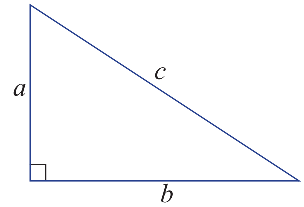
CHAPTER

2

Measurement and Geometry

Pythagoras' theorem and surds

In *ICE-EM Mathematics Year 8*, you learnt about the remarkable relationship between the lengths of the sides of a right-angled triangle. This result is known as **Pythagoras' theorem**.



The theorem states that the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides. In symbols:

$$c^2 = a^2 + b^2$$

The converse of this theorem is also true. This means that if we have a triangle in which the square of one side equals the sum of the squares of the other two sides, then the triangle is right-angled, with the longest side being the hypotenuse.

Pythagoras' theorem leads to the discovery of certain irrational numbers, such as $\sqrt{2}$ and $\sqrt{3}$. These numbers are examples of **surds**. In this chapter, we investigate the arithmetic of surds.

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2A Review of Pythagoras' theorem and applications

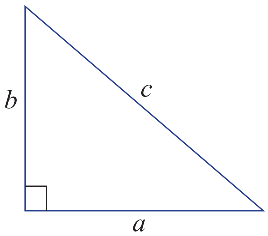
In Year 8, we used Pythagoras’ theorem to solve problems related to right-angled triangles.

Pythagoras’ theorem and its converse

- In any right-angled triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.

  $c^2 = a^2 + b^2$

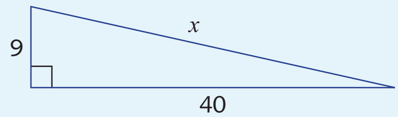
- A triangle with side lengths of a, b, c which satisfy $c^2 = a^2 + b^2$ is a right-angled triangle. The right angle is opposite the side of length c.



Example 1

Find the value of the unknown side in each triangle.

a



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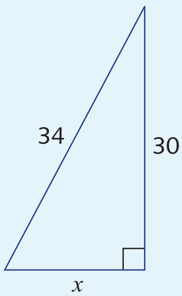

Solution

a

$$\begin{aligned} x^2 &= 9^2 + 40^2 \\ &= 1681 \\ x &= \sqrt{1681} \\ &= 41 \end{aligned}$$

b

$$\begin{aligned} x^2 + 30^2 &= 34^2 \\ x^2 &= 34^2 - 30^2 \\ &= 256 \\ x &= \sqrt{256} \\ &= 16 \end{aligned}$$

Calculators and rounding

In Example 1 above, the values for x^2 are perfect squares and so we could take the square root easily. This is not always the case.

In Year 8, we found the approximate square roots of numbers that are not perfect squares by looking up a table of square roots.

Instead of doing this, from now on we are going to use a calculator.

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When using a calculator to find a square root, try to have in mind a rough idea of what the answer should be. For example, $\sqrt{130}$ should be close to 11, since $11^2 = 12$ and $12^2 = 144$.

Note: $\sqrt{130} \approx 11.402$ correct to 3 decimal places.

A calculator gives the approximate value of a square root to a large number of decimal places, far more than we need. We often round off a decimal to a required number of decimal places.

The method for rounding to 2 decimal places is as follows.

- Look at the digit in the third decimal place.
- If the digit is less than 5, take the two digits to the right of the decimal point. For example, 1.764 becomes 1.76, correct to 2 decimal places.
- If the digit is more than 4, take the two digits to the right of the decimal point and increase the second of these by one. For example, 2.455 becomes 2.46, correct to 2 decimal places.

Example 2

a A calculator gives $\sqrt{3} \approx 1.732\,050\,808$. State the value of $\sqrt{3}$ correct to 2 decimal places.

b A calculator gives $\sqrt{5} \approx 2.236\,067\,977$. State the value of $\sqrt{5}$ correct to 2 decimal places.

c State the value of 1.697, correct to 2 decimal places.

Solution

a $\sqrt{3} \approx 1.73$, since the digit, 2, in the third decimal place is less than 5.

b $\sqrt{5} \approx 2.24$, since the digit, 6, in the third decimal place is more than 4.

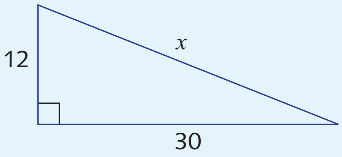
c $1.697 \approx 1.70$, since the digit, 7, in the third decimal place is more than 4 and 69 rounds to 70.

The symbol ≈ means ‘is approximately equal to’.

When approximating, we should state the number of decimal places to which we have rounded our answer.

Example 3

Find the length of the unknown side, correct to 2 decimal places.



Solution

$$x^2 = 12^2 + 30^2$$
$$= 1\,044$$
$$x = \sqrt{1\,044}$$
$$= 32.31 \text{ (correct to 2 decimal places)}$$

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In the example on the previous page, $\sqrt{1044}$ is the exact length while 32.31 is an approximation to the length of the side.

Example 4

Determine whether or not the three side lengths given form the sides of a right-angled triangle.

a 24, 32, 40 **b** 14, 18, 23

Solution

a The square of the length of the longest side $= 40^2$

$= 1600$

The sum of the squares of the lengths of the other sides $= 24^2 + 32^2$

$= 576 + 1024$

$= 1600$

Since $40^2 = 24^2 + 32^2$, the triangle is right-angled.

b The square of the length of the longest side $= 23^2$

$= 529$

The sum of the squares of the lengths of the other sides $= 14^2 + 18^2$

$= 196 + 324$

$= 520$

Since $23^2 \neq 14^2 + 18^2$, the triangle is not right-angled.

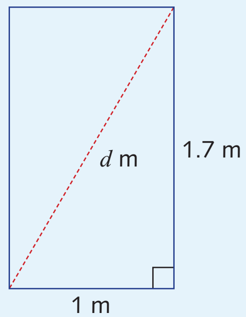
Example 5

A door frame has height 1.7 m and width 1 m. Will a square piece of board 2 m by 2 m fit through the doorway?

Solution

Let d m be the length of the diagonal of the doorway.

Using Pythagoras' theorem,



$$d^2 = 1^2 + 1.7^2$$
$$= 3.89$$
$$d = \sqrt{3.89}$$
$$= 1.97 \text{ (correct to 2 decimal places)}$$

Hence the board will not fit through the doorway.

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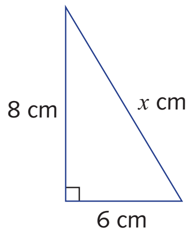
Exercise 2A

Example 1a

1 Use Pythagoras' theorem to find the value of the pronumeral.

a



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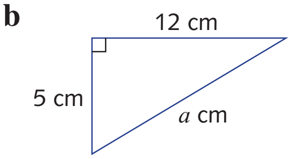

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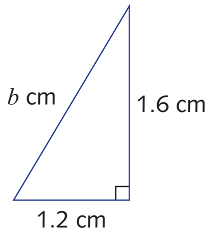
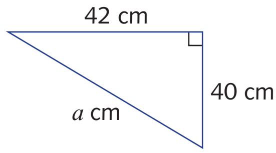


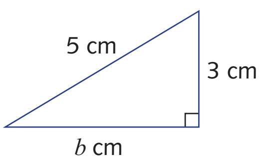
d



Example 1b

2 Use Pythagoras' theorem to find the value of the pronumeral.

a



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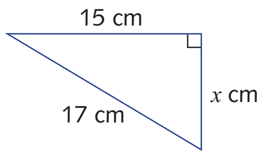

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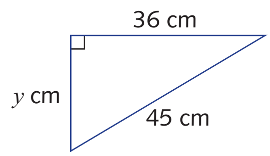

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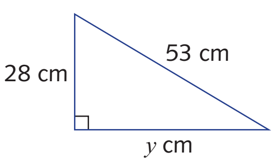

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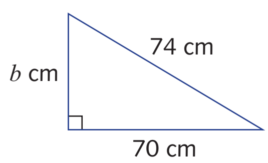

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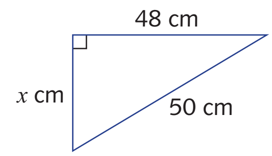


Example 2

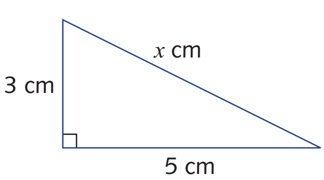
3 Use a calculator to find, correct to 2 decimal places, approximations to these numbers.

a $\sqrt{19}$ **b** $\sqrt{37}$ **c** $\sqrt{61}$ **d** $\sqrt{732}$

Example 3

4 Use Pythagoras' theorem to find the value of the pronumeral. Calculate your answer first as a square root and then correct to 2 decimal places.

a



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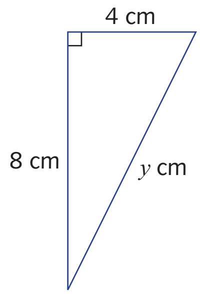

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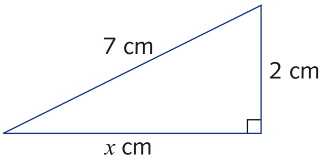

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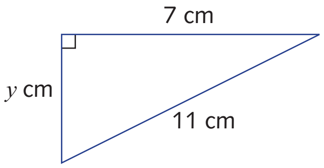

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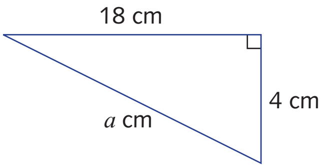

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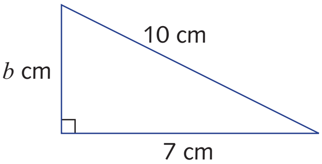

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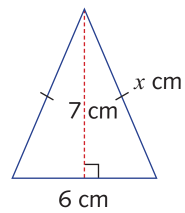

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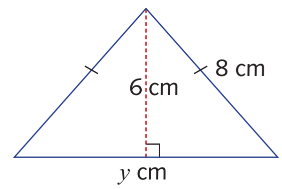


Example 4

5 Determine whether or not the triangle with the three side lengths given is right-angled.

a 16, 30, 34 **b** 10, 24, 26 **c** 4, 6, 7

d 4.5, 7, 7.5 **e** 6, 10, 12 **f** 20, 21, 29

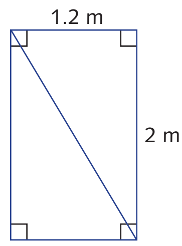
6 In the table below, the side lengths of right-angled triangles are listed. Copy and complete the table, giving answers as a whole number or square root.

	Lengths of the two shortest sides of a right-angled triangle		Length of the hypotenuse
a	3 cm	4 cm	...
b	5 cm	6 cm	...
c	4 cm	...	9 cm
d	6 cm	10 cm	...
e	...	7 cm	10 cm

Example 5

7 A door frame has height 1.8 m and width 1 m. Will a square piece of board 2.1 m wide fit through the opening?

8 A tradesman is making the wooden rectangular frame for a gate. In order to make the frame stronger and to keep it square, the tradesman will put a diagonal piece into the frame as shown in the diagram.



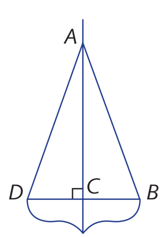
If the frame is 1.2 m wide and 2 m high, find the length of the diagonal piece of wood, in metres, correct to 3 decimal places.

9 A signwriter leans his ladder against a wall so that he can paint a sign. The wall is vertical and the ground in front of the wall is horizontal. The signwriter's ladder is 4 m long.

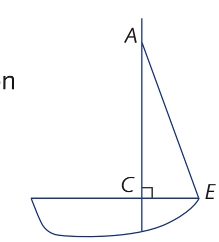
If the signwriter wants the top of the ladder to be 3.8 m above the ground when leaning against the wall, how far, correct to 1 decimal place, should the foot of the ladder be placed from the wall?

10 A boat builder needs to calculate the lengths of the stays needed to support a mast on a yacht. Two of the stays (AB and AD) will be the same length, as they go from a point A on the mast to each side of the boat, as shown in the diagram. The third stay (AE) will be different in length as it goes from the point A on the mast to the front of the boat.

Front elevation



Side elevation



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If $AC = 5$ m, $CB = CD = 1.2$ m and $CE = 1.9$ m, find the length, to the nearest centimetre, of:

a one of the side stays, *AB* or *AD*

b the front stay, *AE*

c stainless steel wire needed to make the three stays

11 As part of a design, an artist draws a circle passing through the four corners (vertices) of a square.

a If the square has side lengths of 4 cm, what is the radius, to the nearest millimetre, of the circle?

b If the circle has a radius of 3 cm, what are the side lengths, to the nearest millimetre, of the square?

12 A parent is asked to make some scarves for the local Scout troop. Two scarves can be made from one square piece of material by cutting on the diagonal. If this diagonal side length is to be 100 cm long, what must be the side length of the square piece of material to the nearest mm?

13 A girl planned to swim straight across a river of width 25 m. After she had swum across the river, the girl found she had been swept 4 m downstream. How far did she actually swim? Calculate your answer, in metres, correct to 1 decimal place.

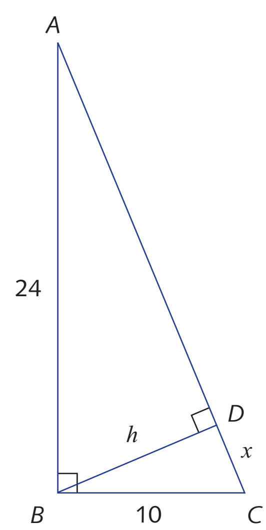
14 A yachtsman wishes to build a shed with a rectangular base to store his sailing equipment. If the shed is to be 2.6 m wide and must be able to house a 4.6 m mast, which is to be stored diagonally across the ceiling, how long must the shed be? Calculate your answer, in metres, correct to 1 decimal place.

15 In triangle *ABC* the line *BD* is drawn perpendicular to *AC*. h is the length of *BD* and x is the length of *CD*.

a Show that the length of *AC* is 26.

b Find the area of triangle *ABC* in two ways to show that $13h = 120$.

c Use Pythagoras' theorem to find x.



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16 In diagrams 1 and 2, we have two squares with the same side length, $a + b$. The sides are divided up into lengths a and b as shown.

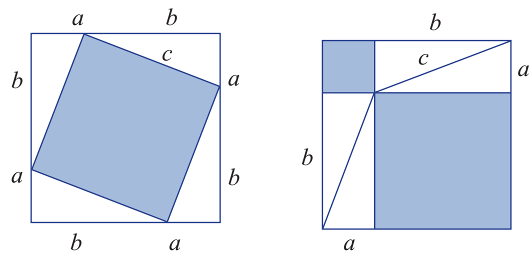

Diagram 1 Diagram 2

a Prove that the shaded part of diagram 1 is a square.

b The two shaded parts of diagram 2 are also squares. Why?

By looking carefully at the two diagrams, show that the area of the shaded square in diagram 1 is the sum of the areas of the shaded squares in diagram 2.

c Use the result of part **b** to prove Pythagoras' theorem.

2B Simplifying surds

This section deals only with surds that are square roots. If a is a positive integer which is not a perfect square then $\sqrt{a}$ is called a **surd**. We will look at surds more generally in Section 2I.

We will now review the basic rules for square roots.

If a and b are positive numbers then:

$$(\sqrt{a})^2 = a$$
$$\sqrt{a^2} = a$$
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$
$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

For example:

$$(\sqrt{11})^2 = 11$$
$$\sqrt{3^2} = 3$$
$$\sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7} = \sqrt{21}$$
$$\sqrt{35} \div \sqrt{5} = \sqrt{35 \div 5} = \sqrt{7}$$

The first two of these rules remind us that, for positive numbers, squaring and taking a square root are **inverse processes**.

When we write $2\sqrt{3}$, we mean $2 \times \sqrt{3}$. As in algebra, we usually do not explicitly write the multiplication sign.

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Example 6

Evaluate:

a $(\sqrt{6})^2$

b $(2\sqrt{6})^2$

Solution

a $(\sqrt{6})^2 = \sqrt{6} \times \sqrt{6}$
$= 6$

b $(2\sqrt{6})^2 = 2\sqrt{6} \times 2\sqrt{6}$
$= 2 \times 2 \times \sqrt{6} \times \sqrt{6}$
$= 4 \times 6$
$= 24$

Example 7

Evaluate:

a $\sqrt{3} \times \sqrt{11}$

b $\sqrt{15} \div \sqrt{3}$

c $\dfrac{\sqrt{42}}{\sqrt{6}}$

d $\sqrt{35} \div \sqrt{10}$

Solution

a $\sqrt{3} \times \sqrt{11} = \sqrt{3 \times 11}$
$= \sqrt{33}$

b $\sqrt{15} \div \sqrt{3} = \sqrt{15 \div 3}$
$= \sqrt{5}$

c $\dfrac{\sqrt{42}}{\sqrt{6}} = \sqrt{\dfrac{42}{6}}$
$= \sqrt{7}$

d $\sqrt{35} \div \sqrt{10} = \sqrt{\dfrac{35}{10}}$
$= \sqrt{\dfrac{7}{2}}$

Example 8

Evaluate:

a $3 \times 7\sqrt{5}$

b $9\sqrt{2} \times 4$

Solution

a $3 \times 7\sqrt{5} = 21\sqrt{5}$

b $9\sqrt{2} \times 4 = 36\sqrt{2}$

Consider the surd $\sqrt{12}$. We can factor out the perfect square 4 from 12, and write:

$$\sqrt{12} = \sqrt{4 \times 3}$$
$$= \sqrt{4} \times \sqrt{3} \qquad \text{(using } \sqrt{ab} = \sqrt{a} \times \sqrt{b}\text{)}$$
$$= 2\sqrt{3}$$

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Hence $\sqrt{12}$ and $2\sqrt{3}$ are equal. We will regard $2\sqrt{3}$ as a simpler form than $\sqrt{12}$ since the number under the square root sign is smaller.

To **simplify** a surd (or a multiple of a surd) we write it so that the number under the square root sign has no factors that are perfect squares other than 1. For example,

$$\sqrt{12} = 2\sqrt{3}$$

In mathematics, we are often instructed to leave our answers in **surd form**. This simply means that we should not approximate the answer using a calculator, but leave the answer, in simplest form, using square roots, cube roots, etc. This is called giving the **exact value** of the answer.

We can simplify surds directly or in stages.

Example 9

Simplify the following.

a $\sqrt{18}$

b $\sqrt{108}$

Solution

a $\sqrt{18} = \sqrt{9 \times 2}$

$= \sqrt{9} \times \sqrt{2}$

$= 3\sqrt{2}$

b $\sqrt{108} = \sqrt{9 \times 4 \times 3}$

$= \sqrt{9} \times \sqrt{4} \times \sqrt{3}$

$= 3 \times 2\sqrt{3}$

$= 6\sqrt{3}$

In some problems, we need to reverse this process.

Example 10

Express as the square root of a whole number.

a $3\sqrt{7}$

b $5\sqrt{3}$

Solution

a $3\sqrt{7} = \sqrt{9 \times 7}$

$= \sqrt{63}$

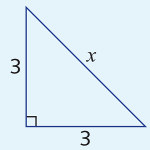
b $5\sqrt{3} = \sqrt{25 \times 3}$

$= \sqrt{75}$

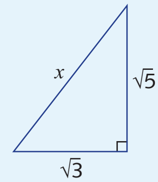
Example 11

Use Pythagoras' theorem to find the value of x. Give your answer as a surd which has been simplified.

a



b



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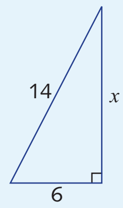

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Solution

a
$$\begin{aligned} x^2 &= 3^2 + 3^2 \\ &= 9 + 9 \\ &= 18 \\ x &= \sqrt{18} \\ &= \sqrt{9 \times 2} \\ &= 3\sqrt{2} \end{aligned}$$

b
$$\begin{aligned} x^2 &= (\sqrt{5})^2 + (\sqrt{3})^2 \\ &= 5 + 3 \\ &= 8 \\ x &= \sqrt{8} \\ &= \sqrt{4 \times 2} \\ &= 2\sqrt{2} \end{aligned}$$

c
$$\begin{aligned} x^2 + 6^2 &= 14^2 \\ x^2 + 36 &= 196 \\ x^2 &= 160 \\ x &= \sqrt{160} \\ &= \sqrt{160 \times 10} \\ &= 4\sqrt{10} \end{aligned}$$

The arithmetic of surds

- If a and b are positive numbers, then:
$$(\sqrt{a})^2 = a$$
$$\sqrt{a^2} = a$$
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$
$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$
- A surd is in its **simplest form** if the number under the square root sign has no factors that are perfect squares, apart from 1.
- To **simplify** a surd, we write it so that the number under the square root sign has no factors that are perfect squares, apart from 1.
  For example, $\sqrt{50} = 5\sqrt{2}$.

Exercise 2B

Example 6a

1 Evaluate:

a $(\sqrt{7})^2$ **b** $(\sqrt{3})^2$ **c** $(\sqrt{11})^2$ **d** $(\sqrt{231})^2$

Example 6b

2 Evaluate:

a $(2\sqrt{3})^2$ **b** $(4\sqrt{3})^2$ **c** $(5\sqrt{2})^2$ **d** $(3\sqrt{5})^2$

e $(2\sqrt{7})^2$ **f** $(7\sqrt{3})^2$ **g** $(11\sqrt{2})^2$ **h** $(2\sqrt{11})^2$

Example 7a

3 Express as a square root of a whole number.

a $\sqrt{3} \times \sqrt{5}$ **b** $\sqrt{2} \times \sqrt{6}$ **c** $\sqrt{5} \times \sqrt{6}$ **d** $\sqrt{11} \times \sqrt{3}$

e $\sqrt{17} \times \sqrt{3}$ **f** $\sqrt{3} \times \sqrt{15}$ **g** $\sqrt{3} \times \sqrt{14}$ **h** $\sqrt{10} \times \sqrt{19}$

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Example 7b, c, d

4 Express as a square root of a number.

a $\sqrt{6} \div \sqrt{2}$ **b** $\sqrt{35} \div \sqrt{7}$ **c** $\dfrac{\sqrt{39}}{\sqrt{3}}$ **d** $\dfrac{\sqrt{46}}{\sqrt{2}}$

e $\dfrac{\sqrt{77}}{\sqrt{11}}$ **f** $\sqrt{40} \div \sqrt{18}$ **g** $\sqrt{6} \div \sqrt{42}$ **h** $\dfrac{\sqrt{14}}{\sqrt{21}}$

Example 8

5 Evaluate the product.

a $2 \times 3\sqrt{2}$ **b** $6 \times 2\sqrt{5}$ **c** $11 \times 3\sqrt{7}$ **d** $6 \times 5\sqrt{7}$

e $15 \times 3\sqrt{2}$ **f** $7 \times 5\sqrt{6}$ **g** $4\sqrt{11} \times 3$ **h** $7\sqrt{3} \times 4$

6 Evaluate:

a $(\sqrt{2})^3$ **b** $(\sqrt{5})^3$ **c** $(\sqrt{5})^2 - (\sqrt{2})^2$

d $(\sqrt{7})^2 - (\sqrt{3})^2$ **e** $(\sqrt{11})^2 + (\sqrt{2})^2$ **f** $(\sqrt{5})^2 + (\sqrt{11})^2$

g $\sqrt{2} \times \sqrt{18}$ **h** $\sqrt{3} \times \sqrt{12}$ **i** $\sqrt{32} \times \sqrt{2}$

Example 9a

7 Simplify each of these surds.

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{45}$ **d** $\sqrt{24}$ **e** $\sqrt{27}$ **f** $\sqrt{44}$

g $\sqrt{50}$ **h** $\sqrt{54}$ **i** $\sqrt{20}$ **j** $\sqrt{98}$ **k** $\sqrt{63}$ **l** $\sqrt{60}$

m $\sqrt{126}$ **n** $\sqrt{68}$ **o** $\sqrt{75}$ **p** $\sqrt{99}$ **q** $\sqrt{28}$ **r** $\sqrt{242}$

Example 9b

8 Simplify each of these surds.

a $\sqrt{72}$ **b** $\sqrt{32}$ **c** $\sqrt{80}$ **d** $\sqrt{288}$ **e** $\sqrt{48}$ **f** $\sqrt{180}$

g $\sqrt{112}$ **h** $\sqrt{216}$ **i** $\sqrt{96}$ **j** $\sqrt{252}$ **k** $\sqrt{160}$ **l** $\sqrt{128}$

m $\sqrt{320}$ **n** $\sqrt{176}$ **o** $\sqrt{192}$ **p** $\sqrt{200}$ **q** $\sqrt{162}$ **r** $\sqrt{243}$

Example 10

9 Express each of these surds as the square root of a whole number.

a $2\sqrt{3}$ **b** $6\sqrt{3}$ **c** $7\sqrt{2}$ **d** $3\sqrt{6}$ **e** $4\sqrt{5}$ **f** $5\sqrt{7}$

g $4\sqrt{3}$ **h** $2\sqrt{13}$ **i** $6\sqrt{11}$ **j** $12\sqrt{10}$ **k** $10\sqrt{7}$ **l** $4\sqrt{11}$

10 Evaluate:

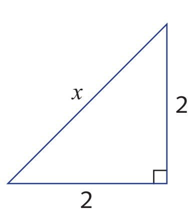
a $\left(\sqrt{\dfrac{2}{3}}\right)^2$ **b** $\sqrt{\dfrac{4}{25}}$ **c** $\sqrt{\dfrac{16}{25}}$ **d** $\sqrt{\dfrac{25}{36}}$

e $\left(\sqrt{\dfrac{5}{11}}\right)^2$ **f** $\sqrt{\dfrac{25}{121}}$ **g** $\left(\sqrt{\dfrac{7}{11}}\right)$ **h** $\sqrt{\dfrac{144}{169}}$

Example 11

11 Use Pythagoras' theorem to find the value of x. Give your answer as a surd which has been simplified.

a



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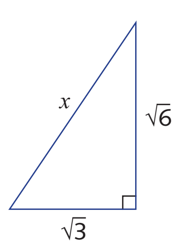

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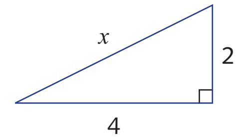

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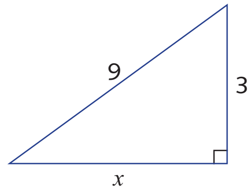
d



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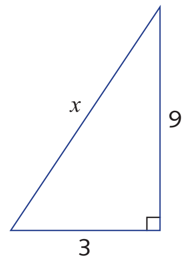

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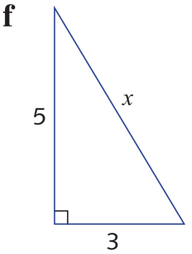

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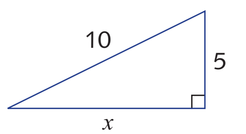
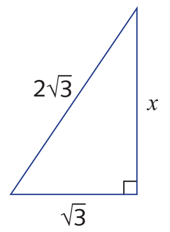
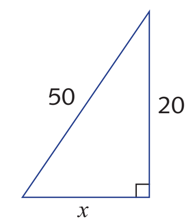


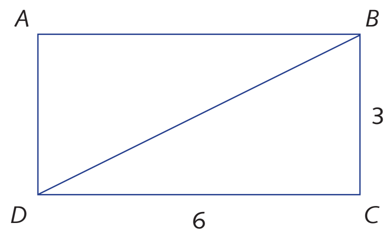
h



i



12 Find the length of the diagonal DB of the rectangle $ABCD$. Express your answer in simplest form.

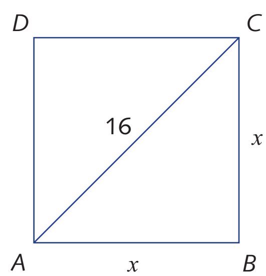


13 A square has side length $2\sqrt{3}$. Find:

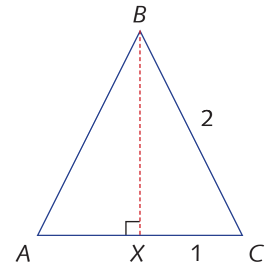
a the area of the square

b the length of the diagonal

14 $ABCD$ is a square. Find the value of x.



15 ABC is an equilateral triangle with side length 2. X is the midpoint of AC. Find the length BX.



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2C Addition and subtraction of surds

We can sometimes simplify sums of surds. The sum $4\sqrt{7} + 5\sqrt{7}$ can be thought of as 4 lots of $\sqrt{7}$ plus 5 lots of $\sqrt{7}$ equals 9 lots of $\sqrt{7}$. This is very similar to algebra, where we write $4x + 5x = 9x$. We say $4\sqrt{7}$ and $5\sqrt{7}$ are **like surds** since they are both multiples of $\sqrt{7}$.

On the other hand, in algebra we cannot simplify $4x + 7y$, because $4x$ and $7y$ are not like terms. Similarly, it is not possible to write the sum of $4\sqrt{2}$ and $7\sqrt{3}$ in a simpler way. They are **unlike surds**, since one is a multiple of $\sqrt{2}$ while the other is a multiple of $\sqrt{3}$.

We can only simplify the sum or difference of like surds.

Example 12

Simplify:

a $4\sqrt{7} + 5\sqrt{7}$

b $6\sqrt{7} - 2\sqrt{7}$

c $2\sqrt{2} + 5\sqrt{2} - 3\sqrt{2}$

d $3\sqrt{5} + 2\sqrt{7} - \sqrt{5} + 4\sqrt{7}$

Solution

a $4\sqrt{7} + 5\sqrt{7} = 9\sqrt{7}$

b $6\sqrt{7} - 2\sqrt{7} = 4\sqrt{7}$

c $2\sqrt{2} + 5\sqrt{2} - 3\sqrt{2} = 4\sqrt{2}$

d $3\sqrt{5} + 2\sqrt{7} - \sqrt{5} + 4\sqrt{7} = 2\sqrt{5} + 6\sqrt{7}$

This cannot be simplified further since $\sqrt{5}$ and $\sqrt{7}$ are unlike surds.

When dealing with expressions involving surds, we should simplify the surds first and then look for like surds.

Example 13

Simplify:

a $\sqrt{8} + 4\sqrt{2} - \sqrt{18}$

b $\sqrt{27} + 2\sqrt{5} + \sqrt{20} - 2\sqrt{3}$

Solution

a
$$\begin{aligned}\sqrt{8} + 4\sqrt{2} - \sqrt{18} &= 2\sqrt{2} + 4\sqrt{2} - 3\sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}\sqrt{27} + 2\sqrt{5} + \sqrt{20} - 2\sqrt{3} &= 3\sqrt{3} + 2\sqrt{5} + 2\sqrt{5} - 2\sqrt{3}\\ &= \sqrt{3} + 4\sqrt{5}\end{aligned}$$

This expression cannot be simplified further.

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Example 14

Simplify:

a $\frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{5}$

b $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{5}$

Solution

a $\frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{5} = \frac{5\sqrt{3}}{10} + \frac{6\sqrt{3}}{10}$ (Use a common denominator.)

$= \frac{11\sqrt{3}}{10}$

b $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{5} = \frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{5}$ (Simplify the surds.)

$= \frac{10\sqrt{2}}{15} - \frac{3\sqrt{2}}{15}$ (Use a common denominator.)

$= \frac{7\sqrt{2}}{15}$

Addition and subtraction of surds

- Simplify each surd first, then look for like surds.
- We can simplify sums and differences of like surds.
- We cannot simplify sums and differences of unlike surds.

Exercise 2C

Example 12

1 Simplify:

a $\sqrt{5} - \sqrt{5}$
b $6\sqrt{2} + 4\sqrt{2}$
c $\sqrt{3} + 4\sqrt{3}$
d $8\sqrt{2} - \sqrt{2}$
e $10\sqrt{7} - 5\sqrt{7}$
f $19\sqrt{5} + 24\sqrt{5}$
g $-13\sqrt{3} + 14\sqrt{3}$
h $3\sqrt{11} - 5\sqrt{11}$
i $-45\sqrt{2} + 50\sqrt{2}$

2 Simplify:

a $2\sqrt{2} + 3\sqrt{2} + \sqrt{2}$
b $-4\sqrt{6} - 3\sqrt{6} - 8\sqrt{6}$
c $10\sqrt{5} - \sqrt{5} - 6\sqrt{5}$
d $-2\sqrt{13} - 5\sqrt{13} + 16\sqrt{13}$
e $\sqrt{7} - 8\sqrt{7} + 5\sqrt{7}$
f $-2\sqrt{10} + 9\sqrt{10} - 7\sqrt{10}$
g $3\sqrt{6} + 4\sqrt{2} - 2\sqrt{6} + 3\sqrt{2}$
h $4\sqrt{5} + 7\sqrt{3} - 2\sqrt{5} - 4\sqrt{3}$

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3 Simplify:

a $3 - \sqrt{3} + 4 - 2\sqrt{3}$ **b** $\sqrt{3} - 2\sqrt{2} + 2\sqrt{3} + \sqrt{2}$

c $-4\sqrt{6} + 11 + 9 - 7\sqrt{6}$ **d** $5\sqrt{14} + 4\sqrt{6} + \sqrt{14} + 3\sqrt{6}$

e $6\sqrt{7} - 2\sqrt{14} + 4\sqrt{14} - 7\sqrt{7}$ **f** $\sqrt{5} - 3\sqrt{2} - 4\sqrt{5} + 7\sqrt{2}$

Example 13

4 Simplify:

a $\sqrt{8} + \sqrt{2}$ **b** $3\sqrt{2} - \sqrt{8}$ **c** $\sqrt{50} + 3\sqrt{2}$

d $\sqrt{18} + 2\sqrt{2}$ **e** $\sqrt{27} + 2\sqrt{3}$ **f** $\sqrt{48} + 2\sqrt{3}$

g $3\sqrt{5} + \sqrt{45}$ **h** $\sqrt{28} + \sqrt{63}$ **i** $\sqrt{8} + 4\sqrt{2} + 2\sqrt{18}$

5 Simplify:

a $\sqrt{72} - \sqrt{50}$ **b** $\sqrt{48} + \sqrt{12}$ **c** $\sqrt{8} + \sqrt{2} + \sqrt{18}$

d $\sqrt{12} + 4\sqrt{3} - \sqrt{75}$ **e** $\sqrt{32} - \sqrt{200} + 3\sqrt{50}$ **f** $4\sqrt{5} - 4\sqrt{20} - \sqrt{45}$

g $\sqrt{54} + \sqrt{24}$ **h** $\sqrt{27} - \sqrt{48} + \sqrt{75}$ **i** $\sqrt{45} + \sqrt{80} - \sqrt{125}$

j $\sqrt{2} + \sqrt{32} + \sqrt{72}$ **k** $\sqrt{1000} - \sqrt{40} - \sqrt{90}$ **l** $\sqrt{512} + \sqrt{128} + \sqrt{32}$

Example 14

6 Simplify:

a $\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{3}$ **b** $\dfrac{2\sqrt{3}}{5} + \dfrac{\sqrt{3}}{2}$ **c** $\dfrac{\sqrt{7}}{2} - \dfrac{\sqrt{7}}{5}$ **d** $\dfrac{3\sqrt{11}}{7} + \dfrac{2\sqrt{11}}{21}$

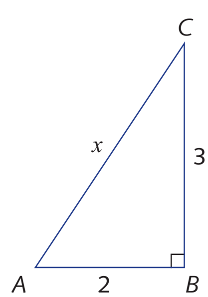
e $\dfrac{7\sqrt{2}}{6} - \dfrac{\sqrt{2}}{3}$ **f** $\dfrac{\sqrt{24}}{2} + \dfrac{\sqrt{6}}{5}$ **g** $\dfrac{\sqrt{27}}{5} - \dfrac{\sqrt{3}}{2}$ **h** $\dfrac{\sqrt{32}}{7} - \dfrac{2\sqrt{2}}{5}$

7 Find the value of x if:

a $\sqrt{63} - \sqrt{28} = \sqrt{x}$ **b** $\sqrt{80} - \sqrt{45} = \sqrt{x}$ **c** $2\sqrt{24} - \sqrt{54} = \sqrt{x}$

8 Find the value of x and the perimeter.

a



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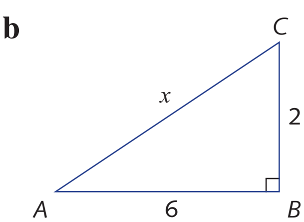

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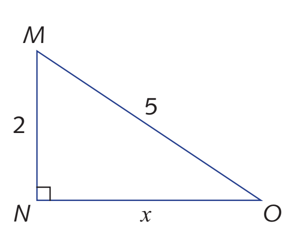
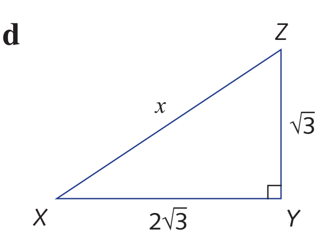


d



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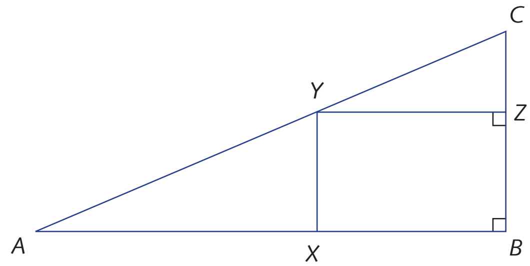
9 In the diagram to the right, $AB = 5\sqrt{3}$, $XB = 2\sqrt{3}$, $CB = \frac{5}{3}\sqrt{5}$, $YX = \sqrt{5}$. Find:

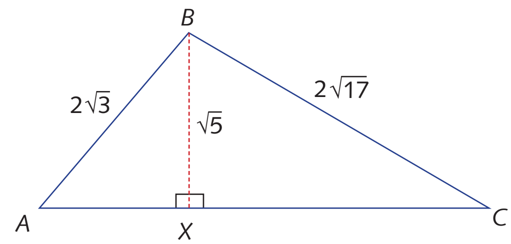
a AX **b** AY

c CZ **d** YC

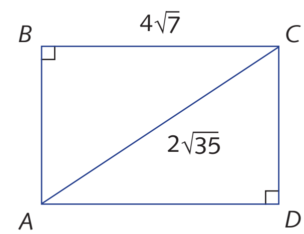


10 In the diagram to the right, find:

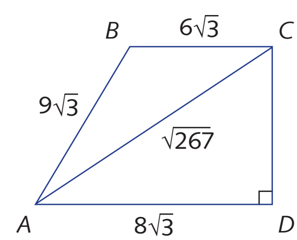
a AX **b** XC **c** AC



11 Find BA and the perimeter of the rectangle.



12 A quadrilateral $ABCD$ has side lengths $AB = 9\sqrt{3}$, $BC = 6\sqrt{3}$ and $AD = 8\sqrt{3}$. $\angle ADC = 90^\circ$ and the diagonal $AC = \sqrt{267}$. Find the perimeter of the quadrilateral.



2D Multiplication and division of surds

When we come to multiply two surds, we simply multiply the numbers outside the square root sign together, and similarly, multiply the numbers under the square root signs. A similar rule holds for division.

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Example 15

Find $4\sqrt{7} \times 2\sqrt{2}$.

Solution

$4\sqrt{7} \times 2\sqrt{2} = 8\sqrt{14}$ $\quad (4 \times 2 = 8,\ \sqrt{7} \times \sqrt{2} = \sqrt{14})$

We can state the procedure we just used as a general rule:

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd},$$

where b and d are positive numbers.

Example 16

Find:

a $15\sqrt{35} \div 5\sqrt{7}$

b $49\sqrt{21} \div 14\sqrt{3}$

Solution

a $15\sqrt{35} \div 5\sqrt{7} = 3\sqrt{5}$ $\quad (15 \div 5 = 3,\ \sqrt{35} \div \sqrt{7} = \sqrt{5})$

b $49\sqrt{21} \div 14\sqrt{3} = \dfrac{7\sqrt{7}}{2}$ $\quad \left(49 \div 14 = \dfrac{7}{2},\ \sqrt{21} \div \sqrt{3} = \sqrt{7}\right)$

We can state the procedure we just used as a general rule:

$$a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}},$$

where b and d are positive numbers and $c \neq 0$.

As usual, we should always give the answer in simplest form.

Example 17

Find $5\sqrt{6} \times 3\sqrt{10}$.

Solution

$$\begin{aligned} 5\sqrt{6} \times 3\sqrt{10} &= 15\sqrt{60} \\ &= 15\sqrt{4 \times 15} \\ &= 30\sqrt{15} \end{aligned}$$

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The distributive law

We can apply the distributive law to expressions involving surds, just as we do in algebra.

Example 18

Expand and simplify $2\sqrt{3}(4 + 3\sqrt{3})$.

Solution

$$\begin{aligned} 2\sqrt{3}(4 + 3\sqrt{3}) &= 2\sqrt{3} \times 4 + 2\sqrt{3} \times 3\sqrt{3} \\ &= 8\sqrt{3} + 6\sqrt{9} \\ &= 8\sqrt{3} + 18 \end{aligned}$$

In algebra, you learnt how to expand brackets such as $(a + b)(c + d)$. These are known as **binomial products**. You multiply each term in the second bracket by each term in the first, then add. This means you expand out $a(c + d) + b(c + d)$ to obtain $ac + ad + bc + bd$. We use this idea again when multiplying out binomial products involving surds.

Example 19

Expand and simplify:

a $(3\sqrt{7} + 1)(5\sqrt{7} - 4)$

b $(5\sqrt{2} - 3)(2\sqrt{2} - 4)$

c $(3\sqrt{2} - 4\sqrt{3})(5\sqrt{3} - \sqrt{2})$

d $(1 - \sqrt{2})(3 + 2\sqrt{2})$

Solution

a $$\begin{aligned} (3\sqrt{7} + 1)(5\sqrt{7} - 4) &= 3\sqrt{7}(5\sqrt{7} - 4) + 1(5\sqrt{7} - 4) \\ &= 15\sqrt{49} - 12\sqrt{7} + 5\sqrt{7} - 4 \\ &= 105 - 7\sqrt{7} - 4 \\ &= 101 - 7\sqrt{7} \end{aligned}$$

b $$\begin{aligned} (5\sqrt{2} - 3)(2\sqrt{2} - 4) &= 5\sqrt{2}(2\sqrt{2} - 4) - 3(2\sqrt{2} - 4) \\ &= 20 - 20\sqrt{2} - 6\sqrt{2} + 12 \\ &= 32 - 26\sqrt{2} \end{aligned}$$

c $$\begin{aligned} (3\sqrt{2} - 4\sqrt{3})(5\sqrt{3} - \sqrt{2}) &= 3\sqrt{2}(5\sqrt{3} - \sqrt{2}) - 4\sqrt{3}(5\sqrt{3} - \sqrt{2}) \\ &= 15\sqrt{6} - 6 - 60 + 4\sqrt{6} \\ &= 19\sqrt{6} - 66 \end{aligned}$$

d $$\begin{aligned} (1 - \sqrt{2})(3 + 2\sqrt{2}) &= 3 + 2\sqrt{2} - 3\sqrt{2} - 4 \\ &= -1 - \sqrt{2} \end{aligned}$$

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Multiplication and division of surds

- For positive numbers b and d, $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$.
- For positive numbers b and d, $a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}}$, provided $c \neq 0$.
- Give the answers in simplest form.
- We expand binomial products involving surds just as we do in algebra:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

Exercise 2D

Example 15

1 Simplify:

a $\sqrt{2} \times \sqrt{3}$ **b** $\sqrt{5} \times \sqrt{2}$ **c** $\sqrt{3} \times 2\sqrt{7}$ **d** $4\sqrt{7} \times \sqrt{2}$
e $3\sqrt{2} \times 2\sqrt{3}$ **f** $4\sqrt{5} \times 2\sqrt{2}$ **g** $5\sqrt{7} \times 3\sqrt{2}$ **h** $6\sqrt{3} \times 4\sqrt{7}$

Example 16

2 Simplify:

a $\sqrt{5} \div \sqrt{5}$ **b** $8\sqrt{10} \div \sqrt{10}$ **c** $6\sqrt{3} \div 2\sqrt{3}$
d $\sqrt{21} \div \sqrt{3}$ **e** $4\sqrt{15} \div \sqrt{5}$ **f** $12\sqrt{33} \div 3\sqrt{3}$
g $\sqrt{35} \div \sqrt{7}$ **h** $3\sqrt{2} \div \sqrt{2}$ **i** $36\sqrt{15} \div 4\sqrt{3}$
j $14\sqrt{40} \div 7\sqrt{5}$ **k** $20\sqrt{30} \div 2\sqrt{6}$ **l** $7\sqrt{28} \div 4\sqrt{7}$

Example 17

3 Simplify:

a $\sqrt{6} \times \sqrt{3}$ **b** $\sqrt{6} \times \sqrt{6}$ **c** $\sqrt{7} \times 2\sqrt{7}$ **d** $4\sqrt{5} \times \sqrt{10}$
e $2\sqrt{6} \times 2\sqrt{2}$ **f** $3\sqrt{6} \times 2\sqrt{2}$ **g** $7\sqrt{10} \times 3\sqrt{2}$ **h** $3\sqrt{14} \times 2\sqrt{7}$

Example 18

4 Expand and simplify:

a $\sqrt{3}(\sqrt{6} + \sqrt{3})$ **b** $\sqrt{2}(2\sqrt{2} - \sqrt{6})$ **c** $2\sqrt{5}(\sqrt{2} - \sqrt{5})$
d $3\sqrt{3}(2\sqrt{3} - 1)$ **e** $5\sqrt{5}(4\sqrt{2} - 3)$ **f** $\sqrt{7}(2\sqrt{7} - \sqrt{14})$
g $4\sqrt{5}(2\sqrt{15} - 3\sqrt{3})$ **h** $2\sqrt{3}(3\sqrt{3} - 5)$ **i** $3\sqrt{2}(3 + 4\sqrt{3})$
j $6\sqrt{5}(\sqrt{2} + 1)$ **k** $3\sqrt{7}(2 - \sqrt{14})$ **l** $3\sqrt{5}(\sqrt{15} + 3)$

Example 19

5 Expand and simplify:

a $(4\sqrt{5} + 1)(3\sqrt{5} + 2)$ **b** $(4 + 2\sqrt{6})(2 + 5\sqrt{6})$
c $(3\sqrt{2} + 2)(3\sqrt{2} - 1)$ **d** $(1 + \sqrt{5})(7 - 6\sqrt{5})$
e $(2\sqrt{3} - 4)(3\sqrt{3} + 5)$ **f** $(3\sqrt{7} - 1)(5\sqrt{7} - 2)$
g $(7\sqrt{2} + 5)^2$ **h** $(4\sqrt{3} - 2)^2$

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6 Expand and simplify:

a $(3\sqrt{5} + 2)(\sqrt{2} + 3)$
b $(2\sqrt{3} - 3)(3\sqrt{2} + 1)$
c $(4 + 2\sqrt{3})(2\sqrt{7} - 5)$
d $(8 - 4\sqrt{6})(2 - \sqrt{7})$
e $(\sqrt{7} - \sqrt{5})(\sqrt{7} + \sqrt{5})$
f $(5\sqrt{2} - 3\sqrt{3})(2\sqrt{2} + 2\sqrt{3})$
g $(4\sqrt{7} - \sqrt{5})(2\sqrt{5} + \sqrt{7})$
h $(4\sqrt{2} - \sqrt{5})(2\sqrt{5} + \sqrt{2})$

7 If $x = \sqrt{2} - 1$ and $y = \sqrt{3} + 1$, find:

a xy
b $x + y$
c $x(x + 2)$
d $\sqrt{3}x - \sqrt{2}y$
e $2xy$
f $x + 3y$
g $(x + 1)(y + 1)$
h $\frac{6\sqrt{10}}{x + 1}$
i $\frac{1}{x} + \frac{1}{\sqrt{2} + 1}$
j $\frac{1}{x} - \frac{1}{\sqrt{2} + 1}$
k $x + \frac{1}{x}$

2E Special products

In algebra, you learned the following special expansions. They are called **identities** because they are true for all values of the pronumerals. These are especially important when dealing with surds.

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a - b)(a + b) = a^2 - b^2$$

The first two identities are called the **perfect square** identities. The last identity is known as the **difference of two squares** identity.

You will need to be very confident with these identities and be able to recognise them. Since they are true for all numbers, we can apply them to surds. Recall that $(\sqrt{a})^2 = a$ holds for any positive number a.

Example 20

Expand and simplify $(\sqrt{5} + \sqrt{3})^2$.

Solution

$$\begin{aligned}(\sqrt{5} + \sqrt{3})^2 &= (\sqrt{5})^2 + 2 \times \sqrt{5} \times \sqrt{3} + (\sqrt{3})^2 \\ &= 5 + 2\sqrt{15} + 3 \\ &= 8 + 2\sqrt{15}\end{aligned}$$

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Example 21

Expand and simplify $(3\sqrt{2} - 2\sqrt{3})^2$.

Solution

$$
\begin{aligned}
(3\sqrt{2} - 2\sqrt{3})^2 &= (3\sqrt{2})^2 - 2 \times 3\sqrt{2} \times 2\sqrt{3} + (2\sqrt{3})^2 \\
&= 18 - 12\sqrt{6} + 12 \\
&= 30 - 12\sqrt{6}
\end{aligned}
$$

You should always express your answer in simplest form.

Example 22

Expand and simplify $(2\sqrt{3} + 4\sqrt{6})^2$.

Solution

$$
\begin{aligned}
(2\sqrt{3} + 4\sqrt{6})^2 &= (2\sqrt{3})^2 + 2 \times 2\sqrt{3} \times 4\sqrt{6} + (4\sqrt{6})^2 \\
&= 12 + 16\sqrt{18} + 96 \\
&= 108 + 48\sqrt{2}
\end{aligned}
$$

The following example shows an interesting application of the difference of two squares' identity. We will use this later in this chapter when simplifying surds with a binomial denominator.

Example 23

Expand and simplify:

a $(\sqrt{7} - 5)(\sqrt{7} + 5)$

b $(5\sqrt{6} - 2\sqrt{5})(5\sqrt{6} + 2\sqrt{5})$

Solution

a
$$
\begin{aligned}
(\sqrt{7} - 5)(\sqrt{7} + 5) &= (\sqrt{7})^2 - 5^2 \\
&= 7 - 25 \\
&= -18
\end{aligned}
$$

b
$$
\begin{aligned}
(5\sqrt{6} - 2\sqrt{5})(5\sqrt{6} + 2\sqrt{5}) &= (5\sqrt{6})^2 - (2\sqrt{5})^2 \\
&= 25 \times 6 - 4 \times 5 \\
&= 130
\end{aligned}
$$

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Identities

The following identities are often used in calculations involving surds.

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a-b)(a+b) = a^2 - b^2$

Exercise 2E

Example 20

1 Expand and simplify:

a $(5+\sqrt{3})^2$ **b** $(\sqrt{2}+6)^2$ **c** $(4+2\sqrt{5})^2$

d $(3\sqrt{3}+1)^2$ **e** $(\sqrt{5}+\sqrt{7})^2$ **f** $(2\sqrt{3}+\sqrt{2})^2$

g $(5\sqrt{2}+3\sqrt{3})^2$ **h** $(\sqrt{2}+3\sqrt{5})^2$ **i** $(2\sqrt{5}+4\sqrt{3})^2$

j $(\sqrt{x}+\sqrt{y})^2$ **k** $(a\sqrt{x}+b\sqrt{y})^2$ **l** $(\sqrt{xy}+1)^2$

Example 21

2 Expand and simplify:

a $(\sqrt{7}-2)^2$ **b** $(4-\sqrt{3})^2$ **c** $(2\sqrt{5}-1)^2$

d $(2-3\sqrt{3})^2$ **e** $(\sqrt{5}-\sqrt{3})^2$ **f** $(2\sqrt{3}-\sqrt{2})^2$

g $(\sqrt{2}-4\sqrt{5})^2$ **h** $(4\sqrt{2}-3\sqrt{7})^2$ **i** $(\sqrt{x}-\sqrt{y})^2$

Example 22

3 Expand and simplify:

a $(2\sqrt{10}+4\sqrt{5})^2$ **b** $(5\sqrt{6}+2\sqrt{3})^2$ **c** $(\sqrt{21}+\sqrt{3})^2$

d $(2\sqrt{35}+\sqrt{5})^2$ **e** $(2\sqrt{10}-\sqrt{2})^2$ **f** $(3\sqrt{2}-\sqrt{10})^2$

g $(\sqrt{70}-3\sqrt{10})^2$ **h** $(\sqrt{50}-3\sqrt{5})^2$ **i** $(\sqrt{11}-2\sqrt{22})^2$

j $(\sqrt{10}-\sqrt{5})^2$ **k** $(2\sqrt{3}-\sqrt{6})^2$ **l** $(5\sqrt{14}-3\sqrt{21})^2$

Example 23

4 Expand and simplify:

a $(3-\sqrt{5})(3+\sqrt{5})$ **b** $(\sqrt{6}-1)(\sqrt{6}+1)$

c $(7\sqrt{2}+3)(7\sqrt{2}-3)$ **d** $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$

e $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$ **f** $(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})$

g $(\sqrt{2}-3\sqrt{5})(\sqrt{2}+3\sqrt{5})$ **h** $(4\sqrt{2}-3\sqrt{5})(4\sqrt{2}+3\sqrt{5})$

i $(6\sqrt{2}-2\sqrt{7})(6\sqrt{2}+2\sqrt{7})$ **j** $(2\sqrt{3}-5\sqrt{6})(2\sqrt{3}+5\sqrt{6})$

k $(2\sqrt{5}+7\sqrt{10})(2\sqrt{5}-7\sqrt{10})$ **l** $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$

5 If $x=\sqrt{2}-1$ and $y=\sqrt{2}+1$, find:

a x^2 **b** y^2 **c** x^2+y^2 **d** x^2-y^2

e xy **f** x^2y **g** y^2x **h** $\frac{1}{x}+\frac{1}{y}$

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6 Find:

a $(\sqrt{5}+\sqrt{3})^2+(\sqrt{5}-\sqrt{3})^2$

b $(\sqrt{5}+\sqrt{3})^2-(\sqrt{5}-\sqrt{3})^2$

c $(\sqrt{3}+\sqrt{2})^2+(\sqrt{3}-\sqrt{2})^2$

d $(\sqrt{3}+\sqrt{2})^2-(\sqrt{3}-\sqrt{2})^2$

e $(\sqrt{7}-\sqrt{2})^2+(\sqrt{7}+\sqrt{2})^2$

f $(\sqrt{6}+2)^2-(\sqrt{6}-2)^2$

7 Simplify $(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d})$.

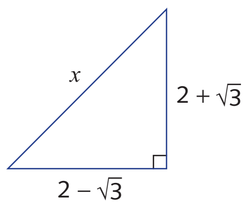
8 For the figures shown, find:

i the value of x

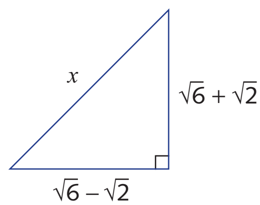
ii the area of the triangle

iii the perimeter of the triangle

a



b



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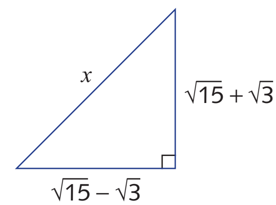
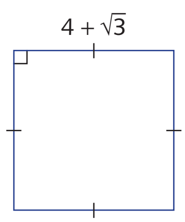


9 For the figures shown, find:

i the perimeter

ii the area

a



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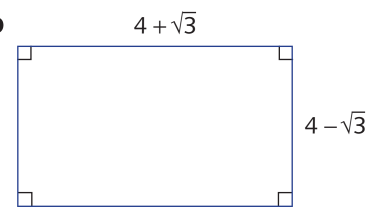

10 An equilateral triangle has sides length $1+\sqrt{3}$. Find:

a the perimeter of the triangle

b the area of the triangle

2F Rationalising the denominator

The expression $\dfrac{2}{\sqrt{3}}+\dfrac{\sqrt{3}}{2}$ is untidy. The first term has a square root in the denominator.

Fractions involving surds are easiest to deal with when they are expressed in simplest form, with a rational denominator.

When we multiply the numerator and denominator of a fraction by the same number, we form an equivalent fraction. The same happens with a quotient involving surds.

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Example 24

Express $\frac{4}{\sqrt{2}}$ with a rational denominator.

Solution

$$\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$ (Multiply top and bottom by the same surd.)

$$= \frac{4\sqrt{2}}{2}$$

$$= 2\sqrt{2}$$

Example 25

Write $\frac{9}{4\sqrt{3}}$ as a quotient with a rational denominator.

Solution

$$\frac{9}{4\sqrt{3}} = \frac{9}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$ (Multiply top and bottom by $\sqrt{3}$.)

$$= \frac{9\sqrt{3}}{12}$$

$$= \frac{3\sqrt{3}}{4}$$

Example 26

Simplify $\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2}$.

Solution

$$\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{3}}{2}$$ (Rationalise the denominator of the first term.)

$$= \frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{2}$$

$$= \frac{4\sqrt{3}}{6} + \frac{3\sqrt{3}}{6}$$ (Use a common denominator.)

$$= \frac{7\sqrt{3}}{6}$$

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Rationalising the denominator

- Rationalising the denominator means converting the denominator into a rational number.
- We use the result

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}, \text{ where } a \text{ is positive,}$$

to rationalise the denominator of a quotient involving surds.

Exercise 2F

Example 24

1 Rationalise the denominator.

a $\frac{1}{\sqrt{5}}$ **b** $\frac{5}{\sqrt{6}}$ **c** $\frac{3}{\sqrt{3}}$ **d** $\frac{14}{\sqrt{7}}$ **e** $\frac{\sqrt{3}}{\sqrt{7}}$

f $\frac{\sqrt{18}}{\sqrt{6}}$ **g** $\frac{\sqrt{5}}{\sqrt{3}}$ **h** $\frac{\sqrt{6}}{\sqrt{3}}$ **i** $\frac{\sqrt{14}}{\sqrt{7}}$ **j** $\frac{2\sqrt{7}}{\sqrt{3}}$

Example 25

2 Rationalise the denominator.

a $\frac{1}{5\sqrt{3}}$ **b** $\frac{7}{3\sqrt{2}}$ **c** $\frac{4}{7\sqrt{2}}$ **d** $\frac{\sqrt{5}}{3\sqrt{7}}$ **e** $\frac{\sqrt{2}}{3\sqrt{10}}$

f $\frac{2\sqrt{5}}{3\sqrt{2}}$ **g** $\frac{4\sqrt{2}}{5\sqrt{7}}$ **h** $\frac{8\sqrt{18}}{2\sqrt{3}}$ **i** $\frac{\sqrt{3}}{4\sqrt{6}}$ **j** $\frac{\sqrt{15}}{3\sqrt{5}}$

3 Given that $\sqrt{2} \approx 1.412$ and $\sqrt{3} \approx 1.732$, find these values, correct to 2 decimal places, without using a calculator. (First rationalise the denominator where appropriate.)

a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{3}}$ **c** $\frac{3}{\sqrt{2}}$ **d** $\frac{5}{\sqrt{3}}$

e $\sqrt{12}$ **f** $\sqrt{18}$ **g** $\frac{1}{\sqrt{12}}$ **h** $\frac{1}{\sqrt{18}}$

Example 26

4 By first rationalising the denominators, simplify each expression.

a $\frac{1}{\sqrt{2}} + \sqrt{2}$ **b** $\frac{2}{\sqrt{3}} + \frac{3}{2\sqrt{3}}$ **c** $\frac{5\sqrt{2}}{3} - \frac{1}{\sqrt{3}}$

d $\frac{4\sqrt{5}}{\sqrt{2}} - \frac{2\sqrt{5}}{\sqrt{2}}$ **e** $\frac{\sqrt{72}}{\sqrt{3}} + \frac{3}{\sqrt{2}} - \frac{2}{2\sqrt{2}}$ **f** $\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{7}{2\sqrt{3}}$

5 If $x = 2\sqrt{14}$ and $y = 4\sqrt{2}$, find and rationalise the denominator.

a $\frac{x}{y}$ **b** $\frac{y}{x}$ **c** $\frac{2x}{y}$ **d** $\frac{\sqrt{2}x}{\sqrt{3}y}$

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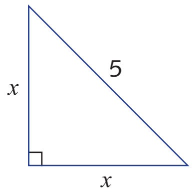
6 If $x = 2\sqrt{3}$, find and rationalise the denominator.

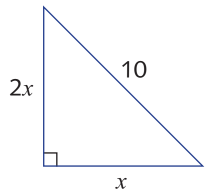
a $x + \frac{1}{x}$ **b** $x - \frac{1}{x}$ **c** $\left(x + \frac{1}{x}\right)^2$ **d** $\left(x - \frac{1}{x}\right)^2$

7 Find the value of x. Express your answer with a rational denominator.

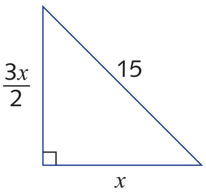
a



b

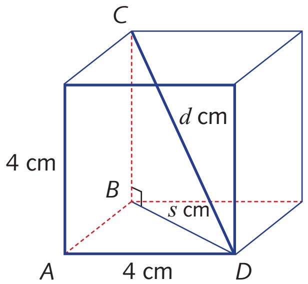


c



2G Applications of Pythagoras' theorem in three dimensions

How can we find the length of the diagonal CD of a cube whose side length is 4 cm? BD is a **face diagonal** and sometimes CD is called a **space diagonal**.



We can apply Pythagoras' theorem to triangle BAD to find the square of the length s cm of the diagonal BD.

$$\begin{aligned} s^2 &= 4^2 + 4^2 \\ &= 32 \end{aligned}$$

We can then apply Pythagoras' theorem again to triangle CBD since $\angle CBD$ is a right angle. The length d cm of the diagonal CD is given by:

$$\begin{aligned} d^2 &= s^2 + 4^2 \\ &= 32 + 16 \\ &= 48 \\ d &= \sqrt{48} = 4\sqrt{3} \\ &\approx 6.93 \quad \text{(correct to 2 decimal places)} \end{aligned}$$

The length of the diagonal CD is 6.93 cm correct to 2 decimal places.

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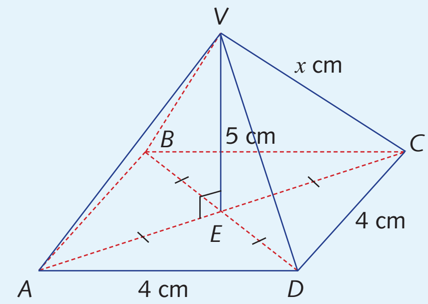
Example 27

A square pyramid has height 5 cm and square base with side length 4 cm. Find the length of the edge *VC* in this diagram.



Solution

The base is a square with side length 4 cm, so:

$$\begin{aligned} AC^2 &= 4^2 + 4^2 \\ &= 32 \\ AC &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$

Hence $EC = 2\sqrt{2}$ (Leave in exact form to maintain accuracy.)

Triangle *VEC* is right-angled, so:

$$\begin{aligned} x^2 &= VE^2 + EC^2 \\ &= 52 + (2\sqrt{2})^2 \\ &= 25 + 8 \\ &= 33 \\ x &= \sqrt{33} \\ &\approx 5.74 \quad \text{(correct to 2 decimal places)} \end{aligned}$$

The lenth of *VC* is 5.74 cm correct to 2 decimal places.

Applications of Pythagoras' theorem in three dimensions

- Pythagoras' theorem can be used to find lengths in three-dimensional problems.
- Always draw a careful diagram identifying the appropriate right-angled triangle(s).
- To maintain accuracy, use exact values and only approximate using a calculator at the end of the problem if required.

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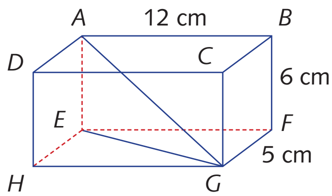
Exercise 2G

Example 27

1 The rectangular prism in the diagram has a length of 12 cm, a width of 5 cm and a height of 6 cm.



a Consider triangle EFG. Find:

i EF **ii** the size of $\angle EFG$

b Find EG.

c Find AG, correct to 1 decimal place.

Note: AG is called the space diagonal of the rectangular prism.

2 Find the length of the space diagonal of the rectangular prism whose length, width and height are:

a 12 cm, 9 cm, 8 cm **b** 12 cm, 5 cm, 8 cm

c 10 cm, 4 cm, 7 cm **d** 8 cm, 6 cm, 4 cm

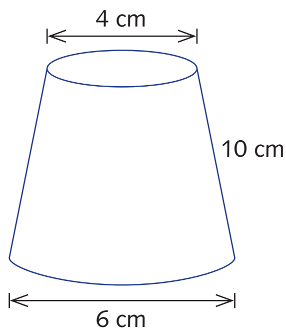
e 7 cm, 2 cm, 3 cm **f** a cm, b cm, c cm

3 Find the length of the longest pencil that can fit inside a cylindrical pencil case of length 15 cm and radius 2 cm.

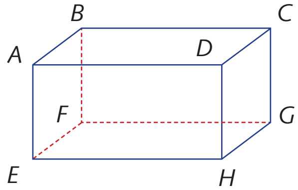
4 A bowl in the shape of a hemisphere of radius length 5 cm is partially filled with water. The surface of the water is a circle of radius 4 cm when the rim of the bowl is horizontal. Find the depth of the water.

5 A builder needs to carry lengths of timber along a corridor in order to get them to where he is working. There is a right-angled bend in the corridor along the way. The corridor is 3 m wide and the ceiling is 2.6 m above the floor. What is the longest length of timber that the builder can take around the corner in the corridor? (*Hint*: Draw a diagram.)

6 A bobbin for an industrial knitting machine is in the shape of a truncated cone. The diameter of the top is 4 cm, the diameter of the base is 6 cm and the length of the slant is 10 cm. Find the height of the bobbin.



7 For the rectangular prism shown opposite, $EH = 4$ cm and $HG = 2$ cm.



a Find the exact length of EG, giving your answer as a surd in simplest form.

b If $AE = \frac{1}{2}EG$, find the exact value of AE.

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c Find the length of:

i BE **ii** BH

d What type of triangle is triangle BEH?

e Show that if $EH = 2a$ cm, $HG = a$ cm and $AE = \frac{1}{2}EG$, then the sides of the triangle BEH are in the ratio 3 : 4 : 5.

2H Binomial denominators

Consider the expression $\frac{1}{\sqrt{7}-\sqrt{5}}$.

How can we write this as a quotient with a rational denominator?

In the section on special products, we saw that:

$$\begin{aligned}(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5}) &= \left(\sqrt{7}\right)^2-\left(\sqrt{5}\right)^2\\ &= 7-5\\ &= 2, \text{ which is rational,}\end{aligned}$$

so

$$\begin{aligned}\frac{1}{\sqrt{7}-\sqrt{5}} &= \frac{1}{\sqrt{7}-\sqrt{5}} \times \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}\\ &= \frac{\sqrt{7}+\sqrt{5}}{7-5}\\ &= \frac{\sqrt{7}+\sqrt{5}}{2}\end{aligned}$$

Similarly,

$$\begin{aligned}(5\sqrt{2}-4)(5\sqrt{2}+4) &= (5\sqrt{2})^2-4^2\\ &= 34, \text{ which again is rational,}\end{aligned}$$

so

$$\begin{aligned}\frac{3}{5\sqrt{2}-4} &= \frac{3}{5\sqrt{2}-4} \times \frac{5\sqrt{2}+4}{5\sqrt{2}+4}\\ &= \frac{15\sqrt{2}+12}{34}\end{aligned}$$

Using the difference of two squares identity in this way is an important technique.

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Example 28

Simplify the following:

a $\dfrac{2\sqrt{5}}{2\sqrt{5}-2}$

b $\dfrac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}}$

Solution

a
$$\begin{aligned}\frac{2\sqrt{5}}{2\sqrt{5}-2} &= \frac{2\sqrt{5}}{2\sqrt{5}-2}\times\frac{2\sqrt{5}+2}{2\sqrt{5}+2}\\ &= \frac{20+4\sqrt{5}}{20-4}\\ &= \frac{20+4\sqrt{5}}{16}\\ &= \frac{4(5+\sqrt{5})}{16}\\ &= \frac{5+\sqrt{5}}{4}\end{aligned}$$

b
$$\begin{aligned}\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} &= \frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}}\times\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}\\ &= \frac{(\sqrt{3}+\sqrt{2})(3\sqrt{2}-2\sqrt{3})}{18-12}\\ &= \frac{(3\sqrt{6}-6+6-2\sqrt{6})}{6}\\ &= \frac{\sqrt{6}}{6}\end{aligned}$$

Binomial denominators

To rationalise a denominator which has two terms, we use the difference of two squares identity:

- In an expression such as $\dfrac{3}{5+\sqrt{3}}$, multiply top and bottom by $5-\sqrt{3}$.
- In an expression such as $\dfrac{\sqrt{2}}{7-3\sqrt{2}}$, multiply top and bottom by $7+3\sqrt{2}$.

The surd $7-\sqrt{3}$ is called the **conjugate** of $7+\sqrt{3}$ and $7+\sqrt{3}$ is the conjugate of $7-\sqrt{3}$.

Exercise 2H

1 Simplify:

a $(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})$

b $(2\sqrt{5}-\sqrt{3})(2\sqrt{5}+\sqrt{3})$

c $(7\sqrt{2}+4\sqrt{3})(7\sqrt{2}-4\sqrt{3})$

d $(4-\sqrt{3})(4+\sqrt{3})$

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Example 28

2 Rationalise the denominator in each expression.

a $\dfrac{1}{\sqrt{6}+1}$ **b** $\dfrac{3}{\sqrt{2}-1}$ **c** $\dfrac{2}{3+\sqrt{5}}$

d $\dfrac{2}{3-\sqrt{5}}$ **e** $\dfrac{4}{\sqrt{5}+\sqrt{2}}$ **f** $\dfrac{1}{\sqrt{7}-\sqrt{5}}$

g $\dfrac{\sqrt{3}}{\sqrt{6}+\sqrt{5}}$ **h** $\dfrac{\sqrt{2}}{\sqrt{2}-\sqrt{3}}$ **i** $\dfrac{\sqrt{2}}{2\sqrt{5}+\sqrt{2}}$

j $\dfrac{\sqrt{5}}{2\sqrt{5}-\sqrt{3}}$ **k** $\dfrac{\sqrt{5}}{3\sqrt{2}+4\sqrt{3}}$ **l** $\dfrac{\sqrt{3}}{2\sqrt{3}-3\sqrt{6}}$

m $\dfrac{2\sqrt{6}}{4\sqrt{2}-3\sqrt{7}}$ **n** $\dfrac{3\sqrt{2}}{6\sqrt{3}+11\sqrt{5}}$ **o** $\dfrac{2\sqrt{3}}{3\sqrt{2}+10\sqrt{3}}$

3 Rationalise the denominators in these expressions and use the decimal approximations $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$ to evaluate them correct to 2 decimal places.

a $\dfrac{1}{\sqrt{2}-1}$ **b** $\dfrac{1}{\sqrt{3}+2}$ **c** $\dfrac{1}{\sqrt{3}+\sqrt{2}}$ **d** $\dfrac{1}{\sqrt{3}-\sqrt{2}}$

4 Find the integers p and q such that $\dfrac{\sqrt{5}}{\sqrt{5}-2} = p + q\sqrt{5}$.

5 Simplify:

a $\dfrac{3}{\sqrt{5}-2} + \dfrac{2}{\sqrt{5}+2}$ **b** $\dfrac{2}{6-3\sqrt{3}} - \dfrac{1}{2\sqrt{3}+3}$

6 Simplify:

a $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} + \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ **b** $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} - \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ **c** $\dfrac{5}{(\sqrt{7}-\sqrt{2})^2}$

2I Irrational numbers and surds

We first recall some facts about fractions and decimals.

Fractions and decimals

We know that some fractions can be written as decimals that **terminate**. For example,

$$\frac{1}{4} = 0.25,\ \frac{3}{16} = 0.1875$$

Some fractions have decimal representations that do not terminate. For example,

$$\frac{1}{3} = 0.33333\ldots \text{ which we write as } 0.\dot{3}.$$

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The dot above the 3 indicates that the digit 3 is repeated forever.

Some fractions have decimal representations that eventually repeat. For example,

$\frac{1}{6} = 0.1666\ldots$ which is written as $0.1\dot{6}$.

Other fractions have decimal representations with more than one repeating digit. For example,

$\frac{1}{11} = 0.090\,909\ldots$ which we write as $0.\dot{0}\dot{9}$,

with a dot above both of the repeating digits.

Another example is $0.12\dot{3}\,45\dot{6} = 0.123\,456\,3456\ldots$

Converting decimals to fractions

It is easy to write terminating decimals as fractions by using a denominator which is a power of 10.

For example,

$$0.14 = \frac{14}{100} = \frac{7}{50}$$

Decimals that have a repeated sequence of digits can also be written as fractions.
For example,

$$0.\dot{1}2\dot{3} = \frac{41}{333} \text{ and } 0.67\dot{1}\dot{2} = \frac{443}{660}$$

A method for doing this is shown in the next two examples.

Example 29

Write $0.\dot{2}$ as a fraction.

Solution

Let $S = 0.\dot{2}$

So $S = 0.2222\ldots$

Then $10S = 2.2222\ldots$ (Multiply by 10.)

$10S = 2 + 0.222\ldots$

Therefore $10S = S + 2$

Hence $9S = 2$

So $S = \frac{2}{9}$

Thus $0.\dot{2} = \frac{2}{9}$

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Example 30

Write $0.\dot{1}\dot{2}$ as a fraction.

Solution

Let $S = 0.\dot{1}\dot{2}$

So $S = 0.121\,212\,12\ldots$

Then $100S = 121\,212\,12\ldots$

Therefore $100S = 12 + S$

Hence $99S = 12$

So $S = \frac{12}{99}$

$= \frac{4}{33}$

Thus $0.\dot{1}\dot{2} = \frac{4}{33}$

Rational numbers

A **rational number** is a number that can be written as a fraction $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

All integers are rational numbers. For example,

$$-3 = \frac{-3}{1}$$

Example 31

Explain why each of these numbers is rational.

a $5\frac{3}{7}$ **b** 2.1237 **c** $0.\dot{2}\dot{4}$

Solution

We will express each number in the form $\frac{p}{q}$.

a $5\frac{3}{7} = \frac{38}{7}$

b $2.1237 = 2\frac{1237}{10\,000}$

$= \frac{21\,237}{10\,000}$

(continued over page)

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c Let $S = 0.\dot{2}\dot{4}$

$= 0.242\,42\ldots$

$100S = 24.242\,42\ldots$ (Multipy by 100.)

$100S = 24 + S$

Hence $99S = 24$

$S = \frac{24}{99}$

$= \frac{8}{33}$

Thus $0.\dot{2}\dot{4} = \frac{8}{33}$

Each number has been expressed as a fraction $\frac{p}{q}$, where p and q are integers, so each number is rational.

Irrational numbers

Mathematicians up to about 600 BCE thought that all numbers were rational. However, when we apply Pythagoras' theorem, we encounter numbers such as $\sqrt{2}$ that are not rational. The number $\sqrt{2}$ is an example of an **irrational number**.

An irrational number is one that is not rational. Hence an irrational number cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Nor can it be written as a terminating or repeating decimal.

In 300 BCE, Euclid proved that making the assumption that $\sqrt{2}$ is rational leads to a contradiction. Hence $\sqrt{2}$ was shown to be irrational. The proof is outlined here.

Assume $\sqrt{2} = \frac{p}{q}$ where p and q are integers with highest common factor 1.

Square both sides of this equation to obtain $2 = \frac{p^2}{q^2}$. We can write $p^2 = 2q^2$. Hence p^2 is even and thus p is even. We can now write, $2 = \frac{4k^2}{q^2}$ For some whole number k. From this, show q is also even which is a contradiction of our assumption that the highest common factor is 1.

The decimal expansion of $\sqrt{2}$ goes on forever but does not repeat. The value of $\sqrt{2}$ can be approximated using a calculator. The same is true of other irrational numbers such as $\sqrt{3}$, $\sqrt{14}$ and $\sqrt{91}$. Try finding these numbers on your calculator and see what you get.

Other examples of irrational numbers include:

$$\sqrt{3},\ \sqrt[3]{2},\ \pi, \pi^3 \text{ and } \sqrt{\pi}$$

There are infinitely many rational numbers because every whole number is rational. We can also easily write down infinitely many other fractions. For example, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

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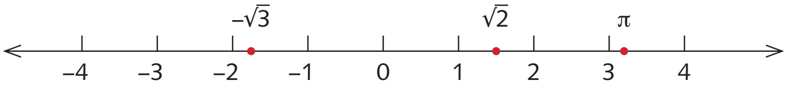
There are infinitely many irrational numbers too. For example, $\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{4}, \ldots$

Every number, whether rational or irrational, is represented by a point on the number line. Conversely, we can think of each point on the number line as a number.



Example 32

Arrange these irrational numbers in order of size on the number line.

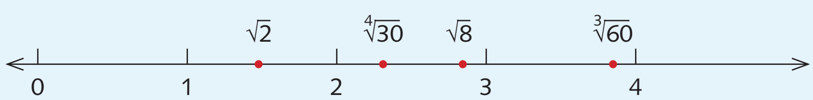
$\sqrt{8}$ $\qquad$ $\sqrt{2}$ $\qquad$ $\sqrt[3]{60}$ $\qquad$ $\sqrt[4]{30}$

Solution

Find on approximation of each number to 2 decimal places using a calculator.

$\sqrt{8} \approx 2.83$ $\qquad$ $\sqrt{2} \approx 1.14$

$\sqrt[3]{60} \approx 3.91$ $\qquad$ $\sqrt[4]{30} \approx 2.34$



Surds

In Year 8, we looked at squares, square roots, cubes and cube roots. For example:

$5^2 = 25$ $\quad$ and $\quad$ $\sqrt{25} = 5$

$5^3 = 125$ $\quad$ and $\quad$ $\sqrt[3]{125} = 5$

The use of this notation can be extended. For example:

$5^4 = 625$ $\quad$ and $\quad$ $\sqrt[4]{625} = 5$

This is read as 'The fourth power of 5 is 625 and the fourth root of 625 is 5'. We also have fifth roots, sixth roots and so on.

In general, we can take the nth root of any positive number a. The **nth root of a** is the positive number whose nth power is a.

The statement $\sqrt[n]{a} = b$ is equivalent to the statement $b^n = a$.

Your calculator will give you approximations to the nth root of a.

Note: For n, a positive integer, $0^n = 0$ and $\sqrt[n]{0} = 0$.

An irrational number which can be expressed as $\sqrt[n]{a}$, where a is a positive whole number, is called a **surd**.

For example, $\sqrt{2}$, $\sqrt{7}$ and $\sqrt[3]{5}$ are all examples of surds, while $\sqrt{4}$ and $\sqrt{9}$ are not surds since they are whole numbers. The number π, although it is irrational, is not a surd. This is also difficult to prove.

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Example 33

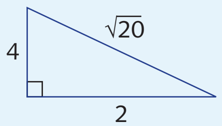
Use Pythagoras' theorem to construct a line of length $\sqrt{20}$.

Solution

Find two perfect squares that sum to 20.

$4^2 + 2^2 = 20$

Draw perpendicular line segments from a common point of lengths 2 units and 4 units. Connect their endpoints with a third line segment.



Irrational numbers and surds

- Every fraction can be written as a terminating, or eventually repeating, decimal.
- Every terminating, or eventually repeating, decimal can be written as a fraction.
- A **rational** number is a number which can be expressed as $\frac{p}{q}$ where p and q are integers and $q \neq 0$. That is, a rational number is an integer or a fraction.
- There are numbers such as π and $\sqrt{2}$ that are **irrational** (not rational).
- Each rational and each irrational number represents a point on the number line.
- Every point on the number line represents a rational or an irrational number.
- An irrational number which can be expressed as $\sqrt[n]{a}$, where n and a are positive whole numbers, is called a **surd**. π is not a surd.

Exercise 2I

Example 29

1 Write each repeating decimal as a fraction.

a $0.\dot{5}$ **b** $0.\dot{7}$ **c** $0.\dot{9}$ **d** $0.1\dot{4}$ **e** $0.2\dot{3}$ **f** $0.6\dot{2}$

Example 30

2 Write each repeating decimal as a fraction.

a $0.\dot{1}\dot{3}$ **b** $0.\dot{0}\dot{7}$ **c** $0.\dot{9}\dot{1}$ **d** $0.\dot{2}4\dot{1}$ **e** $0.\dot{6}1\dot{3}$

f $0.0\dot{1}\dot{6}$ **g** $0.3\dot{2}\dot{4}$ **h** $0.51\dot{2}\dot{6}$ **i** $0.00\dot{1}\dot{2}$

Example 31

3 Show that each number is rational by writing it as a fraction.

a $3\frac{2}{3}$ **b** 5.15 **c** $0.\dot{4}$ **d** $0.\dot{1}\dot{5}$ **e** $5\frac{1}{7}$ **f** 1.3

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4 Which of these numbers are irrational?

a $\sqrt{7}$ **b** $\sqrt{25}$ **c** $0.\dot{6}$ **d** $\frac{\pi}{3}$

Example 32

5 By approximating correct to 2 decimal places, place these real numbers on the same number line.

a $\sqrt{7}$ **b** 2.7 **c** $\sqrt[3]{18}$ **d** 2π **e** $\sqrt{2}$

6 Which of these numbers are surds?

a 3 **b** $\sqrt{5}$ **c** 4 **d** $\sqrt{9}$ **e** $\sqrt{7}$ **f** $\sqrt{16}$

g $\sqrt{10}$ **h** $\sqrt{1}$ **i** $\sqrt{3}$ **j** $\sqrt{15}$ **k** 5 **l** $\sqrt{25}$

Example 33

7 **a** Use the fact that $1^2 + 2^2 = 5$ to construct a length $\sqrt{5}$.

b Use the fact that $4^2 + 5^2 = 41$ to construct a length $\sqrt{41}$.

8 How would you construct an interval of length:

a $\sqrt{73}$? **b** $\sqrt{12}$? **c** $\sqrt{21}$?

9 **a** Show that $\frac{138}{19} = 7 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}$

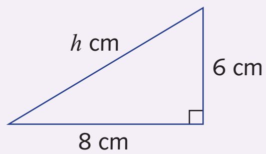
The expression on the right is called a **continued fraction**.

b Express $\frac{153}{11}$ as a continued fraction with all numerators 1.

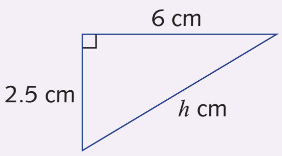
Review exercise

1 For each right-angled triangle, find the value of the pronumeral.

a

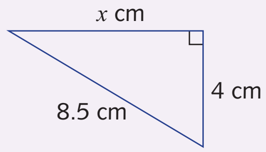


b



2 For each right-angled triangle, find the value of the pronumeral.

a



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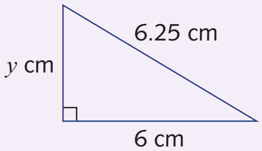

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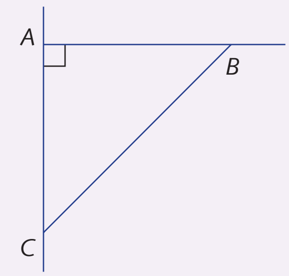
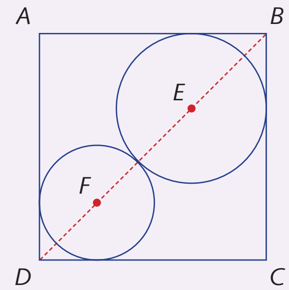
3 A support bracket is to be placed under a shelf, as shown in the diagram. If $AB = AC = 20$ cm, find, correct to the nearest millimetre, the length of BC.



4 A road runs in an east–west direction joining towns A and B, which are 40 km apart. A third town, C, is situated 20 km due north of B. A straight road is built from C, to the road between A and B and meets it at D, which is equidistant from A and C. Find the length of road CD.

5 Circles of radius 6 cm and 3 cm are placed in a square, as shown in the diagram opposite. Find, correct to 2 decimal places:



a EB **b** FD

c the length of the diagonal BD

d the side length of the square

6 Simplify:

a $3\sqrt{2} + 2\sqrt{2}$ **b** $\sqrt{32} - \sqrt{18}$

7 Simplify:

a $2\sqrt{3} \times 5\sqrt{6}$ **b** $3\sqrt{5} \times 2\sqrt{10}$

8 Simplify:

a $2\sqrt{3}(3 + \sqrt{3})$ **b** $5\sqrt{2}(3\sqrt{2} - 2)$

9 Expand and simplify:

a $(2\sqrt{2} + 1)(3\sqrt{2} - 2)$ **b** $(5\sqrt{3} - 2)(2\sqrt{3} - 1)$

10 Simplify:

a $\sqrt{80}$ **b** $\sqrt{108}$ **c** $\sqrt{125}$ **d** $\sqrt{72}$

e $\sqrt{2048}$ **f** $\sqrt{448}$ **g** $\sqrt{800}$ **h** $\sqrt{112}$

11 Simplify:

a $\sqrt{45} + 2\sqrt{5} - \sqrt{80}$ **b** $\sqrt{28} + 2\sqrt{63} - 5\sqrt{7}$

c $\sqrt{44} + \sqrt{275} - 4\sqrt{11}$ **d** $\sqrt{162} - \sqrt{200} + \sqrt{288}$

12 Express with a rational denominator.

a $\dfrac{3}{\sqrt{11}}$ **b** $\dfrac{1}{5\sqrt{15}}$ **c** $\dfrac{4}{7\sqrt{7}}$ **d** $\dfrac{3}{\sqrt{17}}$

e $\dfrac{3}{\sqrt{3}}$ **f** $\dfrac{3}{5\sqrt{15}}$ **g** $\dfrac{14}{\sqrt{7}}$ **h** $\dfrac{11}{\sqrt{3}}$

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13 Express with a rational denominator.

a $\dfrac{3}{3-\sqrt{3}}$ **b** $\dfrac{22}{2+3\sqrt{5}}$ **c** $\dfrac{24}{1-\sqrt{7}}$ **d** $\dfrac{24}{1+\sqrt{17}}$

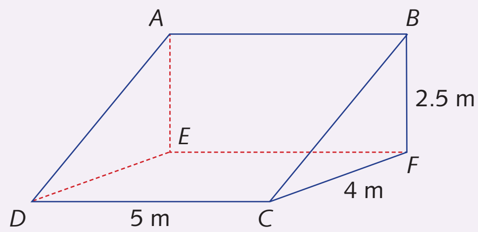
e $\dfrac{3}{2-\sqrt{3}}$ **f** $\dfrac{30}{1-\sqrt{11}}$ **g** $\dfrac{15}{2-\sqrt{7}}$ **h** $\dfrac{10}{2-\sqrt{3}}$

14 **a** Expand and simplify $(2\sqrt{5}-\sqrt{3})^2$.

b Simplify $\dfrac{2}{\sqrt{3}-2}+\dfrac{2}{\sqrt{3}}$, expressing your answer with a rational denominator.

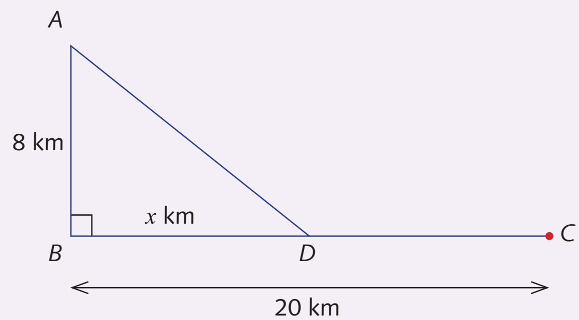
15 The diagram opposite shows part of a skate board ramp. (It is a prism whose cross-section is a right-angled triangle.) Use the information in the diagram to find:

a BC **b** AC



16 Find the length of the space diagonal of a cube with side length 5 cm.

17 A motorist departs from town B, which is 8 km due south from another town, A, and drives due east towards town C, which is 20 km from B. After driving a distance of x km, he notices that he is the same distance away from both towns A and C.

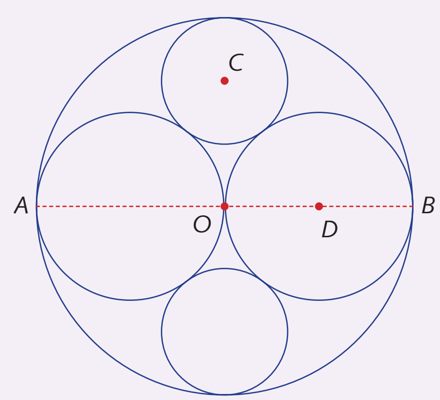


a Express the motorist's distance from A in terms of x (that is, AD).

b Express the motorist's distance from C in terms of x.

c Find the distance he has driven from B.

18 The diagram opposite shows the logo of a particular company. The large circle has centre O and radius 12 cm and AB is a diameter. D is the centre of the middle-sized circle with diameter OB. Finally, C is the centre of the smallest circle.



a What is the radius of the circle with centre D?

b If the radius of the circle with centre C is r cm, express these in terms of r.

i OC **ii** DC

c Find the value of r.

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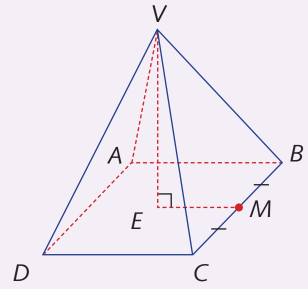
19 In the diagram opposite, $VABCD$ is a square-based pyramid with $AB = BC = CD = DA = 10$ and $VA = VB = VC = VD = 10$.



a What type of triangle is triangle VBC?

b If M is the midpoint of CB, find the exact values of:

i VM

ii VE, the height of the pyramid

20 Write each repeating decimal as a fraction.

a $0.\dot{8}$ **b** $0.1\dot{5}$ **c** $0.7\dot{2}$ **d** $0.0\dot{3}$

e $0.\dot{8}\dot{1}$ **f** $0.\dot{9}\dot{6}$ **g** $0.\dot{1}0\dot{1}$ **h** $0.00\dot{1}\dot{5}$

21 Write $\frac{67}{29}$ as a continued fraction. (See Exercise 2I, Question **9**.)

22 Write $0.\dot{4}79\,28\dot{1}$ as a fraction.

23 How would you construct intervals of the following lengths? Discuss with your teacher.

a $\sqrt{3}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\sqrt{8}$ **d** $\frac{1}{\sqrt{8}}$

24 Find the length of the long diagonal of the rectangular prism whose length, width and height are:

a $3 + \sqrt{2},\ 3 - \sqrt{2},\ 3\sqrt{3}$ **b** $5 + \sqrt{3},\ 5 - \sqrt{3},\ 2\sqrt{2}$

25 For $x = \sqrt{2} = 1$ and $y = \sqrt{5} - 2$, find in simplest form.

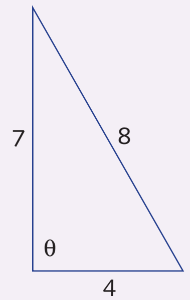
a $\frac{1}{x}$ **b** $\frac{1}{x} + \frac{1}{y}$ **c** $\frac{1}{x^2} + \frac{1}{y^2}$ **d** $\frac{1}{x} - \frac{1}{y}$

Challenge exercise

1 Prove that $\sqrt[3]{2}$ is irrational.

2 In the triangle below, $4^2 + 7^2 = 65 > 8^2$.

Is the angle θ greater or less than 90°? Explain your answer.



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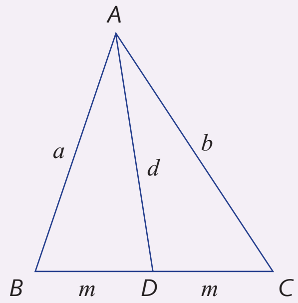
3 **Apollonius' theorem** states that in any triangle, if we join a vertex to the midpoint of the opposite side, and the length of that line is d, then $a^2 + b^2 = 2(d^2 + m^2)$ where $2m = c$.

In words it says: In any triangle, the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median.



Prove this result as follows:

a Drop a perpendicular AE of length h from A to BC and let $ED = p$.

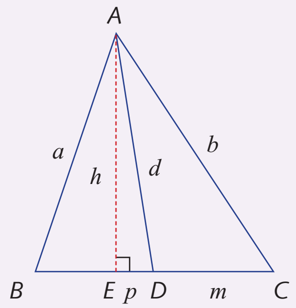
i Write BE in terms of m and p.

ii Write EC in terms of m and p.

b By applying Pythagoras' theorem to the three right-angled triangles in the diagram, complete these statements:

$a^2 =$ .... $b^2 =$ ....

$d^2 =$ ....



c Add the first two equations in part **b** above and simplify.

d Use the third equation in part **b** to deduce Apollonius' theorem.

e What happens when the angle at A is a right angle?

4 A rectangle $ABCD$ has $AB = 15$ cm, $AD = 10$ cm. A point P is located inside the rectangle such that $AP = 14$ cm and $PB = 11$ cm. Find PD in exact form.

5 E is any point inside the rectangle $ABCD$. Let $AE = a$, $EB = b$, $EC = c$ and $ED = x$. Prove that $x^2 = a^2 - b^2 + c^2$.

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6 We use the fact that every integer can be uniquely factorised as a product of primes to give another proof that $\sqrt{2}$ is irrational. We begin by supposing the opposite – that is, we suppose that we can write $\sqrt{2} = \frac{p}{q}$, where p and q are whole numbers with no common factors except 1.

a Explain why q^2 is a factor of p^2.

b Explain why q is a factor of p.

c Explain how this shows that $\sqrt{2}$ is irrational.

7 Prove that $\sqrt{6}$ is irrational.

8 Expand and simplify:

a $(\sqrt{2} + 1)^4$

b $(\sqrt{3} - 1)^4$

9 Prove that there are infinitely many irrational numbers between 0 and 1.

10 Prove that between any two numbers, there are infinitely many rational numbers and infinitely many irrational numbers.

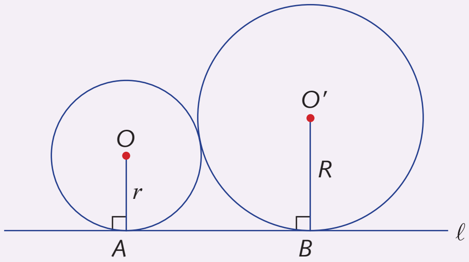
11 Rationalise the denominator of:

a $\dfrac{1}{(1 + \sqrt{2}) + \sqrt{5}}$

b $\dfrac{1}{\sqrt{7} + \sqrt{3} - 2}$

12 Show that $\sqrt[7]{7} < \sqrt[5]{5} < \sqrt{2} < \sqrt[3]{3}$ without using a calculator.

13 Two circles with centres O and O' and radii r and R meet at a single point. A and B are the points where the circles touch line ℓ. Find the distance between points A and B.



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