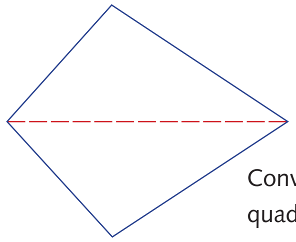
CHAPTER

13

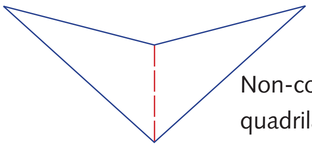
Measurement and Geometry

Congruence and special quadrilaterals

In any quadrilateral, whether it is convex or non-convex, the sum of the interior angles is 360º.



Convex quadrilateral



Non-convex quadrilateral

To show this, we divide the quadrilateral into two triangles and use the fact that the angle sum of each triangle is 180°.

The sum of the exterior angles is 360°.

However, if the quadrilateral has extra properties then new features emerge. Special quadrilaterals include squares, rectangles, parallelograms, rhombuses, trapeziums, kites and cyclic quadrilaterals. In this chapter we will begin to investigate the first four of these and prove special results about them using parallel lines, isosceles triangles and congruence.

We will study the trapezium and the kite in *ICE-EM Mathematics Year 9*, Chapter 7 and cyclic quadrilaterals in *Year 10*, Chapter 13.

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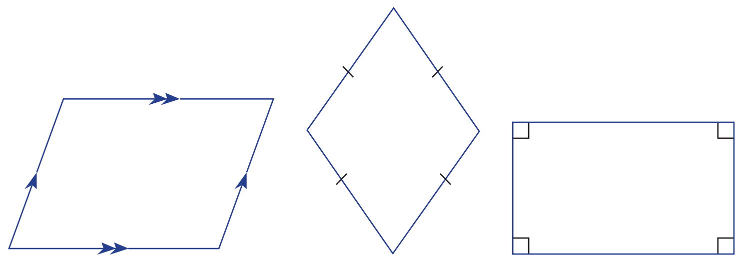
Investigation on diagonals

A **parallelogram** is a quadrilateral with two pairs of parallel sides, a **rhombus** has four equal sides and a **rectangle** has four right angles.

Draw accurate diagrams of a parallelogram, a rhombus and a rectangle in your workbook.



Join the diagonals of each quadrilateral.

1 In which figures are the diagonals of equal length?

2 In which figures do the diagonals bisect each other?

3 In which figures do the diagonals meet at right angles to each other?

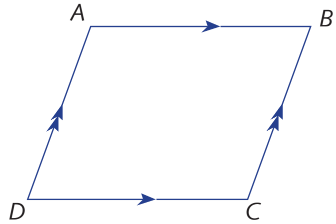
4 A **square** is a quadrilateral that is both a rectangle and a rhombus. What properties do its diagonals have?

In the following sections, we will give geometric proofs of the results you have just discovered by measurement, and we will do much more.

13A Parallelograms and their properties

A **parallelogram** is a quadrilateral with opposite sides that are parallel. Thus the quadrilateral $ABCD$ shown opposite is a parallelogram because $AB \parallel DC$ and $DA \parallel CB$.

The word 'parallelogram' comes from Greek words meaning 'parallel lines'.



Definition of a parallelogram

A **parallelogram** is a quadrilateral with opposite sides that are parallel.

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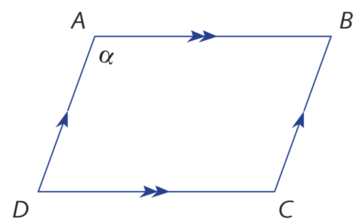
Properties of a parallelogram

First property: the opposite angles of a parallelogram are equal



Theorem: The opposite angles of a parallelogram are equal.

Proof: Let $ABCD$ be a parallelogram.
Let $\angle A = \alpha$.
We need to prove that $\angle C = \alpha$.

$\angle D = 180° - \alpha$ (co-interior angles, $AB \parallel DC$)

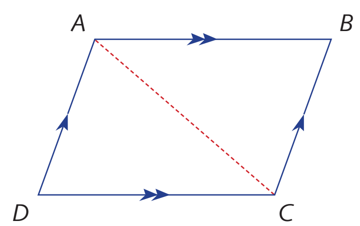
$\angle C = 180° - \angle D$ (co-interior angles, $AD \parallel BC$)

$= 180° - (180° - \alpha)$

$= \alpha$

Hence, the opposite angles of a parallelogram are equal.

Second property: the opposite sides of a parallelogram are equal



Theorem: The opposite sides of a parallelogram are equal.

Proof: Let $ABCD$ be a parallelogram.

We need to prove that $AB = BC$ and $AD = BC$.

Join the diagonal AC.

In the triangles ABC and CDA:

$\angle BAC = \angle DCA$ (alternate angles, $AB \parallel DC$)

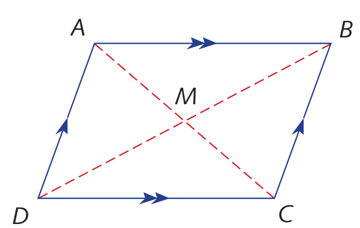
$\angle BCA = \angle DAC$ (alternate angles, $AD \parallel BC$)

AC is common.

So, $\Delta ABC \equiv \Delta CDA$ (AAS)

Hence, $AB = CD$ and $BC = AD$ (matching sides of congruent triangles)

Third property: the diagonals of a parallelogram bisect each other



Theorem: The diagonals of a parallelogram bisect each other.

Proof: Let $ABCD$ be a parallelogram.

Let the diagonals AC and BD meet at M.

We need to prove that $AM = CM$ and $DM = BM$.

In the triangles ABM and CDM:

$\angle BAM = \angle DCM$ (alternate angles, $AB \parallel DC$)

$\angle ABM = \angle CDM$ (alternate angles, $AB \parallel DC$)

$AB = CD$ (opposite sides of parallelogram)

Hence, $\Delta ABM \equiv \Delta CDM$ (AAS)

So $\Delta AM = CM$ and $DM = BM$ (matching sides of congruent triangles)

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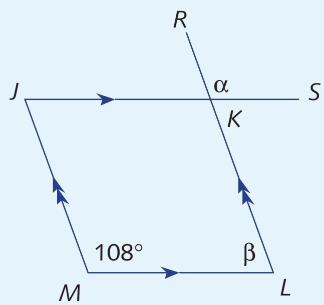
Properties of a parallelogram

- The opposite angles of a parallelogram are equal.
- The opposite sides of a parallelogram are equal.
- The diagonals of a parallelogram bisect each other.

We have just proved three properties of a parallelogram. You can now use these properties to help you solve problems, as shown in the examples below. Where it is possible to do so, name the particular parallelogram being used.

Example 1

Find α and β in the diagram below.



Solution

$\angle JKL = 180°$ (opposite angles of parallelogram $JKLM$)

so $\alpha = 108°$ (vertically opposite angles at K)

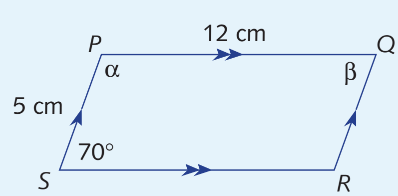
$\beta + 108° = 180°$ (co-interior angles, $JM \parallel KL$)

so $\quad \beta = 72°$

Example 2

a Calculate the perimeter of the parallelogram opposite.

b Find the values of α and β.



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Solution

a $RS = 12$ cm (opposite sides of parallelogram $PQRS$)

and $QR = 5$ cm (opposite sides of parallelogram $PQRS$)

Hence, perimeter $= 12 + 12 + 5 + 5$

$= 34$ cm

b $\alpha + 70° = 180°$ (co-interior angles, $PQ \parallel SR$)

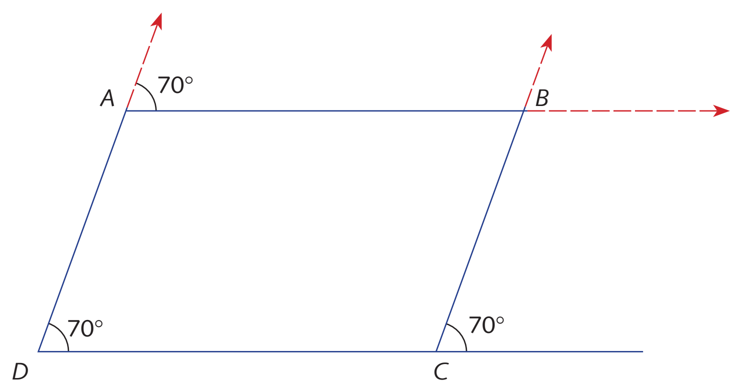
$\alpha = 110°$

$\beta = 70°$ (opposite angles of parallelogram $PQRS$)

The converses of each of the properties are true. They are proved in the exercises.

Exercise 13A

1 Construct a parallelogram $ABCD$ as follows. Start with the interval DC.



Step 1: Use your protractor to construct parallel rays upwards and to the right from D and from C at an angle of 70°. (We have chosen 70°, but any reasonable angle will do in place of 70°.)

Step 2: Choose a point A on the left-hand ray some distance from D.

Step 3: Use your protractor to construct another angle of 70° at the point A in a position corresponding to $\angle D$. This will give a line parallel to DC.

Step 4: Extend this line through A to form the quadrilateral $ABCD$.

a Why is $ABCD$ a parallelogram?

b Check that the opposite sides are equal in length.

c Draw the diagonals AC and BD, and check that their intersection M is the midpoint of each diagonal.

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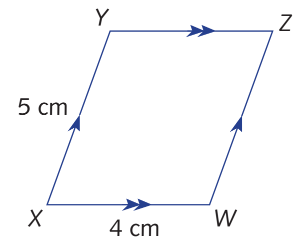
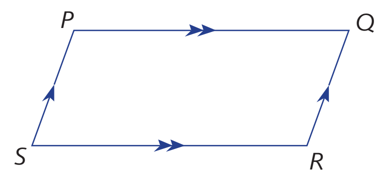
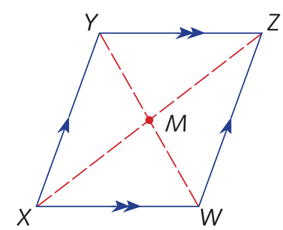
2 In the diagram to the right, $XYZW$ is a parallelogram. What are the lengths of WZ and YZ? Give reasons for your answer.



3 In the diagram to the right, $PQRS$ is a parallelogram. Prove that $\angle S = \angle Q$ and $\angle P = \angle R$, as presented earlier.



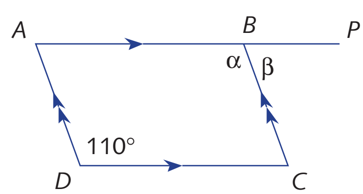
4 In the diagram to the right, $XYZW$ is a parallelogram. M is the point of intersection of XZ and YW. Prove that $XM = ZM$, as presented earlier.



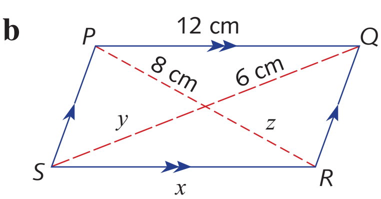
Example 1

5 Use the properties of a parallelogram to find the values of x, y, z, α and β in the diagrams below. Give reasons.

a



b



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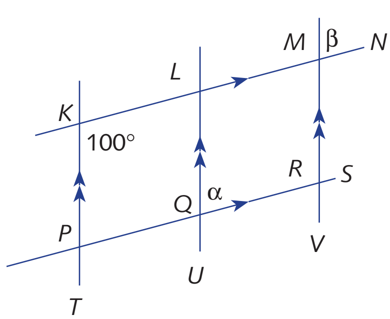

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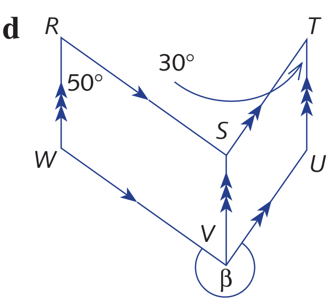

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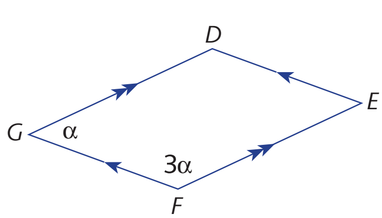

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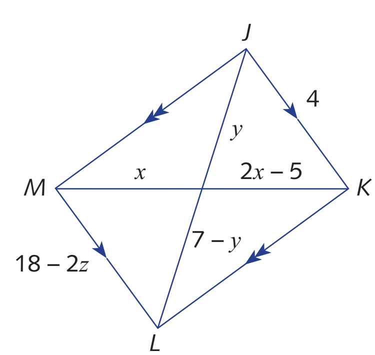

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Example 2

6 Find the perimeter of a parallelogram with side lengths that are:

a 12 cm and 22 cm

b 6.5 km and 12.5 km

c $5\frac{1}{2}$ m and 7 m

d 13 cm and 23 cm

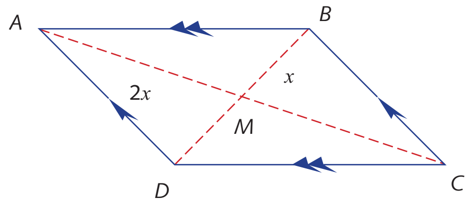
7 Draw a parallelogram using the following steps. First, choose two horizontal lines on your page as opposite sides. Then place your ruler obliquely across the page and use the two edges of the ruler to draw the other two sides. Label the parallelogram *ABCD*.

a Draw the diagonals *AC* and *BD*, and label their intersection *M*.

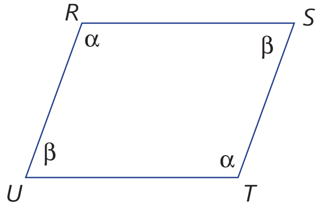
b Draw the circle with centre *M* passing through *A*. Does it go through *C*? What property of the parallelogram does this illustrate?

c Draw a second circle with centre *M* passing through *B*. Does it go through *D*? Why or why not?

8 Show that the diagonal *AC* in the diagram below has twice the length of the diagonal *BD*.



9 In the quadrilateral *RSTU* below, the angles are equal.



Let $\angle R = \angle T = \alpha$ and $\angle S = \angle U = \beta$.

a Use the fact that the angle sum of a quadrilateral is 360° to prove that $\alpha + \beta = 180°$.

b Hence, prove that *RSTU* is a parallelogram.

You have now proved the following test for a parallelogram:

'*If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.*'

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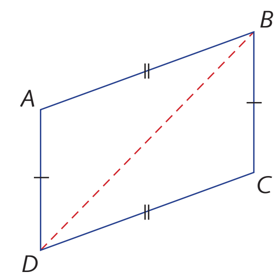
10 In the quadrilateral $ABCD$ to the right, the opposite sides are equal. Join the diagonal BD.



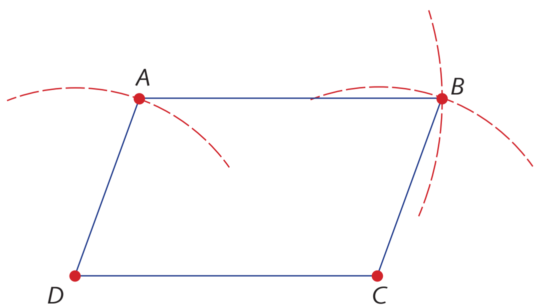
a Prove that $ABD \equiv CDB$.

b Hence, show that $AB \parallel DC$ and $AD \parallel BC$, and hence show that $ABCD$ is a parallelogram.

You have now proved the following test for a parallelogram:

'If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.'

11 Here is another construction of a parallelogram. Start with the interval DC. Make it 8 cm long.



Step 1: Draw a circle with centre D and radius 5 cm. Mark a point A on the circle above DC.

Step 2: Draw a circle with centre A and radius 8 cm (this is the length of DC).

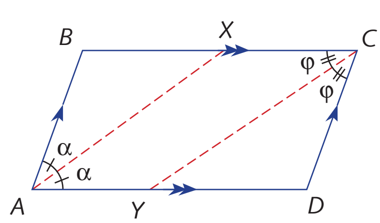
Step 3: Draw a circle with centre C and radius 5 cm. Label as B the point of intersection of the circle from step 2 and the circle from step 3.

Step 4: Join up the quadrilateral $ABCD$.

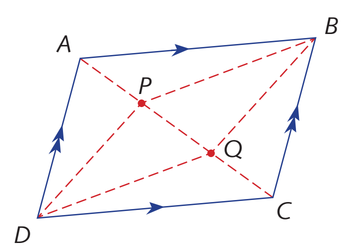
a Explain why $ABCD$ is a parallelogram.

b Use your protractor to check that opposite angles are equal.

12 In the diagram below, $ABCD$ is a parallelogram. The bisector of $\angle A$ meets BC at X. The bisector of $\angle C$ meets AD at Y. Prove AX is parallel to CY.



13 In the diagram to the right, $ABCD$ is a parallelogram. AC is a diagonal and P and Q are points on AC such that $AP = CQ$. Prove that $PBQD$ is a parallelogram.



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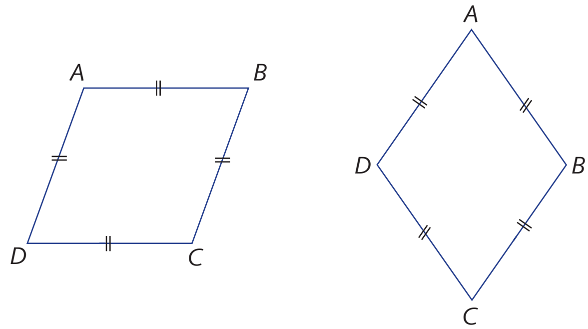
13B Rhombuses and their properties

A **rhombus** is a quadrilateral in which all four sides are equal. Thus the two quadrilaterals $ABCD$ below are both rhombuses because $AB = BC = CD = DA$.



The first rhombus looks like a 'pushed-over square'. You may already be familiar with this description – it is a common-sense way of thinking about a rhombus.

The second rhombus has been rotated so that it looks like the diamond in a pack of cards. It is usually better to think of this as the characteristic shape of a rhombus.

The Greeks took the word *rhombos* from the shape of a piece of wood that was whirled about the head, like a bullroarer, in religious ceremonies.

Definition of a rhombus

A **rhombus** is a quadrilateral in which all four sides are equal.

This section begins with an investigation of some important properties of the diagonals of a rhombus.

A rhombus is a parallelogram

Theorem: A rhombus is a parallelogram.

Proof: Let $ABCD$ be a rhombus.

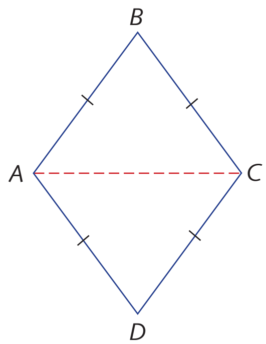
$AB = BC = CD = DA$

Join the diagonal AC.

Since AC is common,

$\Delta ABC \equiv \Delta ADC$ (SSS),

so $\Delta DAC = \angle BCA$ (matching angles of congruent triangles)



Hence, $AD \parallel BC$ (alternate angles)

A similar argument shows that $AB \parallel CD$.

Thus $ABCD$ is a parallelogram.

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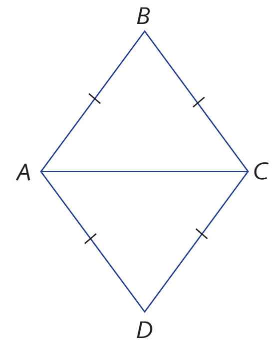
Diagonal properties of a rhombus

Because a rhombus is a parallelogram, its diagonals bisect each other. The diagonals of a rhombus have two further properties.

Theorem: **a** Each diagonal of a rhombus bisects the two interior angles through which it passes.

b The diagonals of a rhombus bisect each other at right angles.

Proof: **a** To prove the first result, join the diagonal AC.



Since AC is common,

$\Delta ABC \equiv \Delta ADC$ (SSS)

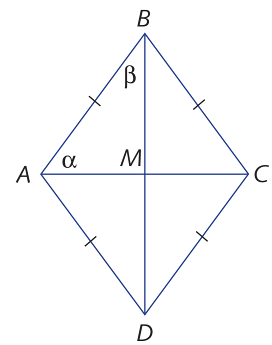
Therefore, $\angle BAC = \angle DAC$ (matching angles of congruent triangles)

Therefore, the diagonal AC bisects the angle at the vertex A.

Similarly, $\angle BCA = \angle DCA$ (matching angles of congruent triangles)

So the diagonal AC bisects the angle at the vertex C.

b To prove the second result, join the other diagonal, BD, and let them meet at M. We must show $\angle BMA = 90°$. Let $\angle BAC = \alpha$ and $\angle ABD = \beta$



Then $\angle DBC = \alpha$ (by part **a**)

and $\angle BCA = \alpha$ (base angles of isosceles ΔABC)

Thus $2\alpha + 2\beta = 180°$ (angle sum of ΔABC)

$\alpha + \beta = 90°$

Therefore, $\angle BMA = 90°$ (angle sum of ΔAMB)

Therefore, the diagonals of rhombus bisect each other at right angles.

These properties can be proved in other ways, such as using congruent triangles.

Properties of a rhombus

- A rhombus is a parallelogram.
- The diagonals of a rhombus bisect each other at right angles.
- Each diagonal of a rhombus bisects the two interior angles through which it passes.

You can now use these diagonal properties of a rhombus to help you solve problems, as shown in the following examples. Where it is possible to do so, name the particular rhombus being used.

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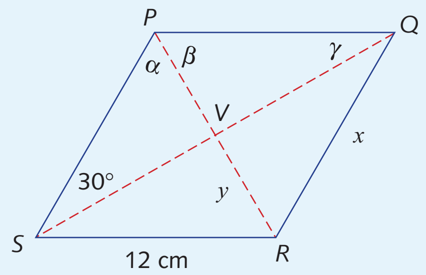
Example 3

Find α, β, γ, x and y in the diagram below, given that $PQRS$ is a rhombus.



Solution

First, $\angle PVS = 90°$ (diagonals of rhombus $PQRS$ meet at right angles)

so $\alpha = 60°$ (angle sum of ΔPVS)

and $\beta = 60°$ (diagonals of rhombus $PQRS$ bisect interior angles)

Second, $\gamma = 30°$ (base angles of isosceles ΔSPQ)

and $x = 12$ cm (sides of rhombus $PQRS$)

Third, $\angle SRP = 60°$ (base angles of isosceles ΔSPR)

so ΔSPR is equilateral, since all angles are 60°.

Hence, $PR = SR = 12$ cm

and $y = 6$ cm (diagonals of rhombus $PQRS$ bisect each other)

Exercise 13B

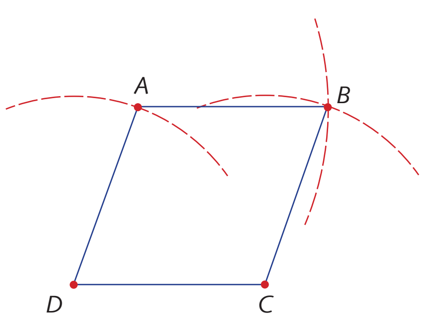
1 Construct a rhombus as follows. Start by drawing an interval DC, 5 cm long.

Step 1: Draw a circle with centre D and radius DC. Mark a point A on the circle above DC.

Step 2: Draw a circle with centre A and the same radius DC.

Step 3: Draw a circle with centre C and the same radius. Label as B the point of intersection of this circle and the circle from step 2.

Step 4: Join up the points A, B, C and D.



Explain why this figure $ABCD$ is a rhombus.

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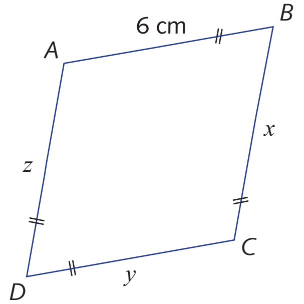
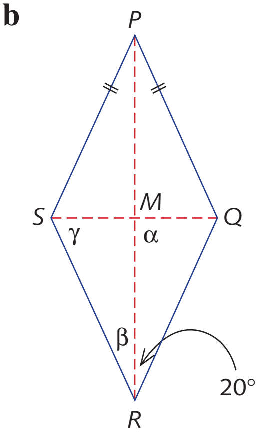
Example 3

2 Find the values of x, y, z, α, β, γ and θ. Give reasons.

a



b



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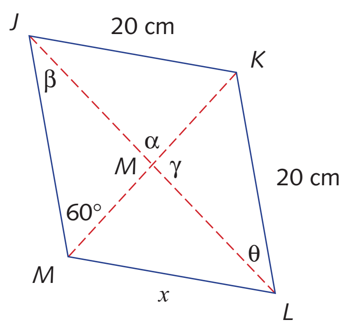

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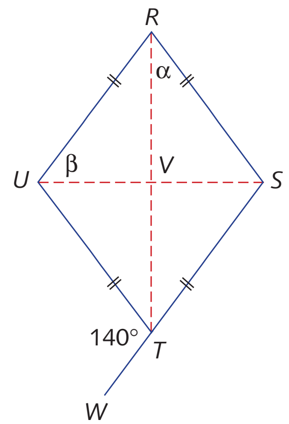

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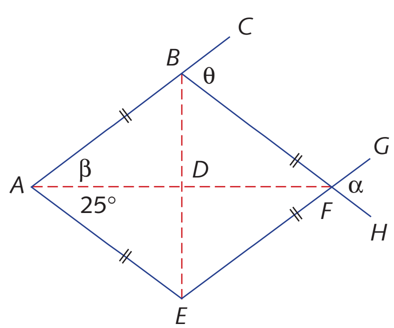

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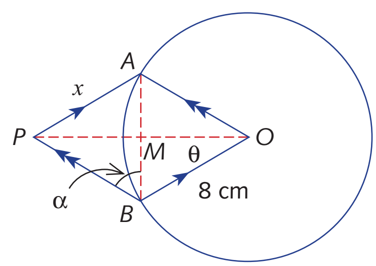
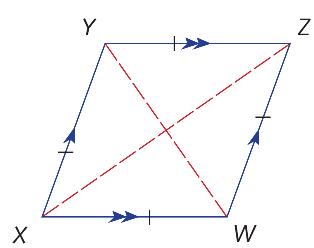


(The point O is the centre of the circle.)

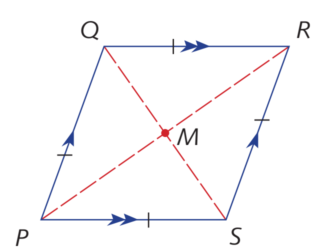
3 In the diagram to the right, $XYZW$ is a rhombus.

Prove that XZ bisects $\angle X$ as presented earlier.



4 In the diagram to the right, $PQRS$ is a rhombus. Let M be the point of intersection of the diagonals QS and PR.

Prove that $QM = MS$, $PM = MR$ and $\angle QMR$ is a right angle, as presented earlier.



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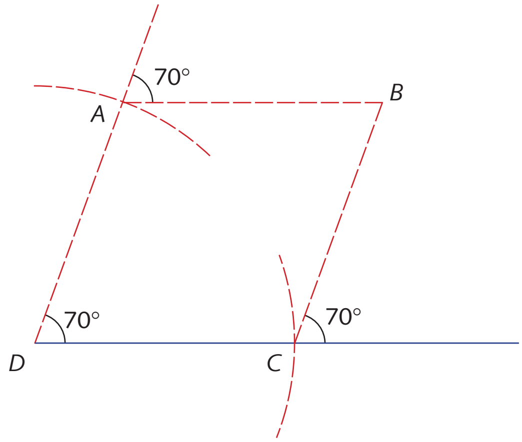
5 Construct a rhombus *ABCD* as follows. Start with the interval *DC*.



a Why is *ABCD* a rhombus?

b Join the diagonals and check with a protractor that they bisect the vertex angles and are perpendicular.

Step 1: Use your protractor to construct parallel rays upwards and to the right from *D* and from *C* at an angle of 70°. (We have chosen 70°, but any reasonable angle will do in place of 70°.)

Step 2: Place the point of your compasses on *D*. Draw an arc that passes through *C* and cuts the ray at a point *A*.

Step 3: Use your protractor to construct another angle of 70° at the point *A* in a position corresponding to $\angle D$. This will give a line parallel to *DC*.

Step 4: Extend this line through *A* to form the rhombus *ABCD*.

6 **a** Join a rhombus as follows. First, use the two edges of your ruler to draw two parallel lines. Then rotate your ruler to draw a second pair of parallel lines. Label the rhombus *ABCD*.

b Draw the diagonals *AC* and *BD*, and label their intersection *M*.

c Use your protractor to check that the diagonals are perpendicular.

d Check with your protractor that the diagonals bisect the vertex angles through which they pass.

e Draw the circle with centre *M* passing through *A*, then draw the circle with centre *M* passing through *B*. Do these circles go through *C* and *D*, respectively? What property of a rhombus do you need to know to answer these questions?

f Prove that this construction does indeed produce a rhombus.

(This is a difficult question.)

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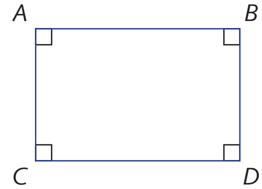
13C Rectangles and squares and their properties

Rectangles

A **rectangle** is a quadrilateral in which all four angles are right angles. The quadrilateral *ABCD* is a rectangle because $\angle A = \angle B = \angle C = \angle D = 90°$.

A rectangle *ABCD* is a parallelogram because each pair of co-interior angles is supplementary.



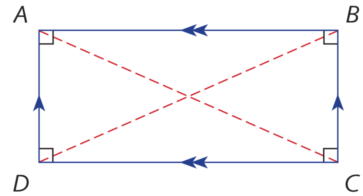
Definition of a rectangle

A **rectangle** is a quadrilateral in which all four angles are right angles.

The word 'rectangle' comes from Latin and means 'right-angled'.

Since a rectangle is a parallelogram, it has all the properties of a parallelogram. This means that:

- the opposite sides are equal, and
- the diagonals bisect each other.

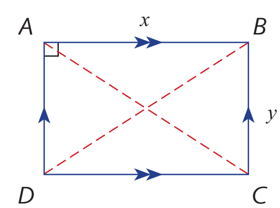


The diagonals of a rectangle, however, have another important property – they are equal in length.

The proof below uses Pythagoras' theorem.

Theorem: The diagonals of a rectangle are equal.

Proof: Let *ABCD* be a rectangle, with diagonals *AC* and *BD*.
Let $AB = x$ and $BC = y$.



We need to prove that $AC = BD$.

First, $DC = x$ (opposite sides of rectangle)

Using Pythagoras' theorem in ΔABC:

$$AC^2 = AB^2 + BC^2$$
$$= x^2 + y^2$$

Using Pythagoras' theorem in ΔBCD:

$$BD^2 = DC^2 + CB^2$$
$$= x^2 + y^2$$

Hence, $AC = BD$

An alternative proof using congruence is developed in question **7** of Exercise 13C.

Diagonal properties of a rectangle

The diagonals of a rectangle are equal and bisect each other.

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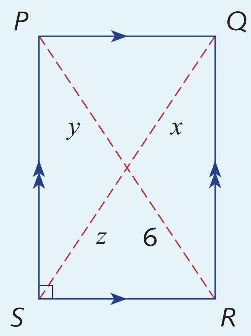
You can now use these diagonal properties of a rectangle to help you solve problems, as shown in the example below.

Example 4

Find the values of x, y and z.



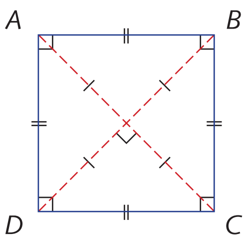
Solution

PQRS is a rectangle.

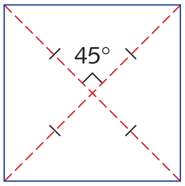
So $x = y = z = 6$ (diagonals of a rectangle are equal and bisect each other)

Squares

A **square** is a rectangle with all sides equal. That is, a square is a quadrilateral that is both a rhombus and a rectangle.



Because a square is a rectangle, it has all the properties of a rectangle. Thus its diagonals are equal and bisect each other.



Because a square is also a rhombus, its diagonals meet at right angles. Also, each diagonal bisects the right angles at the vertices through which it passes, and so meets each side at 45°.

Squares and their properties

A **square** is a rectangle with all sides equal. Thus a square is both a rectangle and a rhombus, so:

- all sides are equal and all angles are right angles
- the opposite sides are parallel
- the diagonals are equal in length, the diagonals bisect each other at right angles, and each diagonal meets each side at 45°.

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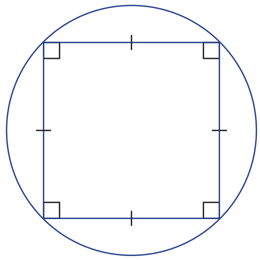
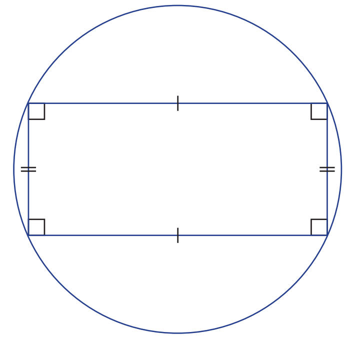
Because the diagonals of both rectangles and squares are equal in length and bisect each other, a circle can be drawn with centre the point of intersection of the diagonals and that passes through the four vertices. This is called the **circumcircle** of the rectangle or square.

The word 'square' comes originally via the Old French from the Latin word *quattuor*, meaning 'four'.



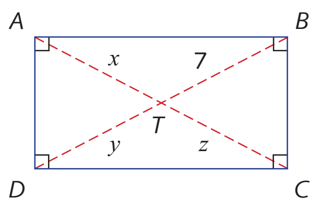


Exercise 13C

Example 4

1 Use the properties of rectangles and squares to find the values of $x, y, z, t, \alpha, \beta, \gamma$ and θ.

a



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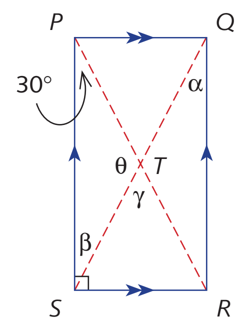

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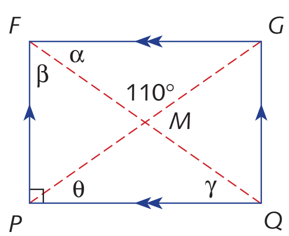

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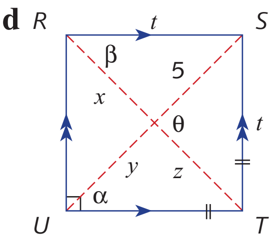
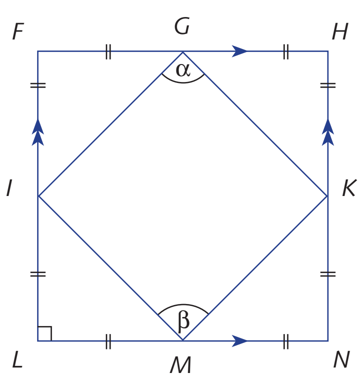
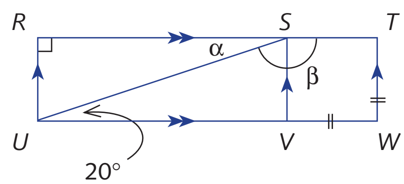


e



f



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2 Draw a rectangle using the following steps.

a First, use two horizontal lines on your page as opposite sides. Then construct two vertical lines which form the other two sides. Call the rectangle $ABCD$.

b Draw the two diagonals and label their intersection M.

c Construct the circle with centre M passing through A. What property of a rectangle tells us that it must pass through B, C and D?

3 Draw a square using the following steps.

a First, draw a horizontal line. Construct a vertical line meeting the horizontal line at D. With centre D and radius 5 cm, draw a circle meeting the vertical line at A and the horizontal line at C. With centres A and C, draw arcs of radius 5 cm meeting at B. Join AB and BC. Explain why $ABCD$ is a square.

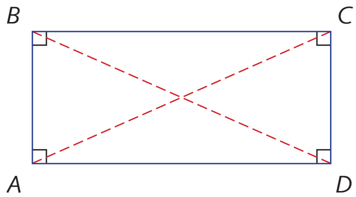
b Draw the two diagonals, and label their intersection M.

c Use your protractor to check that the diagonals intersect at right angles, and that they each meet the sides at 45°.

d Construct the circle with centre M passing through A. Notice that it goes through B, C and D. What property of a square tells us that it must pass through B, C and D?

4 Prove that if a parallelogram has one right angle then it is a rectangle.

5 Here is an alternative proof, using congruence, that the diagonals of a rectangle are equal. Let $ABCD$ be a rectangle, with diagonals AC and BD.



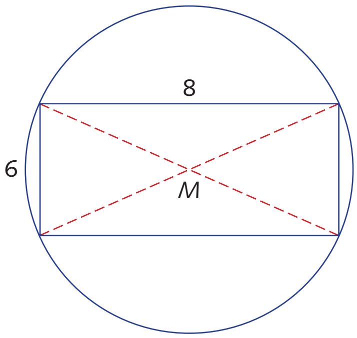
a Prove that $\Delta ACD \equiv \Delta BDC$.

b Hence, prove that $AC = DB$.

6 Find the radius of the circumcircle of:

a a rectangle with side length 6 and 8

b a square of side length 7



7 A quadrilateral has all four sides equal and two equal diagonals. Prove it is a square.

8 **a** $ABCD$ is a square, Q is a point on DC, and R is a point on BC. Suppose that AQ is perpendicular to DR. Prove $AQ = DR$.

b Suppose that two lines m and n are perpendicular and that each line intersects opposite edges of the square $ABCD$. Show that the intervals cut out by the square on m and n are equal in length.

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Review exercise

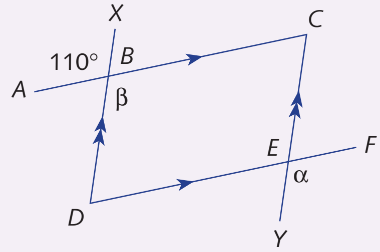
1 List all the properties of the diagonals of:

a a parallelogram **b** a rhombus

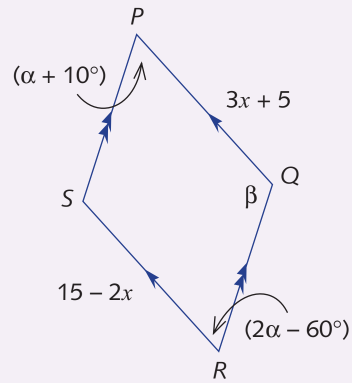
c a rectangle **d** a square

2 Use the properties of a parallelogram to find the values of x, y, z, α and β in the diagrams below. Give reasons.

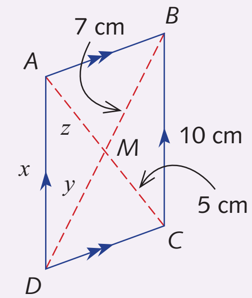
a



b

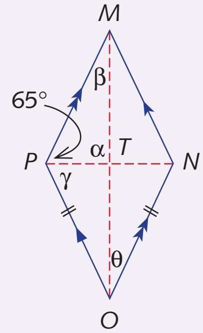


c



3 Use the properties of a rhombus to find the values of x, α, β, γ and θ in the diagrams below, giving reasons.

a



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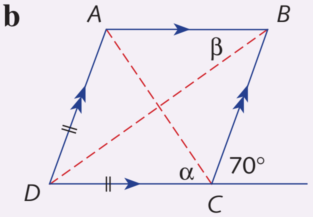

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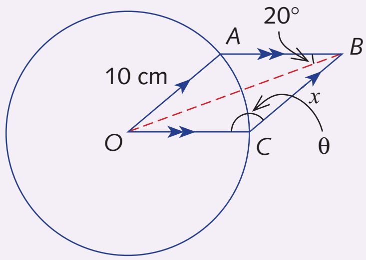
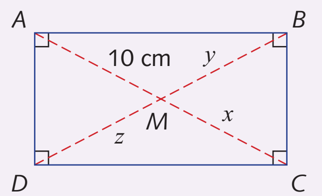


(The point O is the centre of the circle.)

4 Use the properties of rectangles and squares to find the values of x, y, z, α, β, γ and θ in the diagrams below, giving reasons.

a



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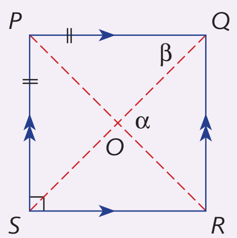

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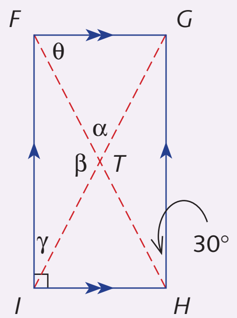

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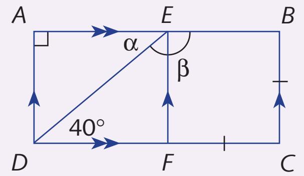

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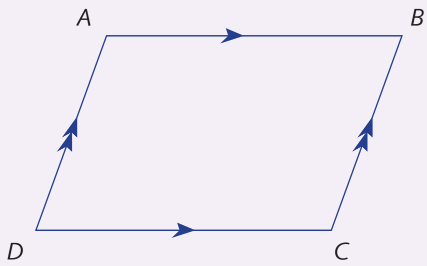
Note: The next three questions are intended as reviews of the proofs in this chapter.

5 The figure *ABCD* below is a parallelogram.

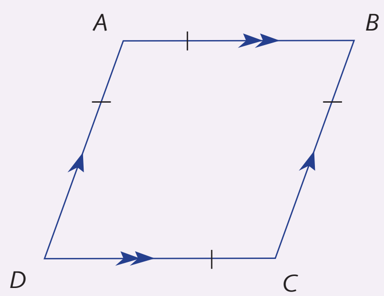


a Prove that the opposite angles are equal.

b Prove that the opposite sides are equal.

c Prove that the diagonals bisect each other.

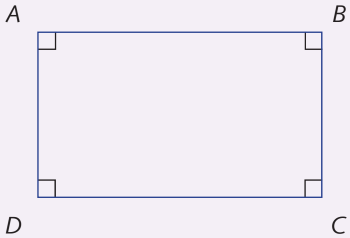
6 The figure *ABCD* below is a rhombus.



a Prove that each diagonal bisects the two vertices through which it passes.

b Prove that the diagonals meet at right angles.

7 The figure *ABCD* below is a rectangle.



Prove that the diagonals are equal.

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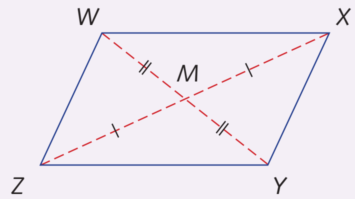
Challenge exercise

1 In the quadrilateral $WXYZ$ below, the diagonals bisect each other at M.



a Prove that $\Delta WMX \equiv \Delta YMZ$.

b Hence show that $WX \parallel ZY$.

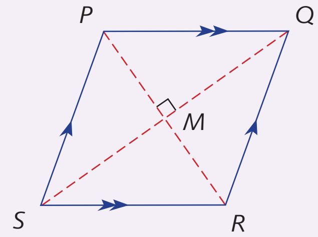
c Prove that $\Delta WMZ \equiv \Delta YMX$.

d Hence show that $WZ \parallel XY$, and hence that $WXYZ$ is a parallelogram.

You have now proved the following test for a parallelogram:

'*If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.*'

2 In the parallelogram $PQRS$ below, the diagonals meet at right angles at M.



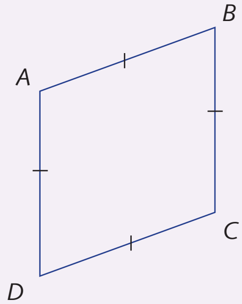
a Prove that $\Delta PMQ \equiv \Delta RMQ$.

b Hence show that $PQ = RQ$, and so $PQRS$ is a rhombus.

You have now proved the following test for a rhombus:

'*If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.*'

3 All sides of the quadrilateral $ABCD$ below are equal.



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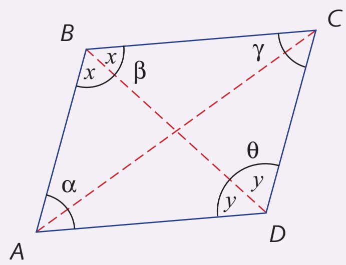
a Prove that $\Delta ADB \equiv \Delta CBD$.

b Hence explain why $AB \parallel DC$ and $AD \parallel BC$, and so $ABCD$ is a parallelogram.

You have now proved that every rhombus is a parallelogram.

4 The diagonals of the quadrilateral $ABCD$ bisect each vertex angle, as shown in the diagram below.



a Use the angle sums of ΔABD and ΔCBD to prove that $\alpha = \gamma$.

b Use a similar argument to prove that $\beta = \theta$.

c Hence explain why $AB \parallel DC$ and $AD \parallel BC$, and so $ABCD$ is a rhombus.

You have now proved the following test for a rhombus:

'If each diagonal of a quadrilateral bisects the two vertex angles through which it passes, then the quadrilateral is a rhombus.'

5 A parallelogram $ABCD$ has diagonals of equal length.

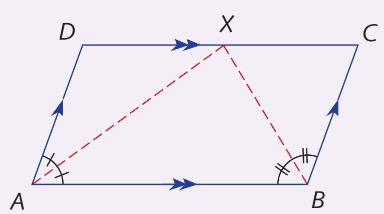
a Prove that $\Delta ABD \equiv \Delta BAC$.

b Hence prove that $\angle A = \angle B = 90°$, and so $ABCD$ is a rectangle.

You have now proved the following test for a rectangle:

'If the diagonals of a parallelogram are equal, then the parallelogram is a rectangle.'

6 In the diagram below, $ABCD$ is a parallelogram such that the bisector of $\angle A$ meets the bisector of $\angle B$ on DC at X. Prove $AB = 2BC$.



7 $ABCD$ is a parallelogram, P is the midpoint of BC, and DP and AB are extended to meet at Q. Prove that $AQ = 2AB$.

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