CHAPTER

7

Measurement and Geometry

Congruence and special quadrilaterals

This chapter reviews work on angles, congruence tests and their applications. Congruence is an extremely useful tool in geometrical arguments. It is used here to prove properties of various types of quadrilaterals, and to develop tests for special quadrilaterals.

This chapter is a revision of all the geometry presented in Years 7 and 8.

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7A Review of angles

Most geometrical arguments rely on the relationship between different angles in the one figure. This section reviews the methods introduced so far, and the next section uses these methods to prove some interesting general results.

Describing angles by their sizes

Angles of three particular sizes have special names:

- An angle of 90° is a **right angle**.
- An angle of 180° is a **straight angle**.
- An angle of 360° is a **revolution**.

Other angles are described by being within a range of angle sizes:

- An **acute angle** is an angle between 0° and 90°.
- An **obtuse angle** is an angle between 90° and 180°.
- A **reflex angle** is an angle between 180° and 360°.

For example, 30° is an acute angle, 140° is obtuse, and 220° is reflex.

Particular pairs of angles also have special names:

- Two angles whose sum is 90° are called **complementary angles**.
- Two angles whose sum is 180° are called **supplementary angles**.

For example, 35° and 55° are complementary angles because 35° + 55° = 90°, and 70° and 110° are supplementary angles because 70° + 110° = 180°.

Angles at a point

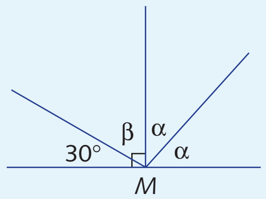
The first group of results concern angles at a point.

- Adjacent angles can be added and subtracted.
- Adjacent angles on a straight line are supplementary.
- Angles in a revolution add to 360°.
- Vertically opposite angles are equal.

Example 1

Find the value of the pronumeral in each figure.

a



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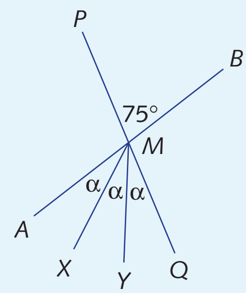

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Solution

a $\beta = 60°$ (complementary angles)

$2\alpha = 90°$ (supplementary angles)

$\alpha = 45°$

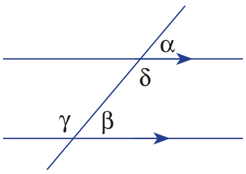
b $3\alpha = 75°$ (vertically opposite angles at M)

$\alpha = 25°$

Angles across transversals

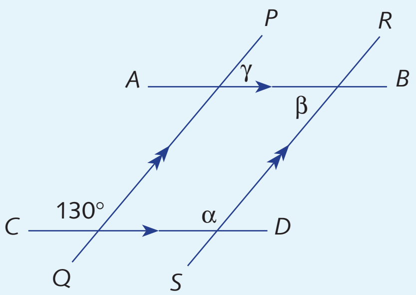
Suppose that a transversal crosses two lines.

- If the lines are parallel, then the alternate angles (for example, δ and γ) are equal.
- If the lines are parallel, then the corresponding angles (for example, α and β) are equal.
- If the lines are parallel, then the co-interior angles (for example, δ and β) are supplementary.



Example 2

Find α, β and γ in the figure to the right.



Solution

$\alpha = 130°$ (corresponding angles, $PQ \parallel RS$)

$\beta = 50°$ (co-interior angles, $AB \parallel CD$)

$\gamma = 50°$ (alternate angles, $PQ \parallel RS$)

Proving that two lines are parallel

The **converses** of the previous three results are tests to prove that two lines are parallel. Suppose that a transversal crosses two lines.

- If two alternate angles are equal, then the lines are parallel.
- If two corresponding angles are equal, then the lines are parallel.
- If two co-interior angles are supplementary, then the lines are parallel.

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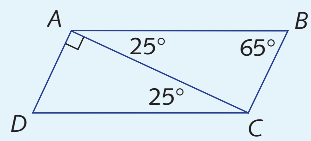
Example 3

Identify every pair of parallel lines in the figure to the right.



Solution

First, $AB \parallel DC$ (alternate angles $\angle BAC$ and $\angle ACD$ are equal).

Secondly, $\angle BAD = 115°$ (adjacent angles at A).

So $AD \parallel BC$ (co-interior angles $\angle BAD$ and $\angle ABC$ are supplementary).

Angle sums of triangles and quadrilaterals

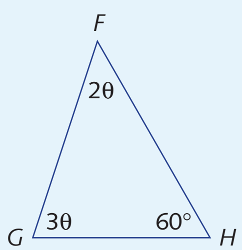
- The sum of the interior angles of a triangle is 180°.
- An exterior angle of a triangle equals the sum of the opposite interior angles.
- The sum of the interior angles of a quadrilateral is 360°.

Proofs of these results are reviewed in Exercise 7B.

Example 4

Find θ in each diagram below.

a



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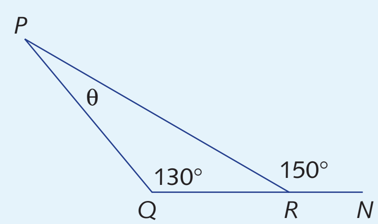

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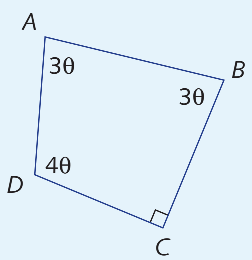

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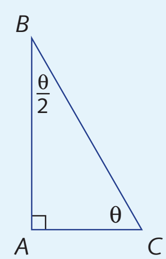

Solution

a $3\theta + 2\theta + 60° = 180°$ (angle sum of ΔFGH)

$5\theta = 120°$

$\theta = 24°$

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b $\theta + 130° = 150°$ (exterior angle of ΔPQR)

$\theta = 20°$

c $3\theta + 3\theta + 4\theta + 90° = 360°$ (angle sum of quadrilateral $ABCD$)

$10\theta = 270°$

$\theta = 27°$

d $\theta + \frac{\theta}{2} + 90° = 180°$ (angle sum of ΔABC)

$\frac{3\theta}{2} = 90°$

$\theta = 60°$

Isosceles and equilateral triangles

A triangle is called **isosceles** if it has two equal sides; it is called **equilateral** if all three sides are equal. Thus all equilateral triangles are also isosceles triangles.

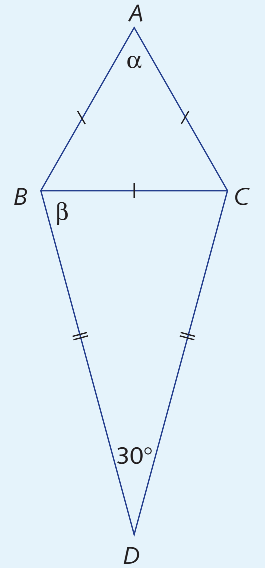
- The base angles of an isosceles triangle are equal.
- Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.
- Each interior angle of an equilateral triangle is 60°.
- Conversely, if the three angles of a triangle are equal, then the triangle is equilateral.

Proofs of these results are reviewed in Exercise 7C.

Example 5

Find α, β, γ and x in the diagrams below.

a



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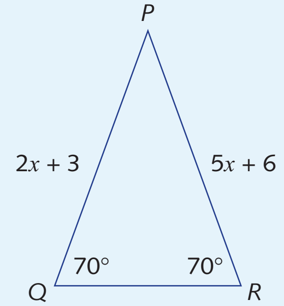

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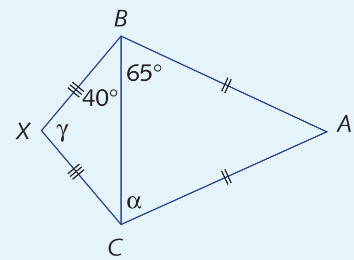

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Solution

a First, $\alpha = 60°$ (ΔABC is equilateral)

Secondly, $\angle BCD = \beta$ (base angles of isosceles ΔBCD)

so $2\beta + 30° = 180°$ (angle sum of ΔBCD)

$\beta = 75°$

b $2x + 3 = 5x + 6$ (opposite angles of ΔPQR are equal)

$-3 = 3x$

$x = -1$

Thus PQ and PR each have length of 1.

c $\alpha = 65°$ (base angles of isosceles ΔABC)

$\angle BCX = 40°$ (base angles of isosceles ΔBXC)

$\gamma = 100°$ (angle sum of ΔXBC)

Exercise 7A

1 **a** Classify these angles using the standard terms.

i 140° **ii** 360° **iii** 33° **iv** 180° **v** 90° **vi** 350°

b Write down the complements of:

i 20° **ii** 72° **iii** 45°

c Write down the supplements of:

i 20° **ii** 172° **iii** 90°

Example 1

2 This question reviews angles at a point. Find the angles marked with pronumerals, giving reasons.

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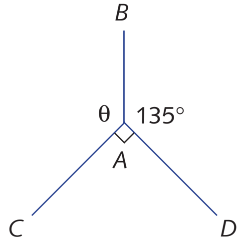

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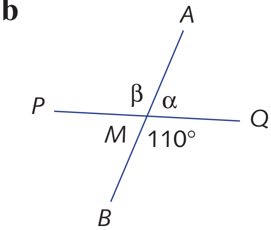

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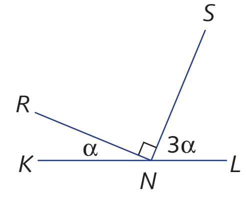

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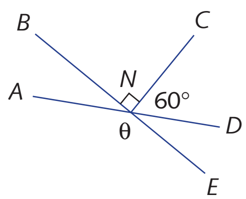

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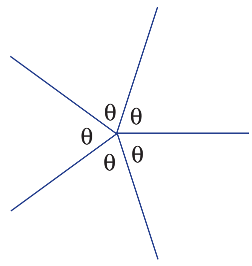

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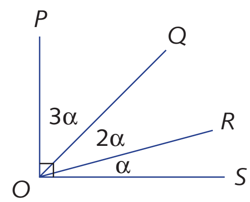

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Example 2

3 This question reviews angles across transversals. Find the angles marked with pronumerals, giving reasons.

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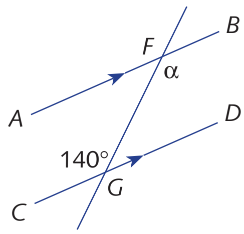

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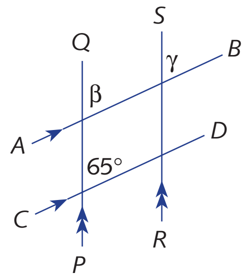

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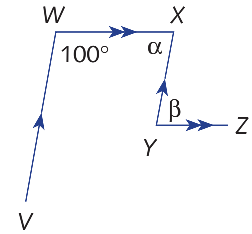

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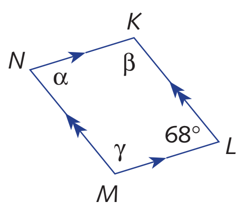

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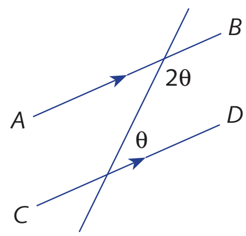

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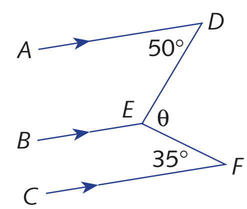

Example 3

4 This question reviews tests for two lines to be parallel. In each part, name all pairs of parallel lines, giving reasons.

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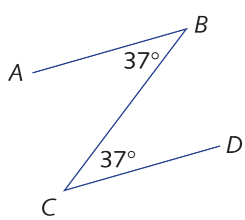

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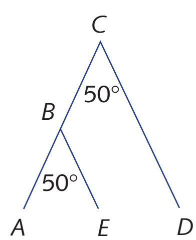

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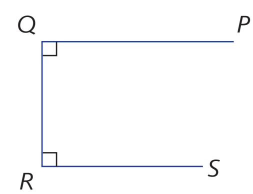

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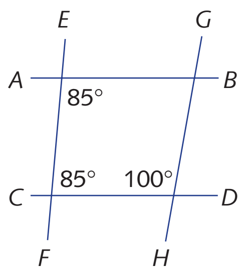

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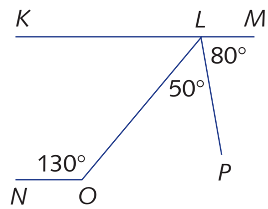

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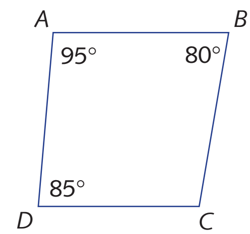

Example 4

5 This question reviews angle sums of triangles and quadrilaterals. Find the angles marked with pronumerals, giving reasons.

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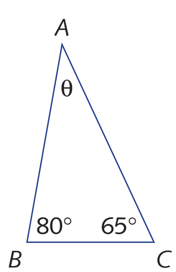

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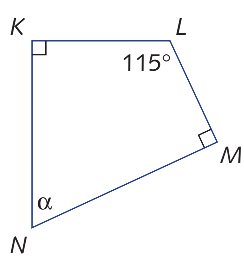

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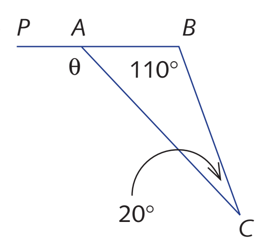

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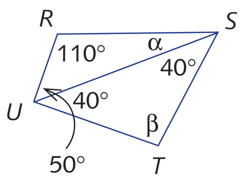
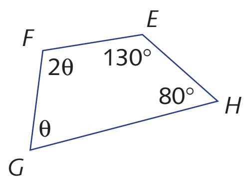
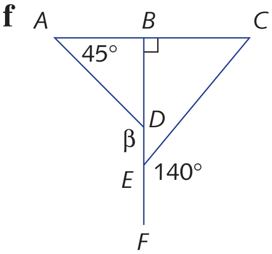
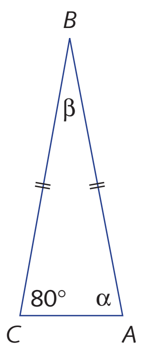
d 

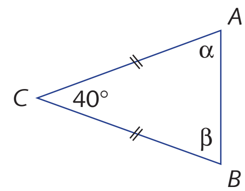
e 

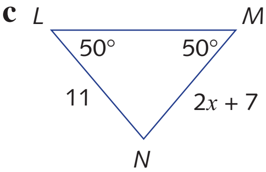
f 

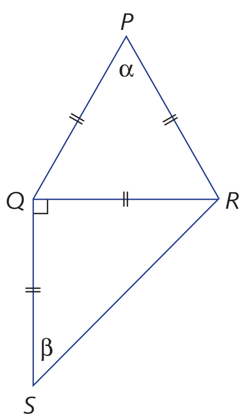
6 This question reviews isosceles and equilateral triangles. Find the angles and sides marked with pronumerals, giving reasons.

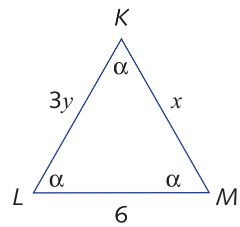
Example 5

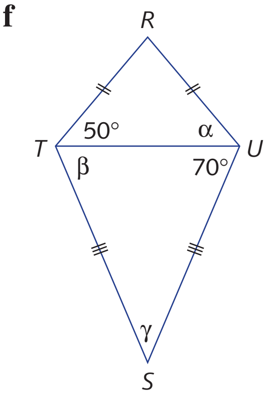
a 

b 

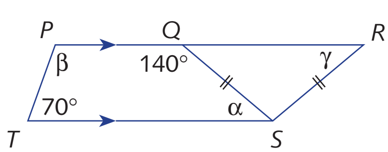
c 

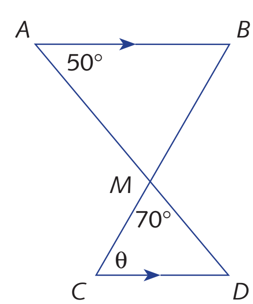
d 

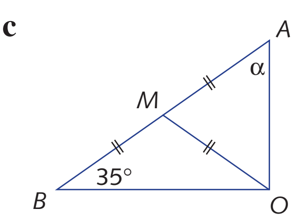
e 

f 

7 Find the values of the pronumerals α, β, γ, θ and x. Different methods are combined in each part of this question.

a 

b 

c 

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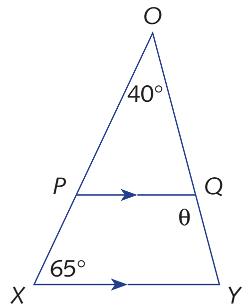

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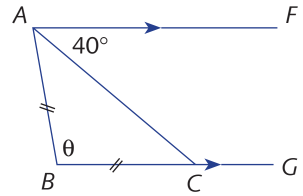

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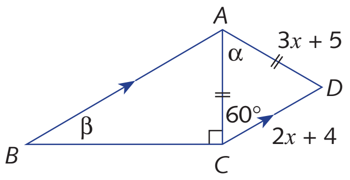

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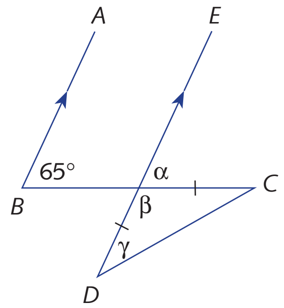

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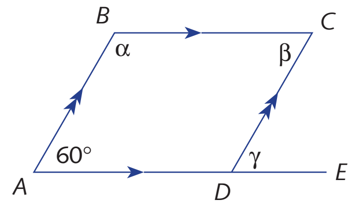

7B Reasoning with angles

This section uses the methods of Section 7A in more general situations. Construction lines are often needed.

Example 6

Find the angle $\angle AOB$ in terms of θ, giving reasons.

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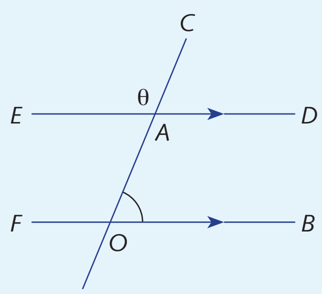

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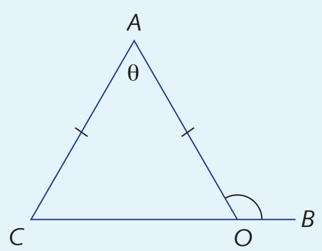

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Solution

a $\angle FOA = \theta$ (corresponding angles, $ED \parallel FB$)

$\angle AOB = 180° - \theta$ (supplementary angle)

b $\angle AOC = \dfrac{180° - \theta}{2}$ (ΔACO is isosceles)

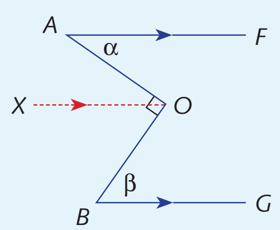
$= 90° - \dfrac{\theta}{2}$

$\angle AOB = 180° - \left(90° - \dfrac{\theta}{2}\right)$ (supplementary angles)

$= 90° + \dfrac{\theta}{2}$

Example 7

Show that $\alpha + \beta = 90°$.



Solution

Draw XO parallel to AF.

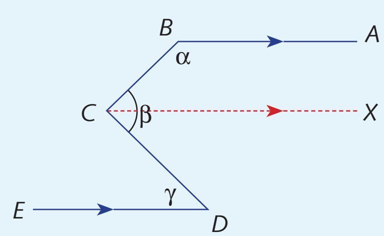
Then $\angle AOX = \alpha$ (alternate angles, $OX \parallel FA$)

and $\angle BOX = \beta$ (alternate angles, $OX \parallel GB$)

so $\alpha + \beta = 90°$ (adjacent angles)

Example 8

Prove that $\alpha + \beta - \gamma = 180°$.



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Solution

Draw CX parallel to BA.

Then $\angle BCX = 180° - \alpha$ (co-interior angles, $BA \parallel CX$)

and $\angle DCX = \gamma$ (alternate angles, $CX \parallel ED$)

Hence $\beta = 180° - \alpha + \gamma$ ($\angle BCD = \angle BCX + \angle DCX$)

so $\alpha + \beta - \gamma = 180°$

Proving theorems

The results of the previous section can be applied to prove many interesting general results, called **theorems**. For the rest of this chapter, each theorem to be proven is stated in italics. Usually the question includes a diagram, introduces the necessary pronumerals and breaks the proof down into a number of steps. Sometimes a construction is required.

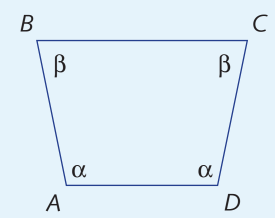
Example 9

Prove that: *A quadrilateral with two pairs of equal adjacent angles is a trapezium.*

In the quadrilateral $ABCD$ to the right, $\angle A = \angle D = \alpha$ and $\angle B = \angle C = \beta$.

a Prove that $\alpha + \beta = 180°$.

b Hence prove that $ABCD$ is a trapezium.



Solution

a $\alpha + \alpha + \beta + \beta = 360°$ (angle sum of quadrilateral $ABCD$)

so $\alpha + \beta = 180°$

b Hence $AD \parallel BC$ (co-interior angles are supplementary)

Therefore $ABCD$ is a trapezium.

Angle sums of polygons

The last four questions of the following exercise deal with interior and exterior angle sums of polygons. These are interesting general results that you may want to remember as part of your known geometric facts.

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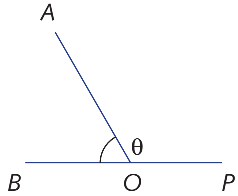
Exercise 7B

Example 6

1 Find the marked angle $\angle AOB$ in terms of θ, giving reasons.

a



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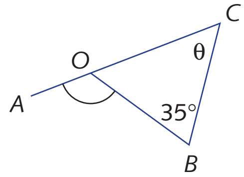

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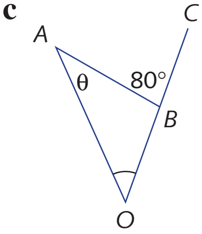

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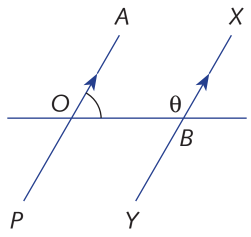

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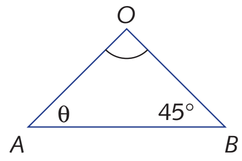

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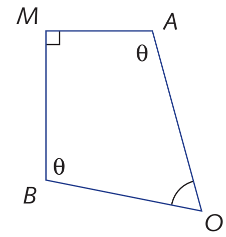

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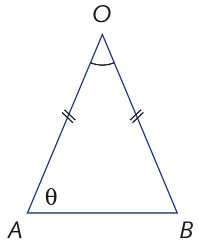

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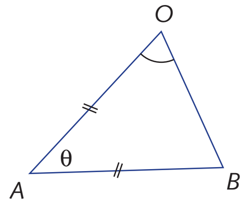
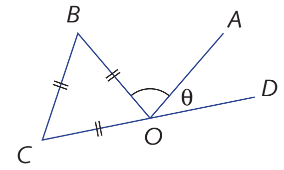
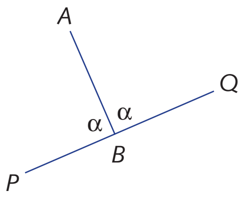


i

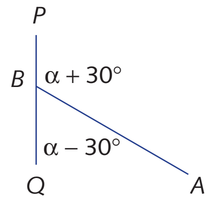


Example 7

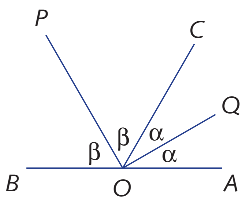
2 **a** Prove that $AB \perp PQ$.



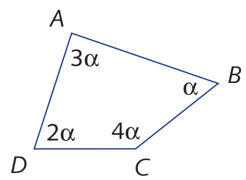
b Prove that $\angle ABP = 2 \times \angle ABQ$.



c Prove that $PO \perp QO$.



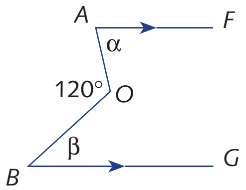
d Prove that $AB \parallel DC$.



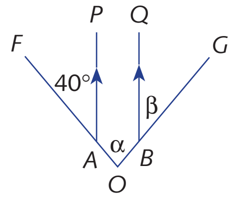
Example 8

3 These questions may require construction lines to be drawn.

a Prove that $\alpha + \beta = 120°$.



b Prove that $\alpha - \beta = 40°$.



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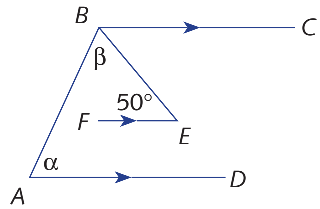
c Prove that $\alpha + \beta = 130°$.



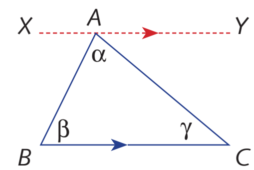
Example 9

4 This question reviews proofs of theorems about angle sums of triangles.

a Prove that: *The sum of the interior angles of a triangle is* 180°.

In ΔABC to the right, $\angle BAC = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$.

The line XAY has been constructed parallel to BC.



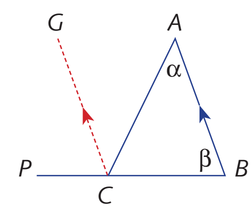
i Explain why $\angle XAB = \beta$ and $\angle YAC = \gamma$.

ii Hence explain why $\alpha + \beta + \gamma = 180°$.

b Prove that: *An exterior angle of a triangle equals the sum of the opposite interior angles.*

In ΔABC to the right, $\angle A = \alpha$ and $\angle B = \beta$.

The line CG has been constructed parallel to BA, and the side BC has been produced to P.



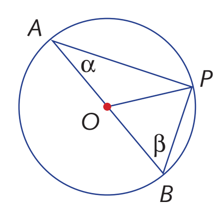
i Explain why $\angle ACG = \alpha$ and $\angle PCG = \beta$.

ii Hence explain why $\angle ACP = \alpha + \beta$.

5 Prove that: *An angle in a semicircle is a right angle.*

In the circle to the right, AOB is a diameter, and P is any other point on the circle.

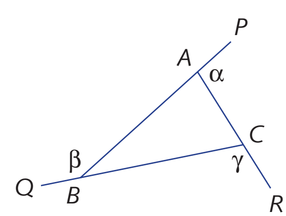
Let $\angle A = \alpha$ and $\angle B = \beta$.



a Prove that $\angle APB = \alpha + \beta$. **b** Hence prove that $\angle APB = 90°$.

6 **a** Prove that: *The sum of the exterior angles of a triangle is* 360°.

In the diagram to the right, α, β and γ are the sizes of the three exterior angles of ΔABC.



i Find the sizes of the three interior angles of ΔABC.

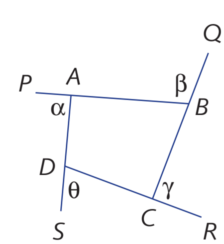
ii Hence show that $\alpha + \beta + \gamma = 360°$.

b Draw a quadrilateral $ABCD$. Use a construction line to help explain why the sum of the interior angles of $ABCD$ is 360°.

c Prove that: *The sum of the four exterior angles of a convex quadrilateral is* 360°.

In the diagram to the right, α, β, γ and θ are the sizes of the four exterior angles of the convex quadrilateral $ABCD$. (All the interior angles of a convex quadrilateral are less than 180°.)

Show that $\alpha + \beta + \gamma + \theta = 360°$.



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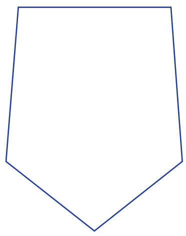
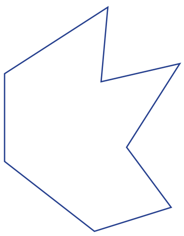
Angle sums of polygons

7 **a** The diagrams below show a pentagon, an octagon and a dodecagon. By dividing each figure into triangles, find the sum of its interior angles.

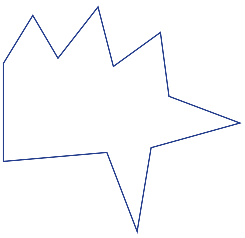
i



ii



iii

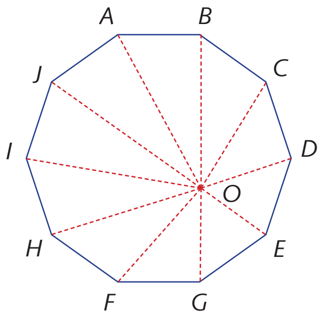


b Using a similar technique, find the sum of the interior angles of:

i a hexagon **ii** a heptagon **iii** a nonagon **iv** a decagon

c Using these examples as a guide, find a formula for the angle sum of an n-sided polygon.

8 In a convex polygon, all the interior angles are less than 180°.



a The diagram to the right shows a convex decagon, and any point O inside the decagon. Intervals are drawn from O to each of the 10 vertices, dissecting the decagon into 10 triangles.

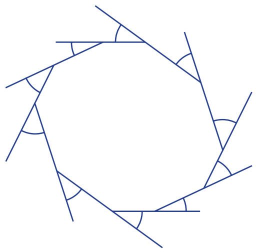
i What is the sum of the interior angles of all 10 triangles so formed?

ii Hence find the sum of the interior angles of the decagon.

b Draw a convex hexagon and repeat the process.

c Using the same technique, find a formula for the sum of the interior angles of a convex polygon with n sides. Confirm that this matches your answer to Question **7c**.

9 **a** In the convex decagon to the right, each side has been produced to form an exterior angle.



i Explain why the sum of the interior angles plus the sum of the exterior angles is 1800°.

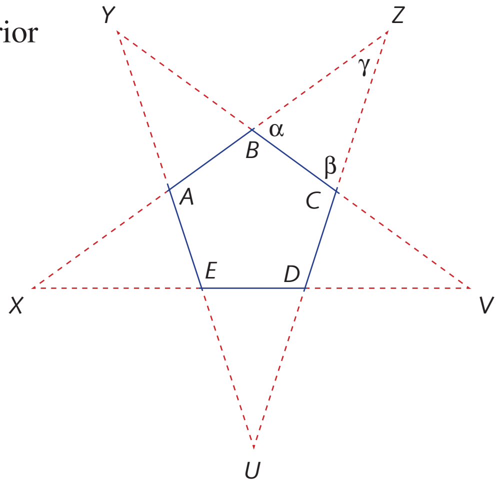
ii Hence find the sum of the 10 exterior angles.

b Draw a convex hexagon and repeat the process.

c Using the same technique, find the sum of the exterior angles of a convex polygon with n sides.

10 Find a formula for the size of each interior angle in a *regular* n-sized polygon.

11 The figure shown is a regular pentagram. It is formed by producing the sides of a regular pentagon.



a Find the size of each interior angle of the pentagon $ABCDE$.

b Find the values of α, β and γ.

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7C Using congruence

In *ICE-EM Mathematics Year 8* we introduced the idea of **congruent figures**.

Congruent figures

- Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.
- Congruent figures have exactly the same shape and size.
- When two figures are congruent, we can match every part of one figure with the corresponding part of the other, so that:
  - matching angles have the same size
  - matching paired intervals have the same length
  - matching paired regions have the same area.

The congruence arguments used in this chapter involve only congruent triangles. In Year 8 we developed four tests for two triangles to be congruent.

The four standard congruence tests for triangles

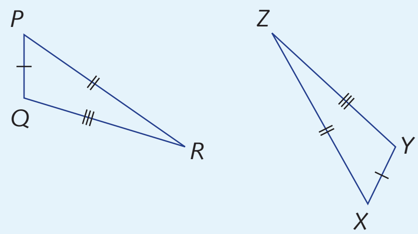
Two triangles are **congruent** if:

- **SSS:** the three sides of one triangle are respectively equal to the three sides of the other triangle, **or**
- **AAS:** two angles and one side of one triangle are respectively equal to two angles and the matching side of the other triangle, **or**
- **SAS:** two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle, **or**
- **RHS:** the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of the other right-angled triangle.

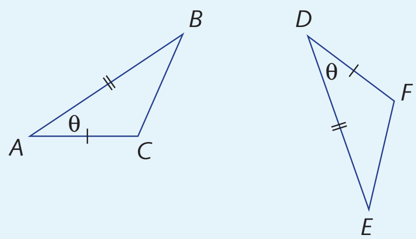
Example 10

In each part, write a congruence statement, giving a test as the reason. Make sure that you write the vertices of the two triangles in matching order.

a



b



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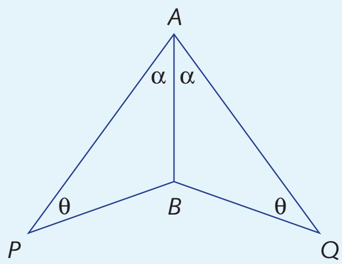
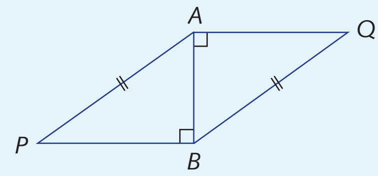
c



d



Solution

a $\Delta PQR \equiv \Delta XYZ$ (SSS)

b $\Delta BAC \equiv \Delta EDF$ (SAS)

c $\Delta ABP \equiv \Delta ABQ$ (AAS)

d $\Delta ABQ \equiv \Delta BAP$ (RHS)

Notice that in parts **c** and **d**, the side AB is **common** to both triangles, and thus provides one of the pairs of equal sides.

Using congruence to prove that two matching sides or angles are equal

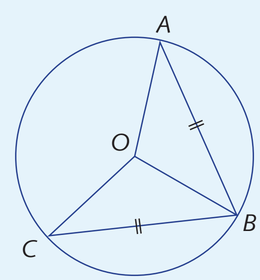
Once we have established that two triangles are congruent, we know that the remaining matching sides and angles are equal, as in the following example.

Example 11

In the diagram to the right, AB and BC are equal chords of a circle with centre O.

a Prove that $\Delta AOB \equiv \Delta COB$.

b Hence prove that $\angle AOB = \angle COB$.



Solution

a In the triangles AOB and COB:

$OA = OC$ (radii)

$OB = OB$ (common)

$AB = CB$ (given)

so $\Delta AOB \equiv \Delta COB$ (SSS)

b Hence $\angle AOB = \angle COB$ (matching angles of congruent triangles)

Using congruence to prove that two lines are parallel

In many situations, congruence is used to prove that two alternate angles, or two corresponding angles, are equal. This allows us to prove that two lines are parallel.

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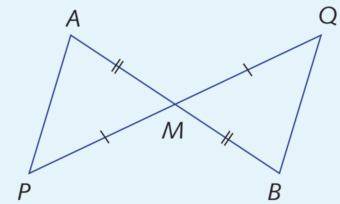
Example 12

In the diagram to the right, $AM = BM$ and $PM = QM$

a Prove that $\Delta AMP \equiv \Delta BMQ$.

b Prove that $AP \parallel BQ$.



Solution

a In the triangles AMP and BMQ:

$AM = BM$	(given)
$PM = QM$	(given)
$\angle AMP = \angle BMQ$	(vertically opposite angles at M)
so $\Delta AMP \equiv \Delta BMQ$	(SAS)

b Hence $\angle APM = \angle BQM$ (matching angles of congruent triangles)

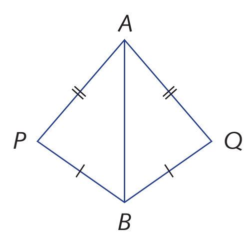
so $AP \parallel BQ$ (alternate angles are equal)

Exercise 7C

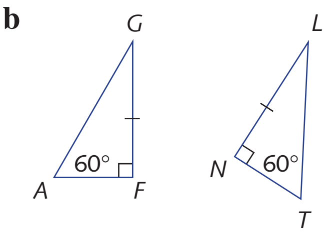
Example 10

1 In each part, write a congruence statement, giving a congruence test as the reason. Make sure that you name the vertices in matching order.

a



b



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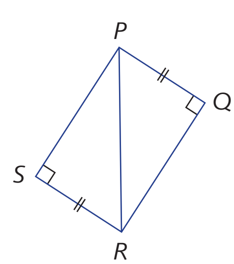

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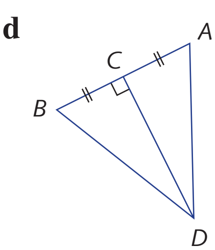
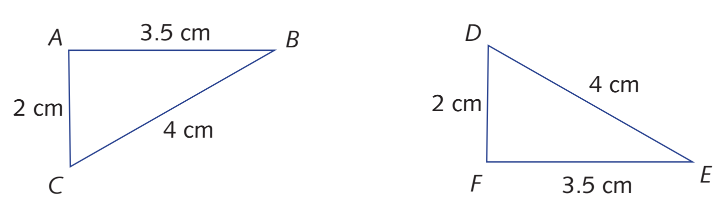


2 In each part, write a congruence statement, giving a congruence test as the reason.

a



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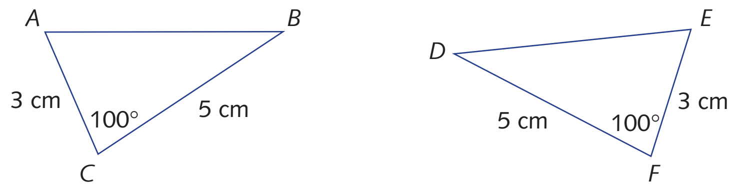
b



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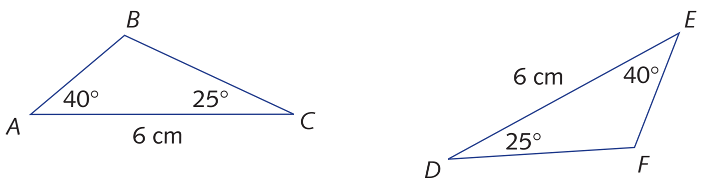
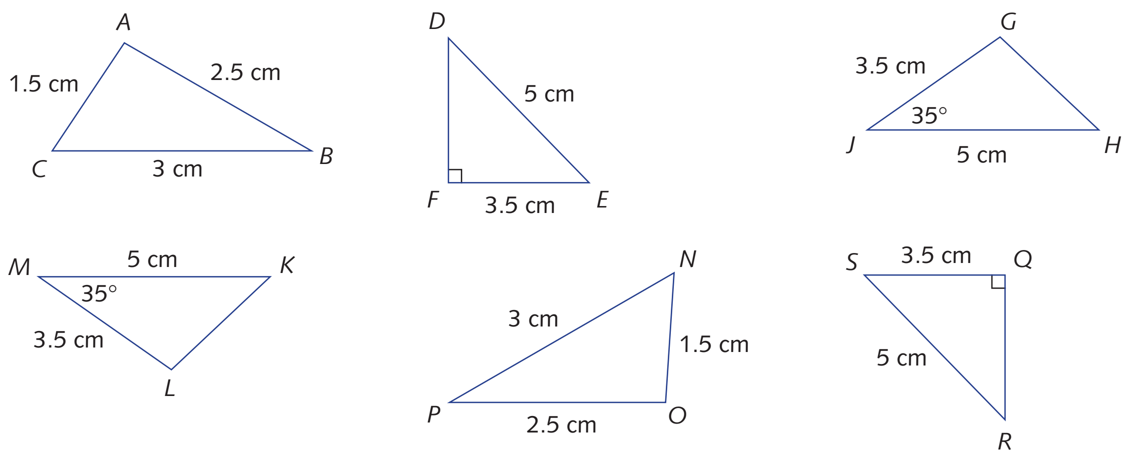
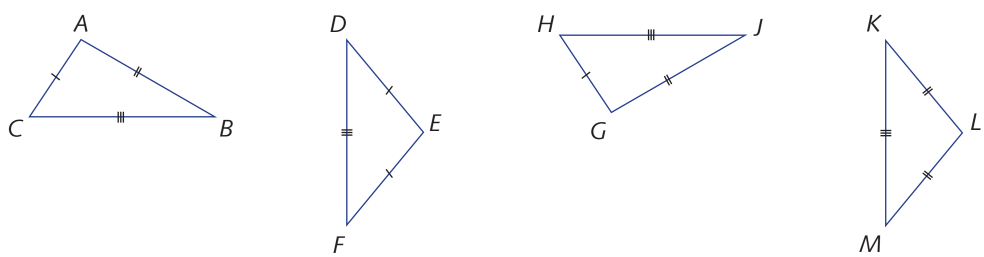


3 From the six triangles drawn, name three pairs of congruent triangles. Give reasons.

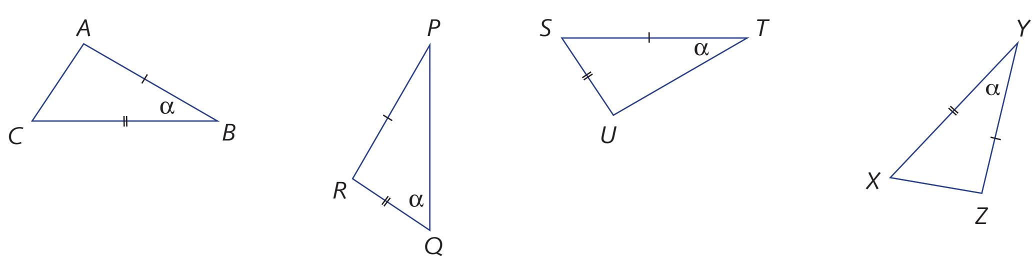


4 In each of the following, state which triangle is congruent to ΔABC, giving the appropriate congruence test as a reason.

a



b



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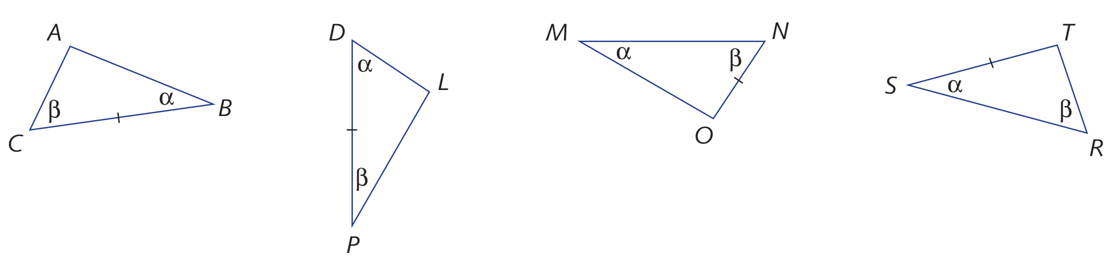

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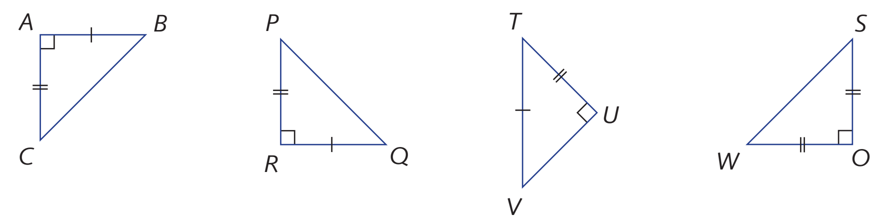
d 

5 $\Delta ABC \equiv \Delta PQR$ (where the vertices are in matching order).

a Which angle in ΔPQR is equal in size to $\angle ABC$?

b Which angle in ΔPQR is equal in size to $\angle CAB$?

c Which side in ΔPQR is equal in length to AC?

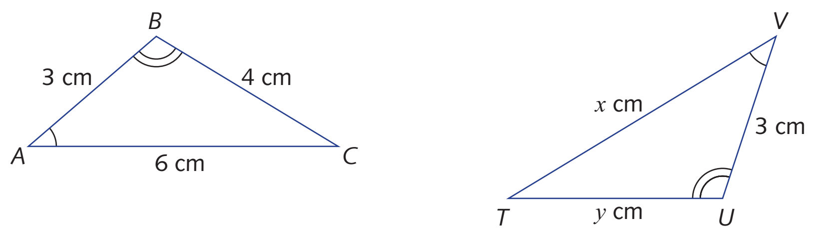
d Which side in ΔABC is equal in length to QR?

6 Find the values of the pronumerals.

a 

b 

c 

d 

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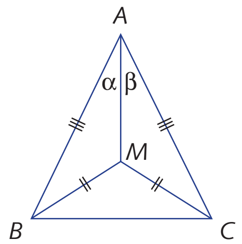
Example 11

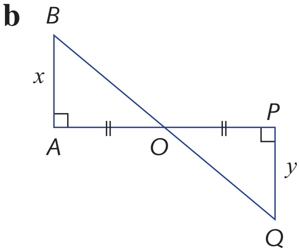
7 **a**



i Prove that $\Delta ABM \equiv \Delta ACM$.

ii Hence prove that $\alpha = \beta$.

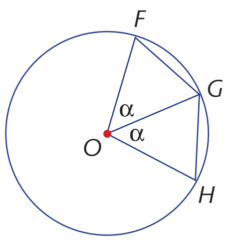
b



i Prove that $\Delta OAB \equiv \Delta OPQ$.

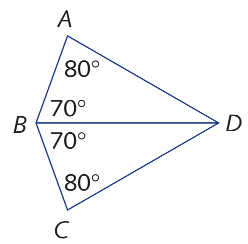
ii Hence prove that $x = y$.

c



Use congruence to prove that $FG = GH$, given that O is the centre of the circle.

d

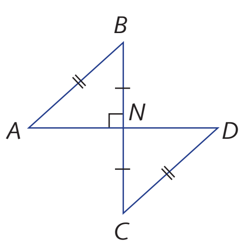


Use congruence to prove that $AD = DC$ and that $AB = BC$.

8 AB is a chord of a circle centre O and M is the midpoint of AB. Prove that OM is perpendicular to AB.

Example 12

9 **a**

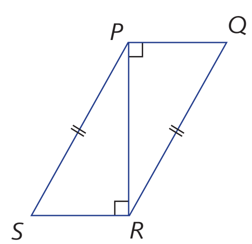


i Prove that $\Delta NAB \equiv \Delta NDC$.

ii Hence prove that $\angle A = \angle D$.

iii Hence prove that $AB \parallel CD$.

b

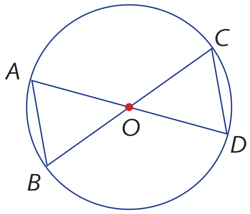


i Give a reason why $PQ \parallel SR$.

ii Prove that $\Delta PRS \equiv \Delta RPQ$.

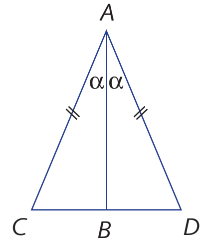
iii Hence prove that $PS \parallel QR$.

c



Use congruence to prove that $AB \parallel CD$, given that O is the centre of the circle.

d



Prove that $\angle ABC = \angle ABD$ using congruence, and hence prove that $AB \perp CD$.

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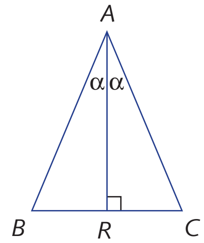
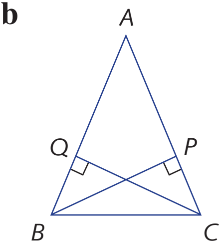
10 **a**



Use congruence to prove that ΔABC is isosceles.

b

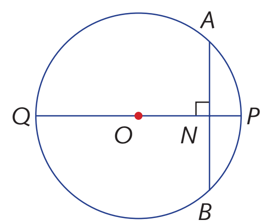


Use congruence to prove that ΔABC is isosceles, given that $BP = CQ$.

11 In the triangle ΔABC, $AB = AC$. The bisectors of angles ABC and ACB meet at X. Prove that $XB = XC$.

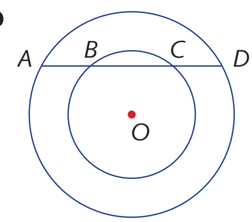
12 In each part, O is the centre of the circle or circles. You will need to construct radii and use congruence in these questions.

a



Prove that PQ bisects AB.

b



Prove that $AB = CD$.

13 This question reviews proofs of several theorems about isosceles and equilateral triangles.

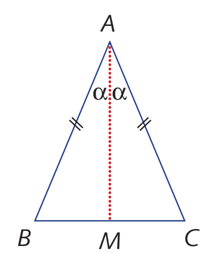
a Prove that: *The base angles of an isosceles triangle are equal.*

The triangle ABC to the right is isosceles, with $AB = AC$.

The bisector of $\angle BAC$ meets the base BC at M.

i Prove that $\Delta ABM \equiv \Delta ACM$.

ii Hence prove that $\angle B = \angle C$.



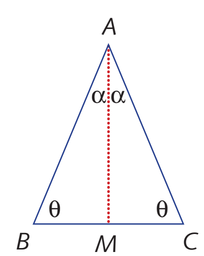
b Conversely, prove that: *If two angles of a triangle are equal, then the sides opposite those angles are equal.*

In the triangle ABC to the right, $\angle B = \angle C$.

The bisector of $\angle BAC$ meets the base BC at M.

i Prove that $\Delta ABM \equiv \Delta ACM$.

ii Hence prove that ΔABC is isosceles.



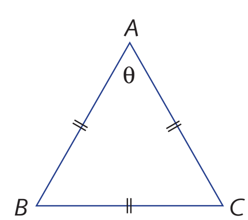
c Prove that: *Each interior angle of an equilateral triangle is* $60°$.

The triangle ABC to the right is equilateral. Let $\angle A = \theta$.

i Give a reason why $\angle B = \theta$.

ii Give a reason why $\angle C = \theta$.

iii Explain why $\theta = 60°$.



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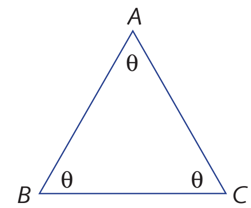
d Conversely, prove that: *If all angles of a triangle are equal, then the triangle is equilateral.*

In the triangle ABC to the right, all the interior angles are equal.

Let $\angle A = \angle B = \angle C = \theta$.

i Give a reason why $AB = AC$.

ii Give a reason why $AC = BC$.



14 This question reviews proofs by congruence of three well-known properties of parallelograms.

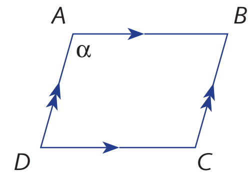
Recall that a **parallelogram** is a quadrilateral whose opposite sides are parallel.

a Prove that: *Opposite angles of a parallelogram are equal.*

A parallelogram $ABCD$ is drawn to the right. Let $\angle A = \alpha$.

i Find the sizes of $\angle B$ and $\angle D$ in terms of α.

ii Hence find the size of $\angle C$ in terms of α.

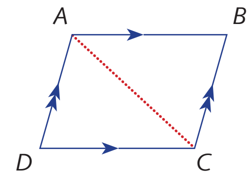


b Prove that: *Opposite sides of a parallelogram are equal.*

Let $ABCD$ be a parallelogram with the diagonal AC joined.

i Prove that $\Delta ABC \equiv \Delta CDA$.

ii Hence prove that $AB = DC$ and $AD = BC$.

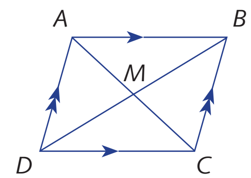


c Prove that: *The diagonals of a parallelogram bisect each other.*

Let the diagonals of the parallelogram $ABCD$ meet at M.

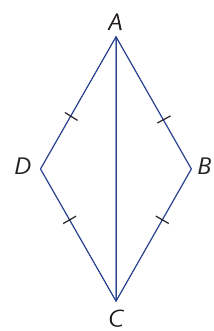
i Prove that $\Delta ABM \equiv \Delta CDM$.

ii Hence prove that $AM = CM$ and $BM = DM$.



15 This question reviews proofs by congruence of some diagonal properties of rhombuses and rectangles.
Recall that a **rhombus** is a quadrilateral with all sides equal.

a Prove that: *A rhombus is a parallelogram.*



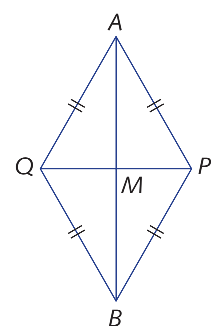
b Prove that: *The diagonals of a rhombus are perpendicular, and bisect the vertex angles through which they pass.*

Let the diagonals of the rhombus $APBQ$ meet at M.

i Prove that $\Delta APB \equiv \Delta AQB$.

ii Hence prove that $\angle PAM = \angle QAM$.

iii Prove also that $AM \perp PQ$.



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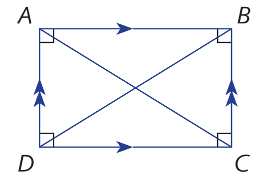
c Prove that: *The diagonals of a rectangle are equal.*

Let $ABCD$ be a rectangle with the diagonals joined.

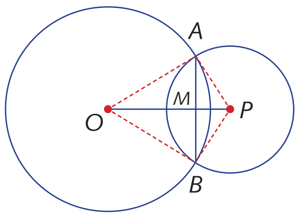
i Prove that $\Delta ADC \equiv \Delta BCD$.

ii Hence prove that $AC = BD$.



16 Prove that: *When two circles intersect, the line joining their centres is the perpendicular bisector of their common chord.*

In the diagram to the right, AB is the common chord of two intersecting circles with centres O and P. The chord AB meets the line OP at M.



a Prove that $\Delta AOP \equiv \Delta BOP$.

b Hence prove that OP bisects $\angle AOB$.

c Prove that $AOM \equiv BOM$.

d Hence prove that $AM = BM$ and $AB \perp OP$.

7D Parallelograms

In Year 8 we defined a parallelogram and discovered some of its important properties.

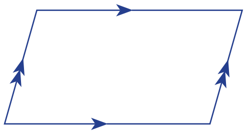
Parallelograms

Definition of a parallelogram

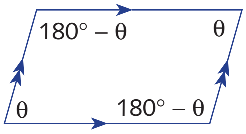
- A **parallelogram** is a quadrilateral whose opposite sides are parallel.

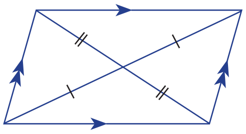
Properties of a parallelogram

- The opposite angles of a parallelogram are equal.
- The opposite sides of a parallelogram are equal.
- The diagonals of a parallelogram bisect each other.



The diagrams below illustrate these properties. Their proofs were reviewed in Exercise 7C, Question **13**.





The purpose of this section is to establish, and subsequently apply, four well-known tests for a quadrilateral to be a parallelogram.

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Four tests for a parallelogram

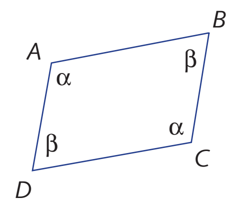
- If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

The first question in the following exercise reviews proofs of these tests.

Exercise 7D

1 This question reviews the proofs of the four tests for a quadrilateral to be a parallelogram.

a Prove that: *If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.*

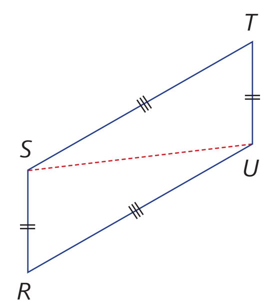


Let $ABCD$ be a quadrilateral with $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$.

i Prove that $\alpha + \beta = 180°$.

ii Hence prove that $AB \parallel DC$ and $AD \parallel BC$.

b Prove that: *If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.*

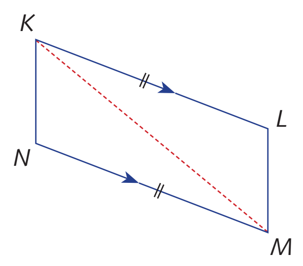


Let $RSTU$ be a quadrilateral with $RS = UT$ and $RU = ST$.

i Prove that $\Delta RSU \equiv \Delta TUS$.

ii Hence prove that $RS \parallel UT$ and $RU \parallel ST$.

c Prove that: *If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.*

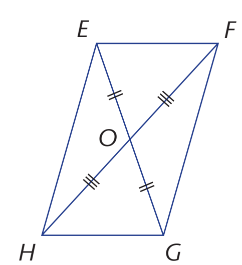


Let $KLMN$ be a quadrilateral with $KL = NM$ and $KL \parallel NM$.

i Prove that $\Delta KLM \equiv \Delta MNK$.

ii Hence prove that $KN \parallel LM$.

d Prove that: *If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.*



Let $EFGH$ be a quadrilateral whose diagonals meet at O and bisect each other.

i Prove that $\Delta EHO \equiv \Delta GFO$.

ii Hence prove that $EH = FG$ and $EH \parallel FG$.

iii Explain why this proves that $EFGH$ is a parallelogram.

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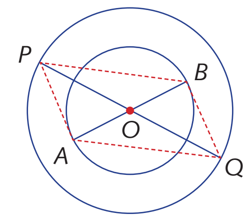
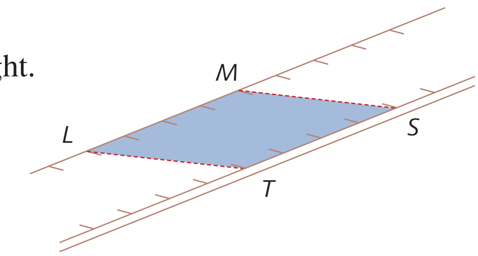
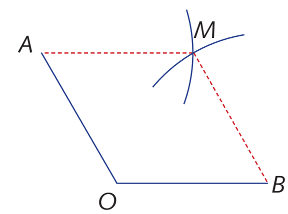
2 **a** In the diagram to the right, both circles have centre O. The interval AOB is a diameter of the smaller circle, and POQ is a diameter of the larger circle. What sort of figure is $APBQ$, and why?



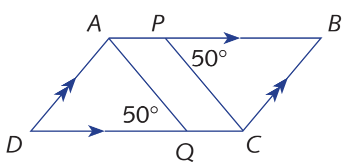
b The opposite edges of my ruler are parallel. I draw intervals LM and ST of length 4 cm on the opposite edges of the ruler, as shown in the diagram to the right. What sort of figure is $LMST$, and why?



c In the diagram to the right, $\angle AOB$ is any given angle. First I draw an arc with centre A and radius OB. Then I draw an arc with centre B and radius OA. The two arcs meet at M, and I claim that $AMBO$ is a parallelogram. Am I correct? Explain why or why not.

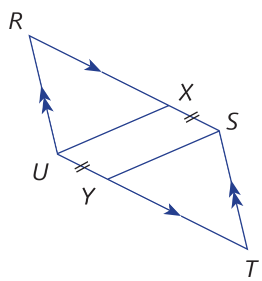


3 **a**



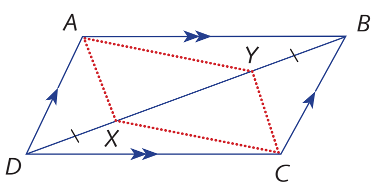
Given that $ABCD$ is a parallelogram, prove that $AP = QC$.

b



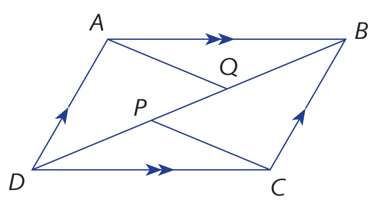
Given that $RSTU$ is a parallelogram, prove that $UX \parallel YS$.

c



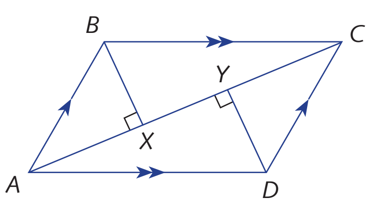
$ABCD$ is a parallelogram with X and Y points on the diagonal BD such that $BY = DX$. Prove that $AYCX$ is a parallelogram.

d



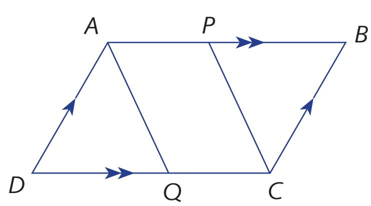
$ABCD$ is a parallelogram and $PB = DQ$. Prove that $AQ = CP$.

e



$ABCD$ is a parallelogram and BX and DY are perpendicular to AC. Prove that $BX = DY$.

f



$ABCD$ is a parallelogram with $BP = DQ$. Prove that $APCQ$ is a parallelogram.

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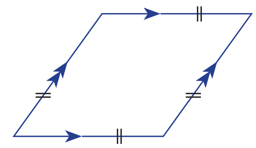
7E Tests for rhombuses, rectangles and squares

Rhombuses, rectangles and squares are special types of parallelograms. Last year we defined these figures and proved some of their properties. This section establishes some important tests for these figures.

Rhombuses

Definition of a rhombus

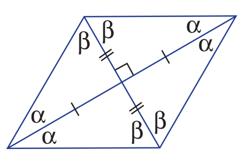
- A **rhombus** is a quadrilateral with four equal sides.

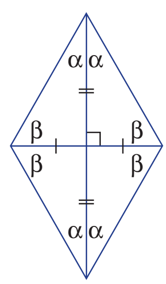


Diagonal properties of a rhombus

- The diagonals of a rhombus bisect each other at right angles.
- The diagonals of a rhombus bisect the vertex angles through which they pass.

These diagonal properties of a rhombus are far clearer when it is drawn in a diamond orientation, with its diagonals vertical and horizontal as illustrated below.





Tests for a rhombus

- If a quadrilateral is a parallelogram with two adjacent sides equal, then the parallelogram is a rhombus.
- If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

These tests are proved in Question **1** of Exercise 7E.

Note: A rhombus is often defined to be a parallelogram with two adjacent sides equal.

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Rectangles

Definition of a rectangle

- A **rectangle** is a parallelogram whose interior angles are right angles.

Diagonal properties of a rectangle

- The diagonals of a rectangle are equal and bisect each other.

Tests for a rectangle

- A parallelogram with one right angle is a rectangle.
- If all angles of a quadrilateral are equal, then the quadrilateral is a rectangle.
- If the diagonals of a quadrilateral are equal and bisect each other, then the quadrilateral is a rectangle.

The three tests for a rectangle mentioned above are proved in Questions **2a**, **2b**, and **1c**, respectively, on the next page as part of Exercise 7E.

Note: A rectangle is often defined to be a parallelogram with one right angle.

Squares

Definition of a square

- A **square** is a quadrilateral that is both a rectangle and a rhombus. Thus a square has all the properties of a rectangle and a rhombus.

Properties of a square

- All angles of a square are right angles.
- All sides of a square are equal.
- The diagonals of a square are equal and bisect each other at right angles.
- Each diagonal meets each side at 45°.

A square is often defined as a regular polygon with four vertices; that is, as a quadrilateral with four equal sides and four equal interior angles. Note that such a figure is a parallelogram, because its opposite sides are equal, so the two definitions agree.

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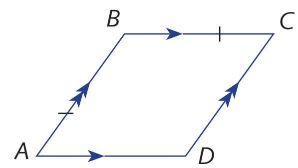
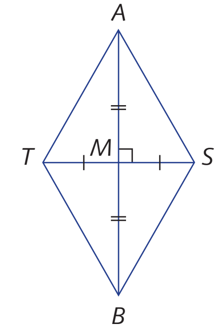
Exercise 7E

1 **a** Prove that: *If a parallelogram has two adjacent sides equal, then the parallelogram is a rhombus.*



b Prove that: *If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.*

The diagonals of the quadrilateral *ASBT* meet at right angles at *M* and bisect each other.



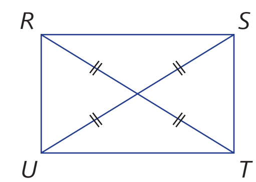
i Why is *ASBT* a parallelogram?

ii Prove that $\Delta AMS \equiv \Delta AMT$.

iii Hence prove that $AS = AT$; that is, *ASBT* is a rhombus.

c Prove that: *If the diagonals of a quadrilateral have equal length and bisect each other, then the quadrilateral is a rectangle.*

The diagonals of the quadrilateral *RSTU* have equal length and bisect each other.



i Why is *RSTU* a parallelogram?

ii Prove that $\Delta URS \equiv \Delta TSR$.

iii Hence prove that $\angle URS = \angle TSR$.

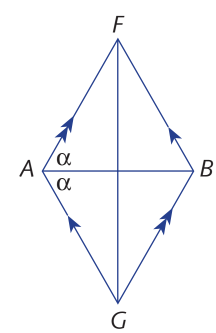
iv Hence prove that $\angle URS = 90°$.

2 **a** Prove that: *A parallelogram with one right angle is a rectangle.*

b Prove that: *If all angles of a quadrilateral are equal, then the quadrilateral is a rectangle.*

3 Prove that: *If one diagonal of a parallelogram bisects one vertex angle, then the parallelogram is a rhombus.*

Let *AFBG* be a parallelogram in which the diagonal *AB* bisects the vertex angle $\angle GAF$.

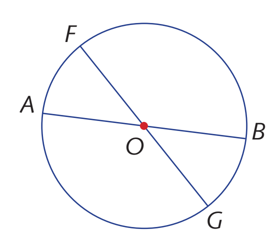


Let $\angle FAB = \angle GAB = \alpha$.

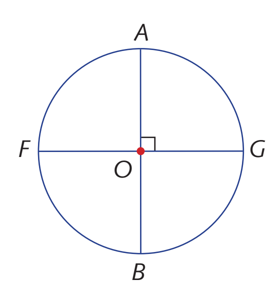
a Prove that $\angle ABF = \alpha$.

b Hence prove that *AFBG* is a rhombus.

4 **a** In the diagram to the right, *O* is the centre of the circle, and *AB* and *FG* are diameters of the circle. What sort of figure is *AFBG*, and why?



b In the diagram to the right, *O* is the centre of the circle, and *AB* and *FG* are perpendicular diameters of the circle. What sort of figure is *AFBG*, and why?



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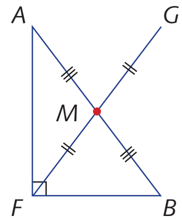
5 Prove that: *The circle with centre the midpoint of the hypotenuse of a right-angled triangle, and passing through the endpoints of the hypotenuse, also passes through the third vertex.*

Let ΔABF be right-angled at F, and let M be the midpoint of the hypotenuse AB. Produce FM to G so that $FM = MG$.



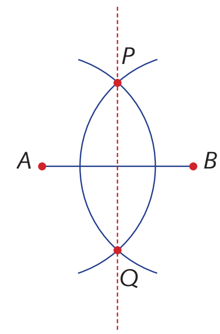
a Explain why $AFBG$ is a rectangle.

b Explain why $AM = BM = FM = GM$.

c Hence explain why the circle with centre M and radius AM passes through F.

6 This question justifies some of the constructions that you learnt in earlier years.

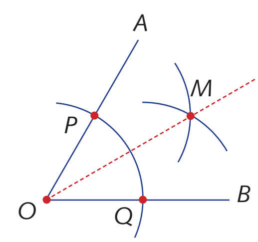
a The diagram shows the standard construction of the perpendicular bisector of a given interval AB. We draw two arcs with centres A and B respectively, equal radius, intersecting at P and Q.



i Explain why $APBQ$ is a rhombus.

ii Hence explain why PQ is the perpendicular bisector of AB.

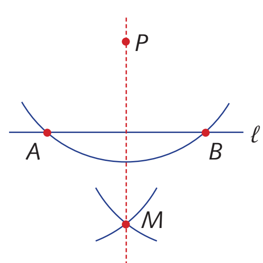
b The diagram shows the standard construction of the bisector of a given angle $\angle AOB$. We draw three arcs with centres O, P and Q respectively, and equal radius.



i Explain why $OPMQ$ is a rhombus.

ii Hence explain why OM is the bisector of $\angle AOB$.

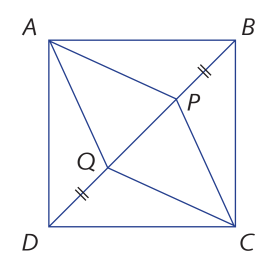
c The diagram shows the standard construction of the perpendicular from a given point P to a given line ℓ. First draw an arc with centre P cutting ℓ at A and B. Then with centres A and B and the same radius, draw arcs meeting at M.



i Explain why $APBM$ is a rhombus.

ii Hence explain why PM is perpendicular to ℓ.

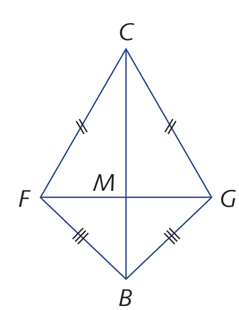
7 The points P and Q are chosen on the diagonal BD of the square $ABCD$ so that $BP = DQ$.



a Prove that the triangles ABP, CBP, ADQ and CDQ are all congruent.

b Hence prove that $APCQ$ is a rhombus.

8 A **kite** is a quadrilateral with two pairs of equal adjacent sides. As with a rhombus, the properties of a kite are best seen when it is drawn in a diamond orientation, as in the diagram.



a Let $CFBG$ be a kite with $CF = CG$ and $BF = BG$. Let the diagonals CB and FG meet at M.

i Show that $\Delta CBF \equiv \Delta CBG$.

ii Hence show that CB bisects $\angle FCG$ and $\angle FBG$.

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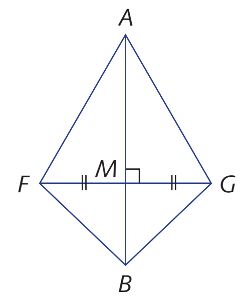
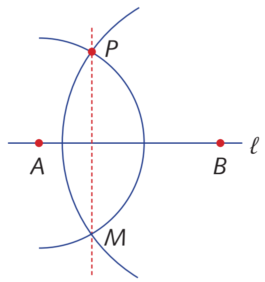
iii Show that $\Delta CMF \equiv \Delta CMG$.

iv Hence show that the diagonals are perpendicular.

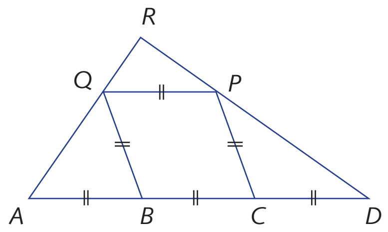
b Conversely, let $AFBG$ be a quadrilateral in which the diagonal AB bisects the diagonal FG and is perpendicular to it. Prove that $AFBG$ is a kite.



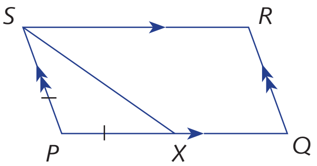
c Let ℓ be a line and P a point not on the line. Choose two points A and B on ℓ. Draw an arc with centre A and radius AP. Then draw a second arc with centre B and radius BP. Let the arcs meet at M. Explain why PM is perpendicular to ℓ.



9 $ABCD$ is a straight line such that $AB = BC = CD$ and $BCPQ$ is a rhombus. The lines produced through AQ and DP meet at R. Prove that $\angle ARD$ is a right angle.



10 $PQRS$ is a parallelogram. X is a point on PQ such that $PX = SP$. Prove that SX bisects $\angle RSP$.



11 A **trapezium** *is a quadrilateral with one pair of parallel lines*.

a The trapezium $ABCD$ has $\angle C = 70°$ and $\angle D = 50°$. Find the sizes of $\angle A$ and $\angle B$.

If the non-parallel sides of a trapezium are equal, then the figure is called an **isosceles trapezium**.

b Prove that the base angles of an isosceles trapezium are equal.

c Prove that the diagonals of an isosceles trapezium are equal in length.

d Prove that if a trapezium has equal diagonals then it is isosceles.

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Review exercise

1 Classify these angles according to their size.

a 78° **b** 180° **c** 90° **d** 340° **e** 360° **f** 162°

2 Write down the complements of:

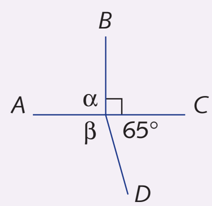
a 56° **b** 67° **c** 47° **d** 15° **e** 26° **f** 46°

3 Write down the supplements of:

a 154° **b** 89° **c** 34° **d** 90° **e** 113° **f** 116°

4 Find the angles marked with pronumerals. Give reasons for your answers.

a



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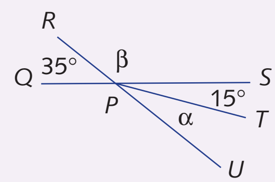

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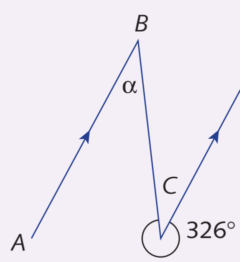

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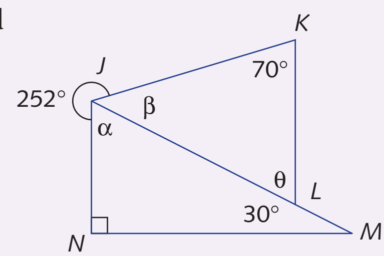

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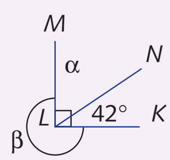

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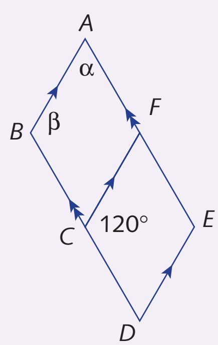

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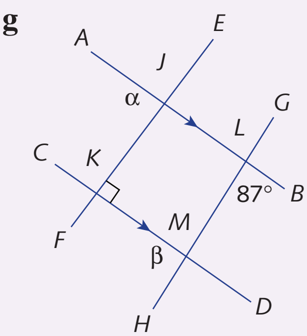
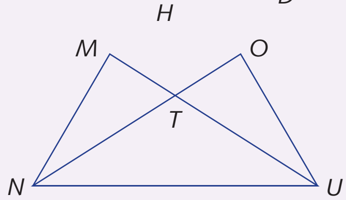


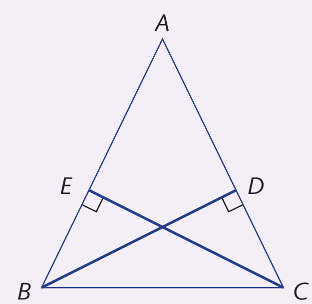
5 In the diagram to the right, $MT = OT$ and $\angle MNT = \angle OUT$. Prove that $NT = UT$.



6 In the diagram to the right, ΔABC is isosceles with $AB = AC$, $BD \perp AC$ and $CE \perp AB$.

a Prove that ΔBDC is congruent to ΔBEC.

b Prove that ΔAED is isosceles.



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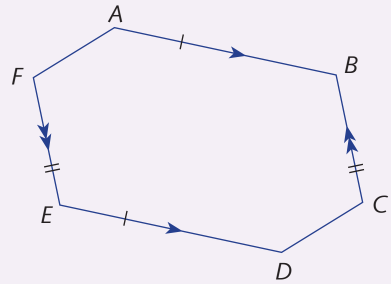
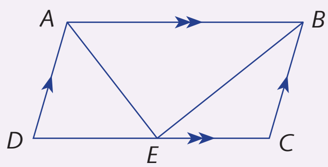
7 *ABCDEF* is a hexagon in which the interval *AB* is parallel and equal in length to the interval *ED*, and the interval *BC* is parallel and equal in length to the interval *FE*.



a Join *B* to *E* and prove that $\angle ABC = \angle FED$.

b Prove that *ACDF* is a parallelogram.

8 In a parallelogram *ABCD*, the point *E* lies on the side *DC* such that *EA* bisects $\angle BAD$ and *EB* bisects $\angle ABC$. Prove that:

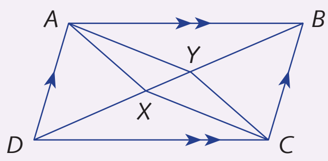


a $DE = EC$

b $DC = 2CB$

c $\angle AEB$ is a right angle

9 In the parallelogram *ABCD*, the points *X* and *Y* are on the diagonal *DB* such that $DX = DA$ and $BY = BC$. Prove that:



a $AX = CY$

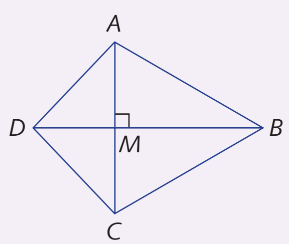
b $AY = CX$

c *AYCX* is a parallelogram

Challenge exercise

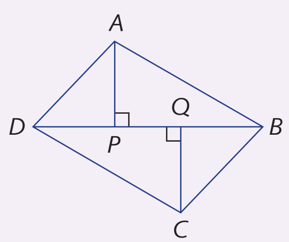
1 **a** Prove that: *If the diagonals of a quadrilateral are perpendicular, then the sums of the squares of opposite sides are equal.*

The diagonals of the convex quadrilateral *ABCD*, to the right, meet at right angles at *M*. Prove that $AB^2 + CD^2 = AD^2 + BC^2$.



b Conversely, prove that: *If the sums of the squares of opposite sides of a quadrilateral are equal, then the diagonals are perpendicular.*

In the convex quadrilateral *ABCD* to the right, $AB^2 + CD^2 = AD^2 + BC^2$. The perpendiculars to the diagonal *BD* from *A* and *C* meet *BD* at *P* and *Q* respectively. Prove that the points *P* and *Q* coincide.



c What happens if the quadrilateral is non-convex?

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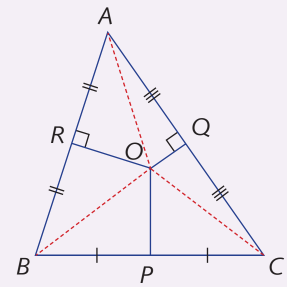
2 The circle passing through all three vertices of a triangle is called a **circumcircle** of the triangle.

The following theorem not only shows that every triangle has a unique circumcircle, but also shows how to construct the circumcircle.

Prove that: *The three perpendicular bisectors of a triangle are concurrent. Their point of intersection (called the circumcentre) is the centre of a circle (called a circumcircle).*

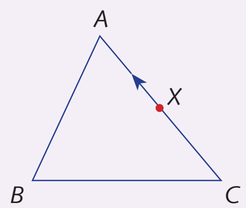


Prove this theorem using congruence. In the diagram above, the midpoints of BC, CA and AB are P, Q and R respectively.

3 In Exercise 7B, you proved that the sum of the exterior angles of a convex polygon is 360°. This question introduces another way of looking at this theorem.

a In each diagram below, imagine that you started at the point X, facing A, and walked around the figure back to X, passing through the vertices A, B, C, . . . in alphabetical order. Through how many complete revolutions would you turn during your journey? Count clockwise rotation as negative, and anticlockwise revolutions as positive.

i



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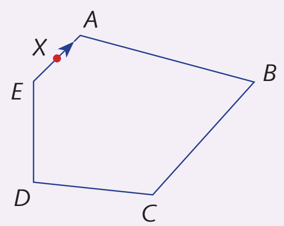

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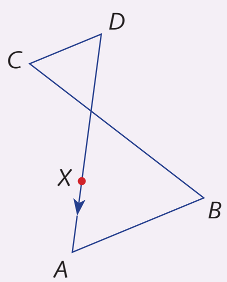

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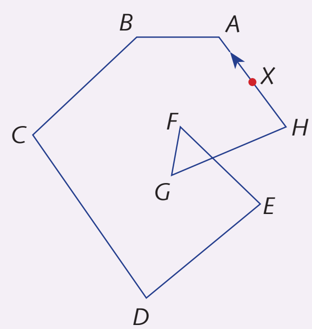

b Use these ideas to develop a proof that the sum of the exterior angles of a convex polygon is 360°.

4 Suppose that you make a journey on the surface of the globe. You start facing east at Singapore, which lies on the Equator. Then you walk east one-quarter of the way around the globe, then turn 90° north and walk to the North Pole, then turn 90° towards Singapore and walk back to Singapore, and finally turn facing east just as you started. Through how many revolutions have you turned during this journey, and what is the sum of the exterior angles of your triangular path?

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5 The diagonals of a convex quadrilateral dissect the quadrilateral into four triangles.

a Prove that: *If the quadrilateral is a trapezium, then a pair of opposite triangles have equal area.*

b Conversely, prove that: *If a pair of opposite triangles have equal area, then the quadrilateral is a trapezium.*

6 **a** Let $ABCD$ be a square. Let Q be a point on and R a point on BC, and let $AQ \perp DR$. Prove that $AQ = DR$.

b Let $ABCD$ be a square. Let ℓ and m be perpendicular lines such that ℓ intersects the sides AB and CD of the square, and m intersects the sides AD and BC. Show that the intervals cut off on ℓ and m by the square are equal in length.

7 Let $ABCD$ be a square of side length 1. Extend the sides AB to E, BC to F, CD to G and DA to H so that $BE = CF = DG = AH = 1$.

a Prove that $EFGH$ is a square, and find its area.

b Show that the centre O of $ABCD$ is also the centre of $EFGH$.

8 Prove that: *Any line through the intersection of the diagonals of a parallelogram dissects the parallelogram into two figures of equal area.*

9 Prove that: *The perpendicular bisector of the base of an isosceles triangle passes through its vertex.*

10 H and K are the midpoint of the sides AB and AC of ΔABC. Points H and K are joined and the line produced to X so that $HK = KX$. Prove that:

a CX is equal and parallel to BH.

b $HK = \frac{1}{2}BC$ and HK is parallel to BC.

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