \alpha$.



Solution

From the diagram, $\Delta FGH \equiv \Delta TSR$ (RHS)

so $\theta = \alpha$ (matching angles of congruent triangles)

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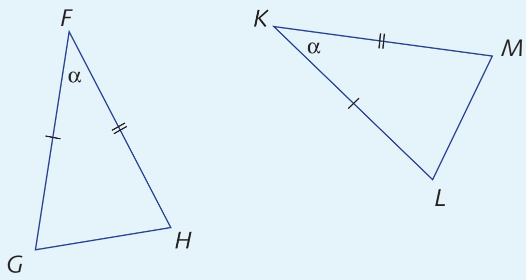
Example 9

Use congruence to show that $GH = LM$.



Solution

From the diagram, $\Delta GFH \equiv \Delta LKM$ (SAS)

so $GH = LM$ (matching sides of congruent triangles)

Exercise 12D

Question **1** is suitable for discussion.

1 **a** Two pairs of compasses, each with legs 12 cm long, are opened so that the arms make an angle of 25°. What congruence test explains why the distance between the point and the pencil lead is the same in both compasses?

b Two 6-metre ladders on horizontal ground rest against a wall, with each ladder reaching 5 metres up the wall. What congruence test explains why the angle between the ladder and the ground is the same for both ladders?

c Pradap attached a rope 9 metres up a vertical pole, then stretched the rope and secured it to the ground near the pole. He measured the length of the rope to be 8.5 metres. Comment on this situation.

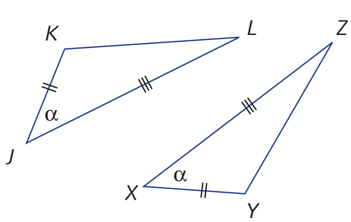
2 **a** Using a ruler and compasses only, construct an isosceles triangle with legs that are 8 cm each and with an apex angle that is 45°. Measure the length of the base.

b Using ruler and compasses only, construct a right-angled triangle ABC in which $\angle A$ is a right angle, $AB = 6$ cm and the hypotenuse $BC = 8$ cm. Use your protractor to measure the size of $\angle B$.

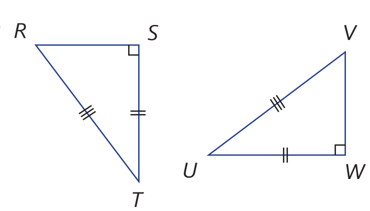
Example 6, 7

3 In each part below, if the two triangles are congruent, write down the congruence statement, giving the appropriate congruence test as a reason.

a



b



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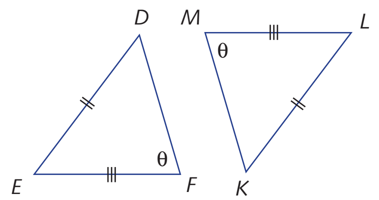
c



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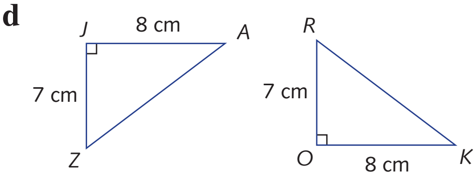

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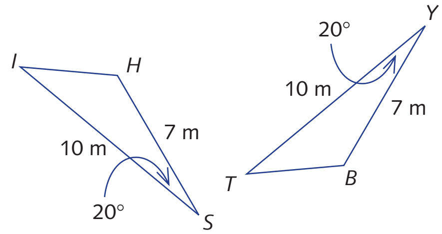

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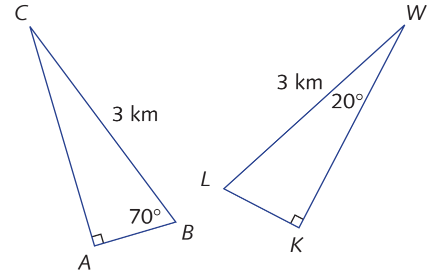
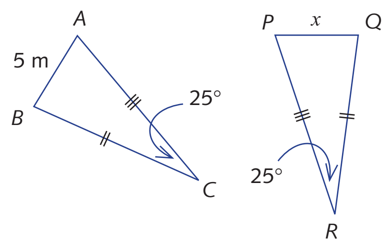


4 If possible, write down a congruence statement (with the appropriate congruence test). Then find the value of x or θ, giving the reason.

a



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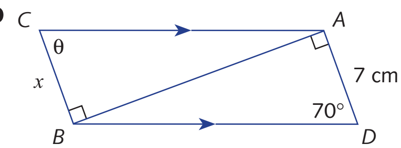

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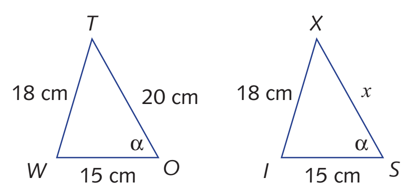

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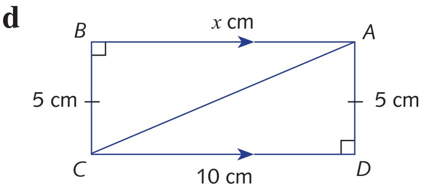
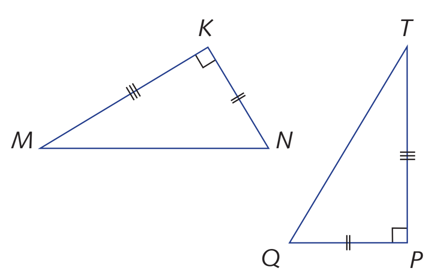


Example 8, 9

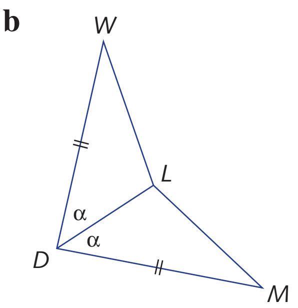
5 In each part, first write a congruence statement. Then prove the required result, giving all reasons.

a



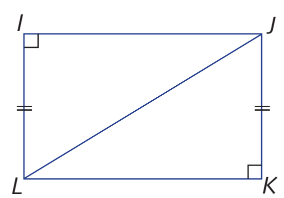
Prove that $\angle M = \angle T$.

b



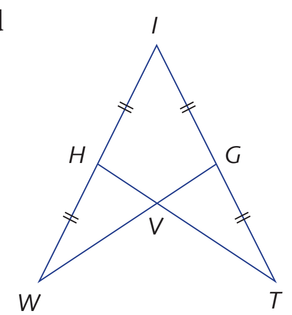
Prove that $WL = ML$.

c



Prove that $IJ = KL$.

d



Prove that $HT = GW$.

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6 The SAS congruence test requires that the angle be the angle included between the two sides. Here is an example where the angle is not included – the construction will produce two triangles that are not congruent. Construct a triangle ABC with $AC = 8$ cm, $BC = 6$ cm and $\angle A = 30°$ as follows.

Step 1: Draw a long horizontal line AX near the bottom of a new page in your exercise book.

Step 2: Construct an angle of 30° at A.

Step 3: Mark C on the sloping arm so that $AC = 8$ cm.

Step 4: With centre C and radius 6 cm, draw an arc cutting the horizontal line at B and B'.

Step 5: Join CB and CB'.

Then ΔABC and $\Delta AB'C$ both fulfil the conditions, but they are clearly not congruent.

12E Using congruence in geometrical problems

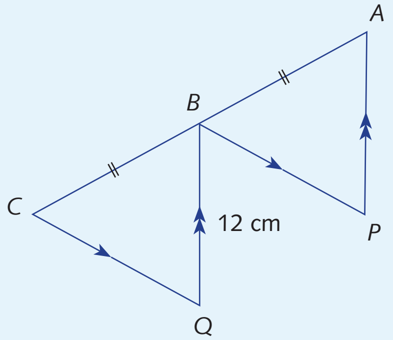
This section shows how to use congruence in geometrical reasoning and problems.

Setting out congruence proofs

Here are three examples showing a recommended way of writing out a solution to a problem involving congruence. Attention to detail is critical.

Example 10

Find the length of AP in the diagram below.



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Solution

In the triangles ABP and BCQ:

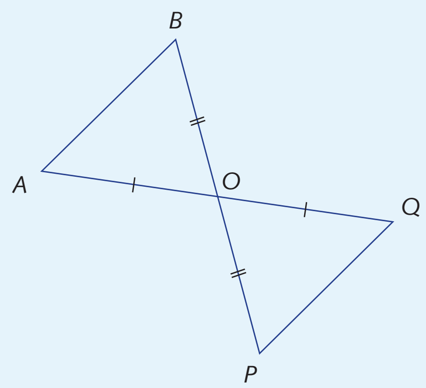
$AB = BC$ (given)
$\angle BAP = \angle CBQ$ (corresponding angles, $AP \parallel BQ$)
$\angle ABP = \angle BCQ$ (corresponding angles, $BP \parallel CQ$)
so $\Delta ABP \equiv \Delta BCQ$ (AAS)
Hence, $AP = 12$ cm (matching sides of congruent triangles)

Proving that two lines are parallel

Congruence is often used to prove that two alternating angles or two corresponding angles are equal. It then follows that the associated lines are parallel.

Example 11

Prove that $AB \parallel PQ$ in the diagram below.



Solution

In the triangles AOB and QOP:

$AO = QO$ (given)
$BO = PO$ (given)
$\angle AOB = \angle QOP$ (vertically opposite at O)
so $\Delta AOB \equiv \Delta QOP$ (SAS)
Hence $\angle A = \angle Q$ (matching angles of congruent triangles)
so $AB \parallel PQ$ (alternate angles are equal)

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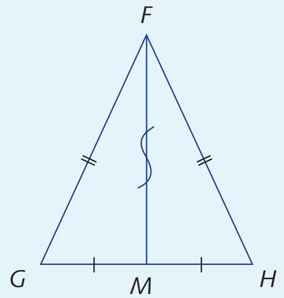
Proving that two lines are perpendicular

If two angles are equal and add to 180°, they must both be right angles. This idea is often used in problems to prove that two lines are perpendicular.

Example 12

In the diagram opposite, FM joins the apex F of the isosceles ΔFGH to the midpoint M of its base. Use congruence to prove that $FM \perp GH$; that is, FM is perpendicular to GH.



Solution

In the triangles GFM and HFM:

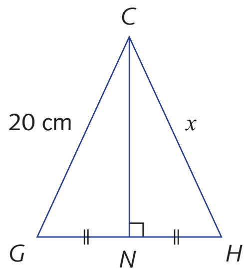
$$\begin{aligned} FM &= FM \text{ (common)} \\ FG &= FH \text{ (given)} \\ GM &= HM \text{ (given)} \\ \text{so } \Delta GFM &\equiv \Delta HFM \text{ (SSS)} \\ \text{Hence } \angle GMF &= \angle HMF \text{ (matching angles of congruent triangles)} \\ \text{But } \angle GMF + \angle HMF &= 180^\circ \text{ (straight angle at } M\text{)} \\ \text{so } \angle GMF &= \angle HMF \\ &= 90^\circ \end{aligned}$$

Exercise 12E

Example 10

1 In each part below, prove that two triangles are congruent. Hence, find the value of x or θ. The point O is always the centre of the circle.

a



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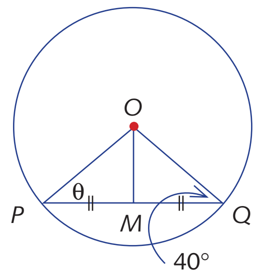
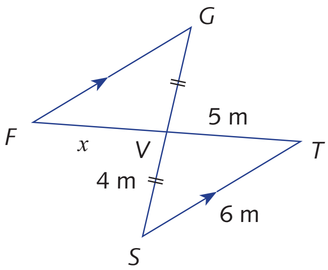


c



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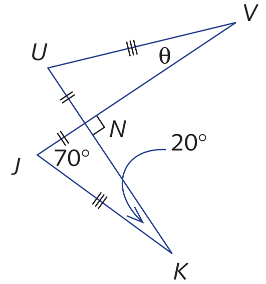
d



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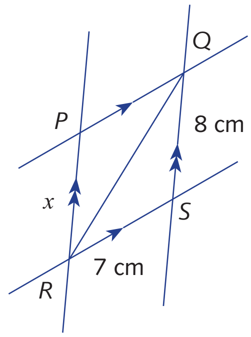

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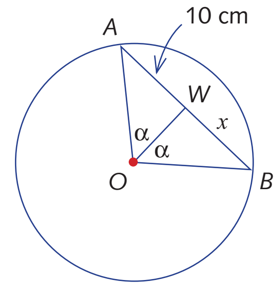
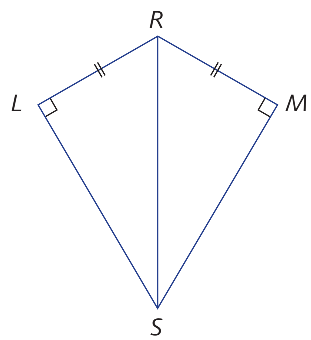


2 In these problems, the points O and Z are always the centres of circles.

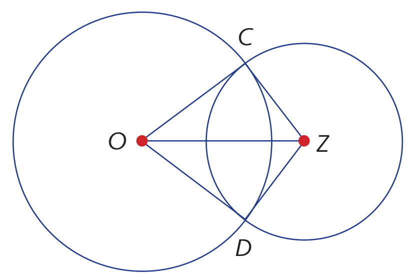
a



i Prove that $\Delta RLS \equiv \Delta RMS$.

ii Hence, show that $LS = MS$.

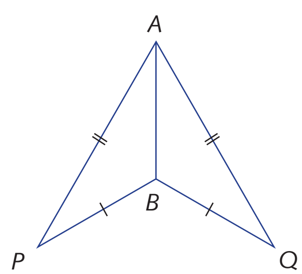
b



i Prove that $\Delta COZ \equiv \Delta DOZ$.

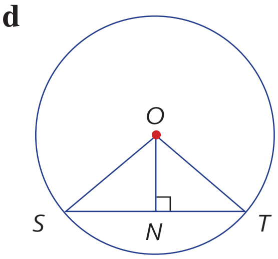
ii Hence, show that $\angle OCZ = \angle ODZ$.

c



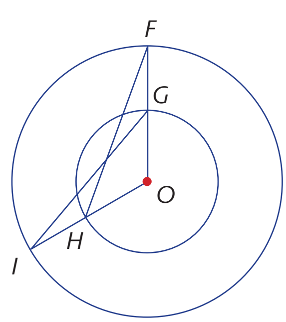
Prove that $\angle P = \angle Q$.

d



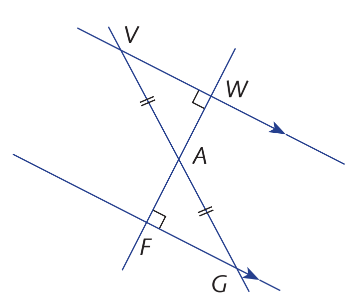
Prove that $SN = TN$.

e



Prove that $\angle OFH = \angle OIG$.

f



Prove that $VW = FG$.

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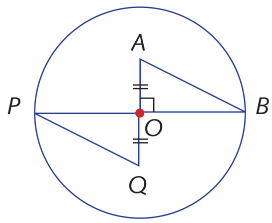
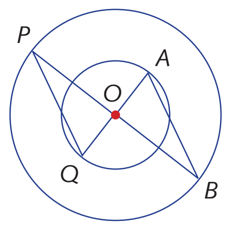
Example 11

3 In each part below, use congruent triangles to prove that the lines AB and PQ are parallel. The point O is always the centre of the circle.

a



b



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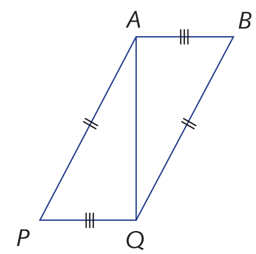
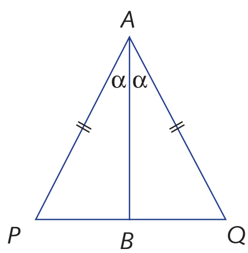


Example 12

4 In each part below, use congruent triangles to prove that the lines AB and PQ are perpendicular. The point O is always the centre of the circle.

a



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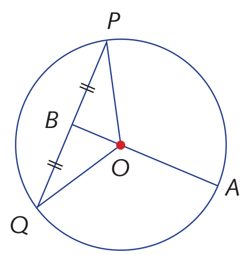

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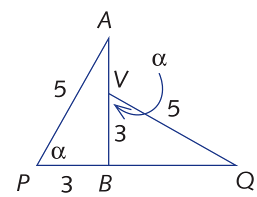
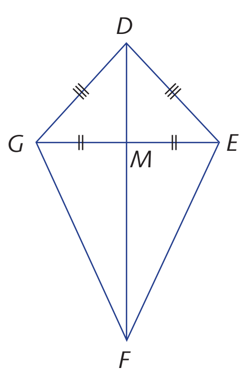


5 Each of these problems requires two applications of congruence. The points O and Z are the centres of circles.

a



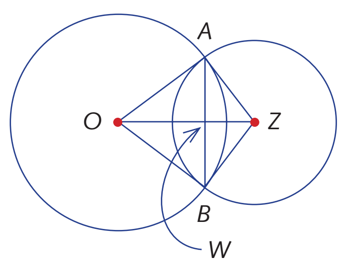
i Prove that $\Delta GDM \equiv \Delta EDM$.

ii Hence, show that $DM \perp GE$.

iii Prove that $\Delta GMF \equiv \Delta EMF$.

iv Hence, show that $GF = EF$. $DEFG$ is called a kite.

b



i Prove that $\Delta OAZ \equiv \Delta OBZ$.

ii Hence, show that $\angle AOW = \angle BOW$.

iii Prove that $\Delta AOW \equiv \Delta BOW$.

iv Hence, show that OZ is the perpendicular bisector of AB.

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12F Congruence and special triangles

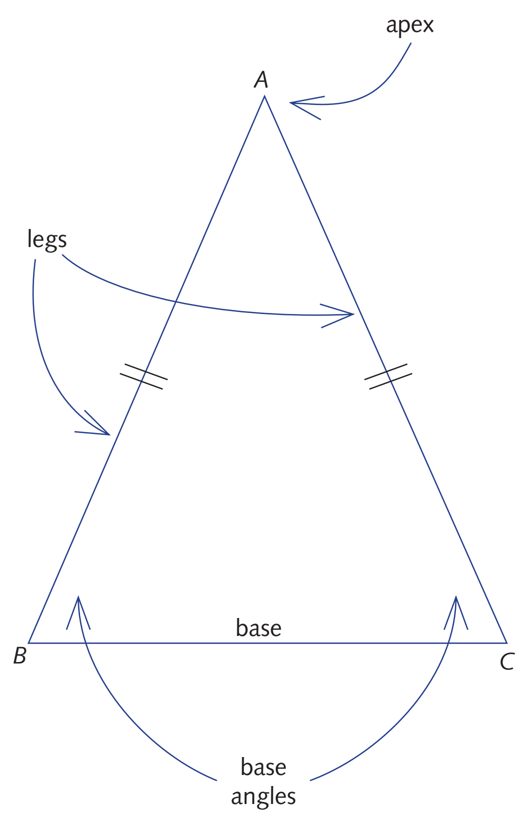
Isosceles and equilateral triangles were introduced in Chapter 13 of *ICE-EM Mathematics Year 7*. Congruence allows us to give proper proofs of three basic theorems about these special triangles.

First, here is a reminder of the definitions of isosceles and equilateral triangles.

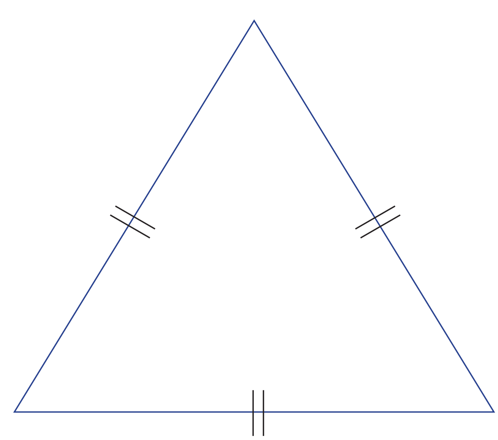
Isosceles and equilateral triangles

- An **isosceles triangle** is a triangle with two (or more) sides equal.
  - The equal sides are called the **legs** and the third side is called the **base**.
  - The legs meet at the **apex** and the other two angles are the **base angles**.
- An **equilateral triangle** is a triangle with all three sides equal.

The word ‘isosceles’ comes from Greek and means ‘equal legs’. The word ‘equilateral’ comes from Latin and means ‘equal sides’.



Isosceles triangle



Equilateral triangle

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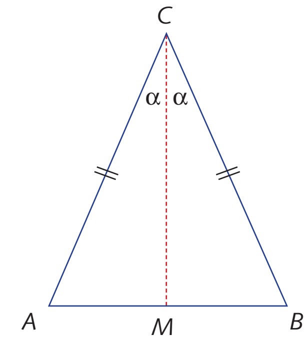
The base angles of an isosceles triangle are equal

Notice that the proof is based on congruent triangles.



Theorem: The base angles of an isosceles triangle are equal.

Proof: Let ΔABC be isosceles, with $CA = CB$.

We need to prove that $\angle A = \angle B$.

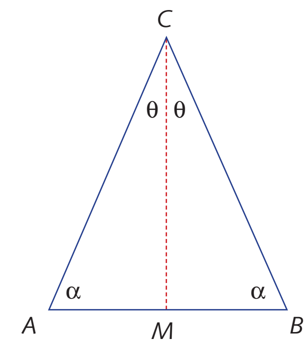
Draw the bisector of $\angle ACB$, and let it meet AB at M.

In the triangles ACM and BCM:

$CA = CB$ (given)
$CM = CM$ (common side)
$\angle ACM = \angle BCM$ (construction)
so $\Delta ACM \equiv \Delta BCM$ (SAS)
Hence $\angle A = \angle B$ (matching angles of congruent triangles)

If two angles of a triangle are equal, then the sides opposite those angles are equal

This proof is also based on congruence.



Theorem: If two angles of a triangle are equal, then the sides opposite those angles are equal. (That is, the triangle is isosceles.)

Proof: Let ΔABC be a triangle with $\angle A = \angle B$.

We need to prove that $CA = CB$.

Draw the bisector of $\angle ACB$, and let it meet AB at M.

In the triangles AMC and BMC:

$\angle CAM = \angle CBM$ (given)
$\angle ACM = \angle BCM$ (construction)
$CM = CM$ (common side)
so $\Delta AMC \equiv \Delta BMC$ (AAS)
Hence $CA = CB$ (matching sides of congruent triangles)

That is, the triangle is isosceles.

This second theorem is the **converse** of the first theorem.

- The first theorem says: 'If two sides of a triangle are equal, then the angles opposite those sides are equal'.
- The second theorem says: 'If two angles of a triangle are equal, then the sides opposite those angles are equal'.

We have now proved that both the theorem and its converse are true.

Isosceles triangles

- The base angles of an isosceles triangle are equal.
- Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal. (That is, the triangle is isosceles.)

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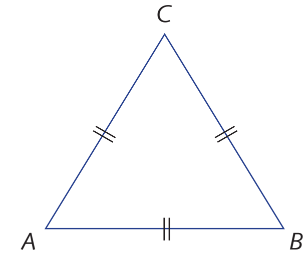
The angles of an equilateral triangle are all 60°

We can prove this result, by applying the earlier result that the base angles of an isosceles triangle are equal. This result is a simple corollary of the earlier result that the base angles of an isosceles triangle are equal.



Theorem: The interior angles of an equilateral triangle are all 60°.

Proof:

$AC = AB$ (given)
so $\angle C = \angle B$ (angles opposite equal sides)
Also $CA = CB$ (given)
so $\angle A = \angle B$ (angles opposite equal sides)

Therefore $\angle A = \angle B = \angle C$

Since the angles add to 180°, each angle must be $180 \div 3 = 60°$.

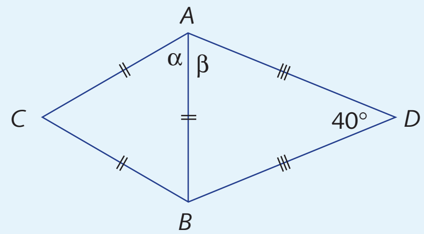
Equilateral triangles

- Each interior angle of an equilateral triangle is 60°.
- Conversely, if all the angles of a triangle are equal, then the triangle is equilateral.

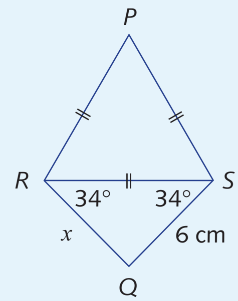
The last dot point is proven in Question **2** of Exercise 12F.

Example 13

a Find α and β in the diagram below.



b Find x and $\angle PSQ$ in the diagram below.



Solution

a First, $\alpha = 60°$ (equilateral ΔABC)
Also, $\angle DBA = \beta$ (base angles of isosceles ΔABD)
so $\beta + \beta + 40° = 180°$ (angle sum of ΔDAB)
Hence $\beta = 70°$

b First, $\angle PSR = 60°$ (equilateral ΔABC)
Also, $\angle PSQ = 94°$ (adjacent angles)
so $RQ = SQ$ (opposite sides of ΔQRS are equal),
Hence $x = 6$ cm

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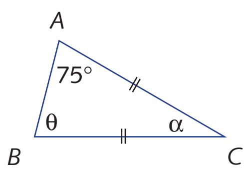
Exercise 12F

Example 13

1 Use the results of this section about isosceles and equilateral triangles to find the values of x, y, α, β, γ and θ. Give all reasons in your solutions. The point O in part **n** is the centre of the circle.

a


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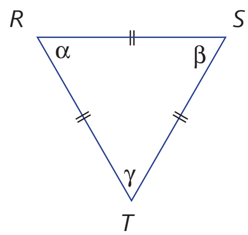

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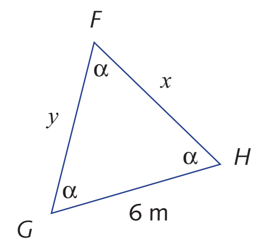

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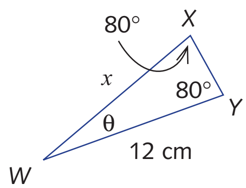

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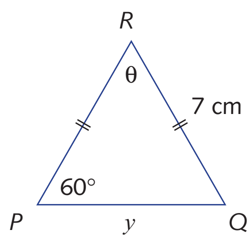

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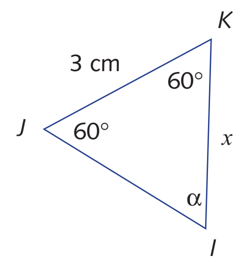
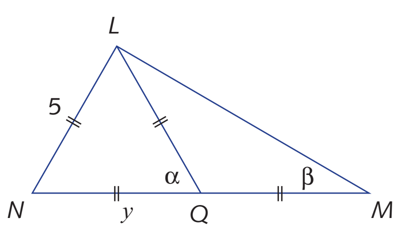


g


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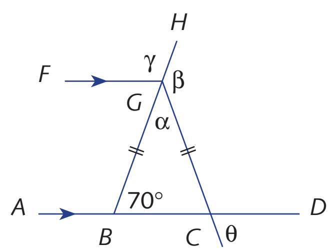
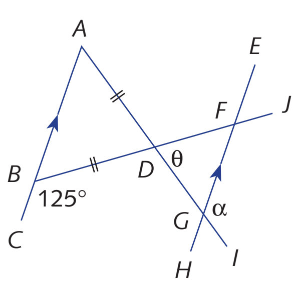


i


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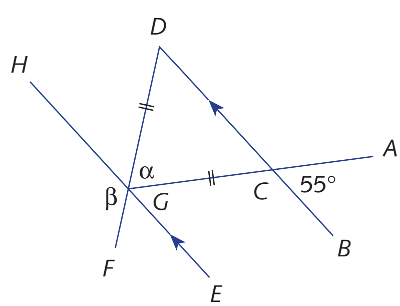

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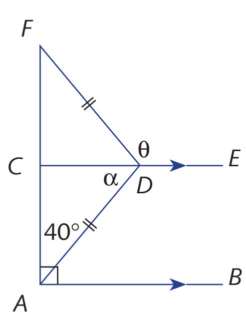

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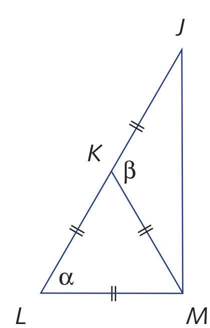

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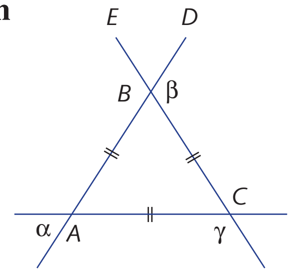
m



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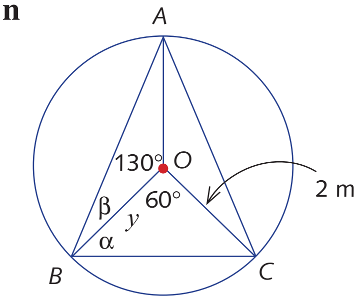
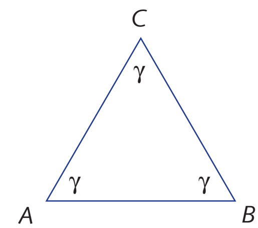


2 In ΔABC opposite, all angles are equal.

a Explain why all the angles are 60°.

b Explain why the triangle is equilateral.

You have now proven a test for an equilateral triangle:
'*If all angles of a triangle are equal, then the triangle is equilateral*'.

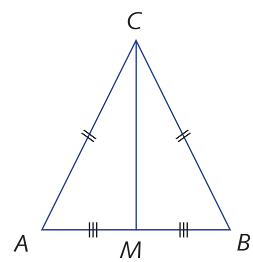


3 Let ΔABC be isosceles, with $AC = BC$, and join CM, where M is the midpoint of the base AB.

a Show that $\Delta AMC \equiv \Delta BMC$.

b Hence, show that $\angle A = \angle B$.

This is a second proof of '*the base angles of an isosceles triangle are equal*'.

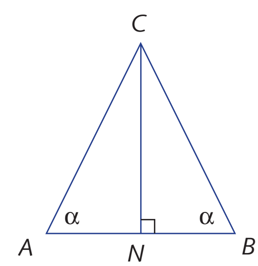


4 Let ABC be a triangle with $\angle A = \angle B$. We will prove that $AC = BC$.

Construct the line through C perpendicular to the base AB.
Let this line meet AB at N.

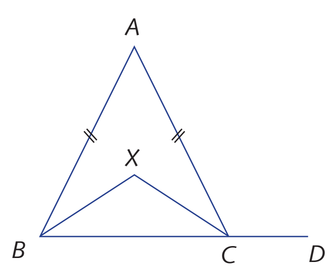
a Show that $\Delta ANC \equiv \Delta BNC$.

b Hence, show that $AC = BC$.



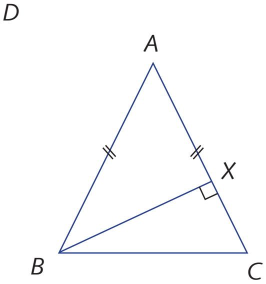
This is another proof of the result '*If the two base angles of a triangle are equal then the sides opposite those angles are equal*'.

5 ΔABC is an isosceles triangle with $AB = AC$.
The angle bisector of $\angle B$ intersects the angle bisector of $\angle C$ at X.
Prove that $\angle BXC = \angle ACD$.



6 ΔABC is an isosceles triangle with $AB = AC$. X is the point on AC such that BX is perpendicular to AC.

Prove that $\angle XBC = \frac{1}{2}\angle BAC$.



7 The bisector of an angle of a triangle cuts the opposite side at right angles. Prove that the triangle is isosceles.

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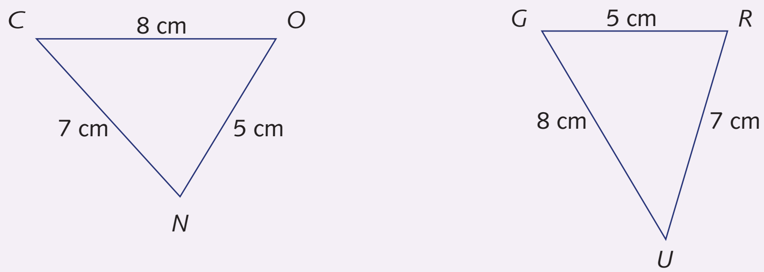
Review exercise

1 In each part write a congruence statement and give a reason.

a



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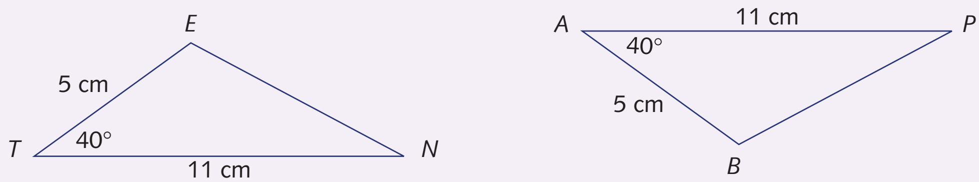

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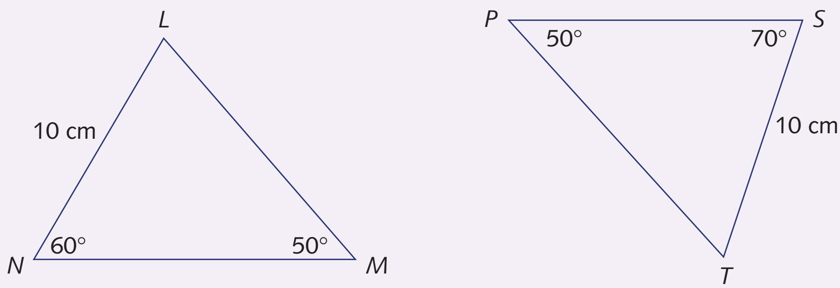
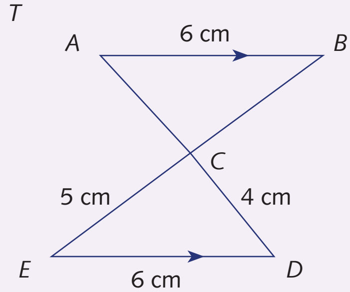


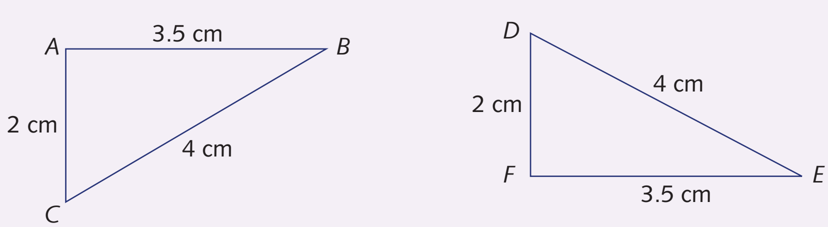
2 **a** In the diagram opposite, name the two congruent triangles and explain why they are congruent.

b Find AC.



3 For each pair, write a congruence statement and give a reason.

a



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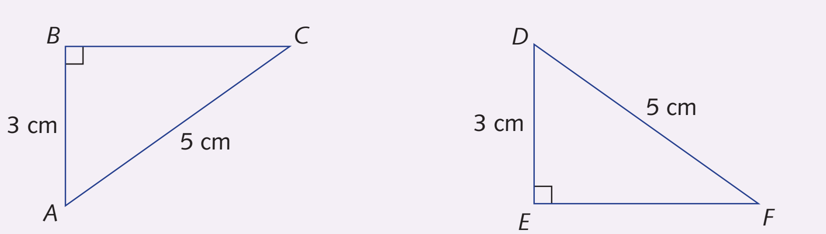

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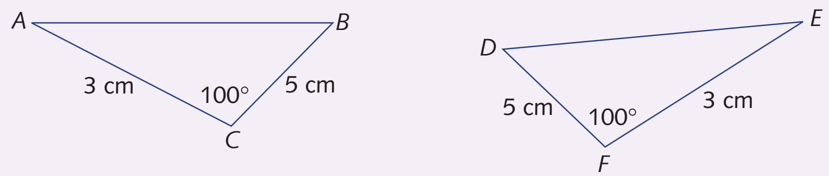

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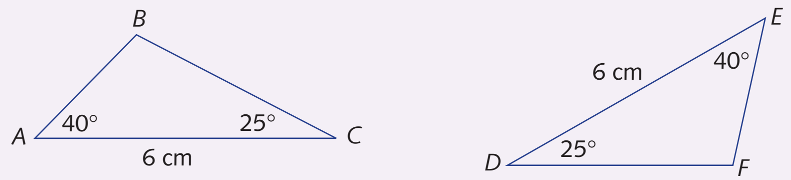
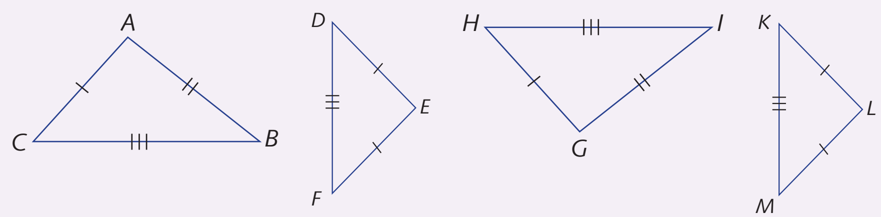


4 In each part, name the triangle congruent to ΔABC.

a



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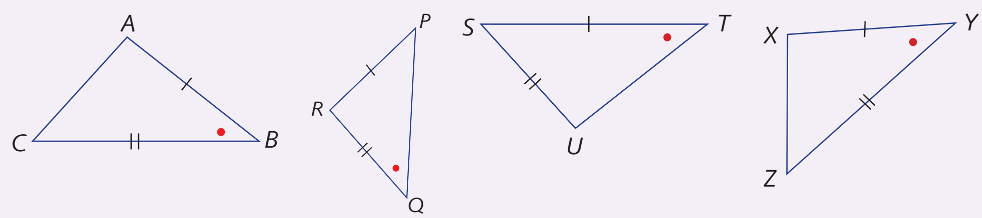

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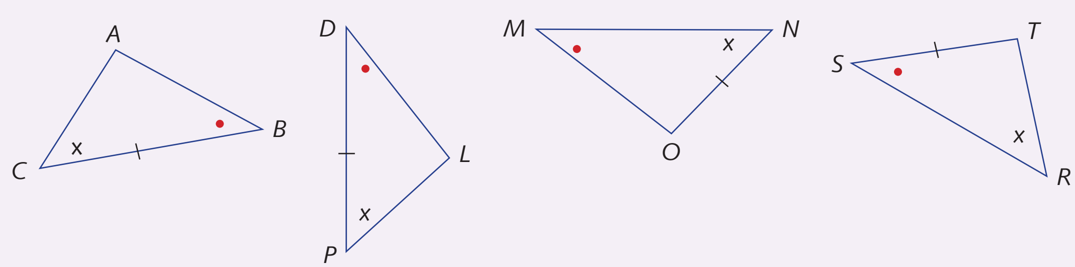
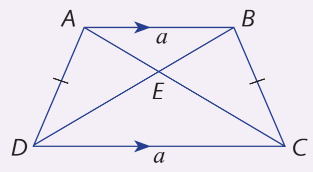


5 In each part, name all the pairs of congruent triangles, giving the abbreviated reason.

a



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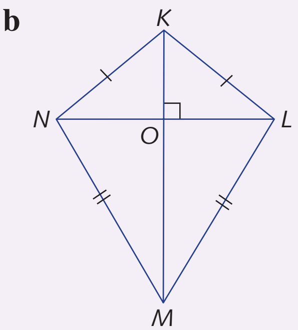

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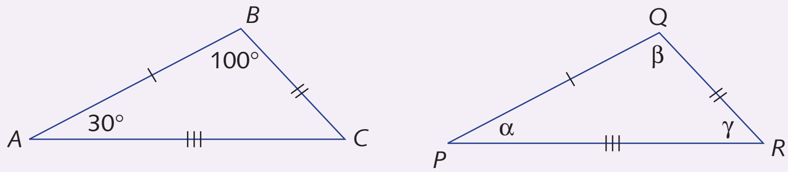
6 Find the values of the pronumerals in each.

a



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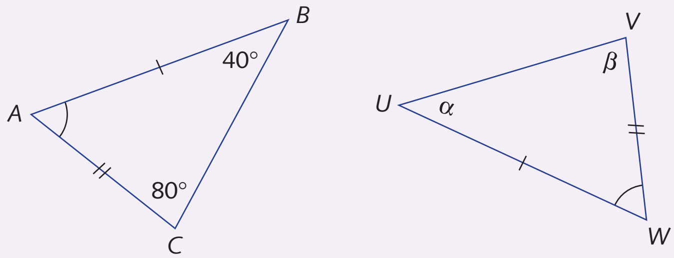

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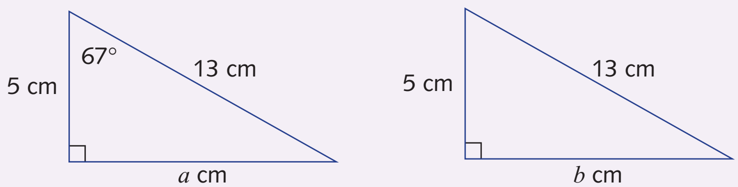

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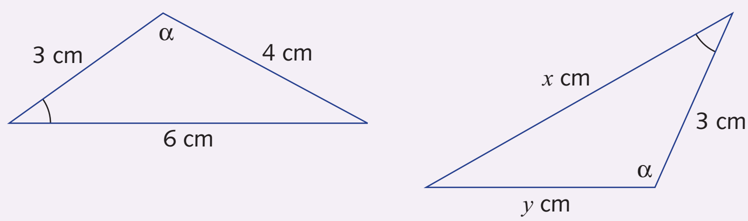

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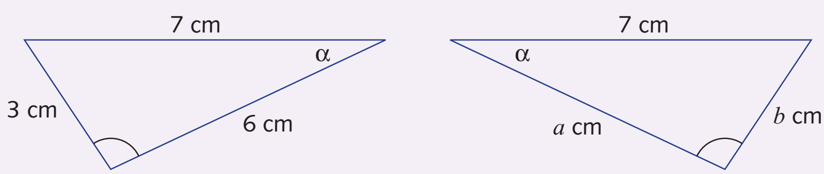
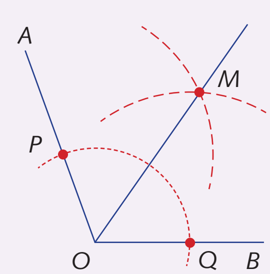


Challenge exercise

1 This question provides a proof of the angle bisector construction. The arcs intersecting at M are centred at P and Q.



a Copy the diagram and join the intervals PM and QM.

b Prove that $\Delta PMO \equiv \Delta QMO$.

c Hence explain why OM is the bisector of $\angle AOB$.

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2 This question provides a proof of the construction of the perpendicular bisector of an interval. The arcs are of the same radius and centred at A and B.

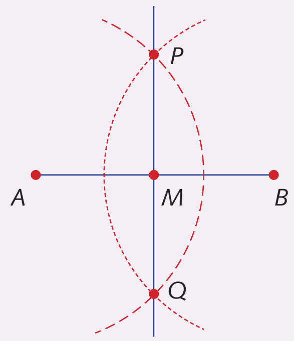
a Copy the diagram and join the intervals AP, BP, AQ and BQ.

b Prove that $\Delta APQ \equiv \Delta BPQ$.

c Hence prove that $\angle APQ = \angle BPQ$.

d Prove that $\Delta APM \equiv \Delta BPM$.

e Hence prove that $AM = BM$ and $PQ \perp AB$.



3 This question provides a proof of the construction of the perpendicular from a given point to a given line.

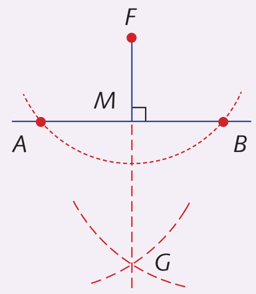
a Copy the diagram and join the intervals AF, BF, AG, BG and FG.

b Prove that $\Delta AFG \equiv \Delta BFG$.

c Hence prove that $\angle AFG = \angle BFG$.

d Prove that $\Delta AFM \equiv \Delta BFM$.

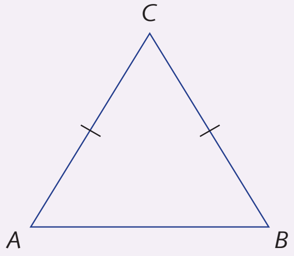
e Hence prove that $FG \perp AB$.



4 In this question, we prove that the base angles of an isosceles triangle are equal by proving that the triangle is congruent to itself. The proof is reputedly due to the Greek mathematician Pappus, who lived in Alexandria from about 290 CE to about 350 CE. Let ΔABC be isosceles, with $AC = BC$.

a Prove that $\Delta ACB \equiv \Delta BCA$. (Note the changed order of the vertices.)

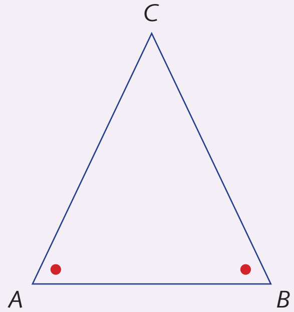
b Hence show that $\angle A = \angle B$.



5 In this question, we give another proof that a triangle with equal base angles is isosceles. This proof involves no construction – the triangle is proven to be congruent to itself. Let ABC be a triangle with $\angle A = \angle B$. We just prove that $AC = BC$.

a Prove that $\Delta ABC \equiv \Delta BAC$.

b Hence show that $AC = BC$.



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6 Explain why the supposed 'proof' outlined below is invalid.

Let ABC be a triangle with $\angle A = \angle B$.

Mark the midpoint M of the base AB, and join CM.

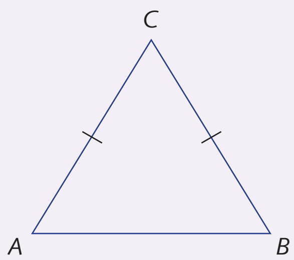
Then $\Delta AMC \equiv \Delta BMC$.

Hence $AC = BC$.

7 The result 'proven' below is clearly false.
Explain why the reasoning is invalid.

Let ΔABC be an isosceles triangle with $AC = BC$.

Let P be any point on the base AB, and join PC.



In the triangles ACP and BCP:

$CP = CP$	(common)
$CA = CB$	(given)
$\angle CAP = \angle CBP$	(base angles of isosceles ΔABC)
so $\Delta ACP \equiv \Delta BCP$	(SAS)
Hence $AP = BP$	(matching sides of congruent triangles)

so P is the midpoint of AB.

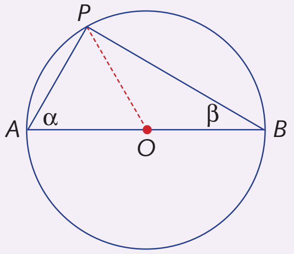
8 There is a valid 'OSS congruence test' (where O stands for 'obtuse angle'):

'If two sides and an obtuse non-included angle of one triangle are equal to two sides and a matching obtuse non-included angle of another triangle, then the triangles are congruent.'

Explain why this result is true.

9 The Greeks said that the oldest geometrical theorem stated and proved was Thales' theorem. Thales' theorem states:

'An angle in a semicircle is a right angle.'



a Let AOB be a diameter of a circle with centre O. Let P be any other point on the circumference. Join AP, BP and OP, and let $\angle PAB = \alpha$ and $\angle PBA = \beta$.

b Use isosceles triangles to prove that $\angle APB = 90°$.

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10 The lines through each vertex of a triangle perpendicular to the opposite side are called the **altitudes** of a triangle. They are concurrent at the **orthocentre** of the triangle. (This can be proved using theorems that you will meet in Year 10.) The orthocentre may lie outside the triangle.

a On a large sheet of paper, draw an obtuse-angled triangle, then construct its orthocentre. You will need to produce two of the sides to perform this construction.

b Let H be the orthocentre of ΔABC. Prove that each of the four points A, B, C and H is the orthocentre of the triangle formed by the other three points. This is true in both cases – when ΔABC is acute-angled and when ΔABC is obtuse-angled.

11 The perpendicular bisectors of the sides of a triangle are concurrent at a point called the **circumcentre**, and this point is the centre of the circle through all three vertices. Here is an outline of a proof. Check the details.

a Take a triangle ABC. Let P be the midpoint of BC, Q be the midpoint of CA, and R be the midpoint of AB. Let the perpendicular bisectors of AB and AC meet at M. Join MA, MB and MC.

b Prove that $\Delta AMR \equiv \Delta BMR$ and that $\Delta AMQ \equiv \Delta CMQ$.

c Hence prove that the circle with centre M and radius MA passes through B and C.

d Prove that MP is perpendicular to BC.

e Under what circumstances does the circumcentre lie outside the triangle?

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