CHAPTER

8

Number and Algebra

Index laws

In Year 7 and Year 8 we introduced powers with whole number indices and developed the index laws for these powers.

We now extend our study to rational (fractional), including negative, indices.

We live in a world of very large and very small numbers. Population sizes, government spending, intergalactic distances and the size of computer memories are examples of very large numbers. Thickness of materials, circuit diagrams and subatomic particles are examples of very small numbers. Some examples of small and large numbers are:

- the time taken by light to travel one metre is approximately 0.000 000 003 seconds
- the radius of a hydrogen atom is approximately 0.000 000 000 025 metres
- the current Big Bang model of astronomy suggests that the Universe is about 13.7 billion years old.

This chapter introduces scientific notation, which is a convenient way of writing such numbers. Significant figures are also discussed.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

8A The index laws

Index notation

We recall the following from *ICE-EM Mathematics Year 8*.

- A **power** is the product of a certain number of factors, all of which are the same.

  For example,

  $2^4 = 2 \times 2 \times 2 \times 2$ is the fourth power of 2.
- The number 2 in 2^4 is called the **base**.
- The number 4 in 2^4 is called the **index** or **exponent**.
- For any number b, $b^1 = b$.
- In general, $b^n = \underbrace{b \times b \times b \times \ldots \times b}_{n}$, where there are n common factors in the product.

  Here b is the **base** and n the **index**.

Example 1

Express as a power or as a product of powers.

a $5 \times 5 \times 5$ **b** $3 \times 3 \times 7 \times 7 \times 7 \times 7$

Solution

a $5 \times 5 \times 5 = 5^3$ **b** $3 \times 3 \times 7 \times 7 \times 7 \times 7 = 3^2 \times 7^4$

Example 2

Express each number as a power of a prime.

a 81 **b** 128

Solution

a $81 = 3 \times 3 \times 3 \times 3$
$= 3^4$

b $128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
$= 2^7$

The *prime decomposition* of these numbers can be found by a process of repeated division (see *ICE-EM Mathematics Year 8*). This process is particularly useful when the decomposition contains more than one type of prime number.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 3

Express as a product of powers of prime numbers.

a 9000 **b** 66 150

Solution

By repeated division:

a $9000 = 2^3 \times 3^2 \times 5^3$

b $66\,150 = 1323 \times 50$
$= 2 \times 3^3 \times 5^2 \times 7^2$

The following laws for indices were discussed in Chapter 3 of *ICE-EM Mathematics Year 8* for powers with whole number indices.

In the following, a and b are integers and m and n are non-zero whole numbers.

Index laws

Index law 1

To multiply powers of the same base, add the indices.

$$a^m a^n = a^{m+n}$$

Index law 2

To divide powers of the same base, subtract the indices.

$$\frac{a^m}{a^n} = a^{m-n} \qquad \text{where } m > n \text{ and } a \neq 0$$

Index law 3

To raise a power to a power, multiply the indices.

$$(a^m)^n = a^{mn}$$

Index law 4

A power of a product is the product of the powers.

$$(ab)^m = a^m b^m$$

Index law 5

A power of a quotient is the quotient of the powers.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad \text{where } b \neq 0$$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 4

Simplify, expressing the answer in index form.

a $3^2 \times 3^4$

b $a^3 \times a^5$

c $3x^2 \times x^3$

d $2a^2b^3 \times 5ab^2$

Solution

a $3^2 \times 3^4 = 3^6$

b $a^3 \times a^5 = a^8$

c $3x^2 \times x^3 = 3x^5$

d $2a^2b^3 \times 5ab^2 = 10 \times a^{2+1} \times b^{3+2}$

$= 10a^3b^5$

Example 5

Simplify in power form, and hence evaluate where appropriate.

a $\dfrac{3^5}{3^2}$

b $\dfrac{9^5}{9^4}$

c $10^6 \div 10^4$

d $a^7 \div a^4$

e $\dfrac{3y^4}{y}$

f $\dfrac{6x^5}{2x^3}$

Solution

a $\dfrac{3^5}{3^2} = 3^{5-2}$

$= 3^3$

$= 27$

b $\dfrac{9^5}{9^4} = 9^1$

$= 9$

c $10^6 \div 10^4 = 10^{6-4}$

$= 10^2$

$= 100$

d $a^7 \div a^4 = a^{7-4}$

$= a^3$

e $\dfrac{3y^4}{y} = 3 \times \dfrac{y^4}{y}$

$= 3 \times y^{4-1}$

$= 3y^3$

f $\dfrac{6x^5}{2x^3} = \dfrac{6}{2} \times \dfrac{x^5}{x^3}$

$= 3 \times x^{5-3}$

$= 3x^2$

Example 6

Simplify, expressing the answer in index form.

a $\dfrac{3x^3y^2}{4xy} \times \dfrac{6x^2y^3}{x^3y^2}$

b $\dfrac{8a^2b^3}{3a^3b} \div \dfrac{4ab^2}{9a^3b^5}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Solution

a $\dfrac{3x^3y^2}{4xy} \times \dfrac{6x^2y^3}{x^3y^2} = \dfrac{18x^5y^5}{4x^4y^3}$

$= \dfrac{9xy^2}{2}$

b $\dfrac{8a^2b^3}{3a^3b} \div \dfrac{4ab^2}{9a^3b^5} = \dfrac{8a^2b^3}{3a^3b} \times \dfrac{9a^3b^5}{4ab^2}$

$= \dfrac{72 \times a^5 \times b^8}{12 \times a^4 \times b^3}$

$= 6ab^5$

Example 7

Simplify by expanding the brackets.

a $\left(\dfrac{2}{3}\right)^2$ **b** $\left(\dfrac{m}{n}\right)^5$ **c** $\left(\dfrac{x^3}{y^2}\right)^2 \times \left(\dfrac{y}{x}\right)^4$ **d** $\left(\dfrac{2x^2}{3}\right)^2 \div \dfrac{4x^3}{9}$

Solution

a $\left(\dfrac{2}{3}\right)^3 = \dfrac{2^3}{3^3}$

$= \dfrac{8}{27}$

b $\left(\dfrac{m}{n}\right)^5 = \dfrac{m^5}{n^5}$

c $\left(\dfrac{x^3}{y^2}\right)^2 \times \left(\dfrac{y}{x}\right)^4 = \dfrac{x^6}{y^4} \times \dfrac{y^4}{x^4}$

$= x^2$

d $\left(\dfrac{2x^2}{3}\right)^2 \div \dfrac{4x^3}{9} = \dfrac{4x^4}{9} \times \dfrac{9}{4x^3}$

$= x$

There are different possible interpretations of the word ‘simplify’. There may be more than one acceptable simplified form.

A number raised to the power zero

Clearly $\dfrac{4^3}{4^3} = 1$. If the index laws are to apply then $4^{3-3} = 4^0 = 1$.

Hence we define $a^0 = 1$, for all non-zero numbers a.

Note: 0^0 is not defined. If x is a pronumeral, $x^0 = 1$ $(x \neq 0)$.

Example 8

Simplify:

a $(5a^3)^0$

b $\dfrac{6x^2y}{xy^2} \times \dfrac{y^3x}{2y^2x^2}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Solution

a $(5a^3)^0 = 1$

b $\frac{6x^2y}{xy^2} \times \frac{y^3x}{2y^2x^2} = \frac{6}{2} \times \frac{x^3}{x^3} \times \frac{y^4}{y^4}$

$= 3x^0y^0$

$= 3 \times 1 \times 1$

$= 3$

Example 9

Simplify:

a $(mn^2)^0$ **b** $(a^4b^2)^3$ **c** $(2a^4)^3$ **d** $2(x^2y)^0 \times (x^2y^3)^3$

Solution

a $(mn^2)^0 = 1$

b $(a^4b^2)^3 = a^{12}b^6$

c $(2a^4)^3 = 2^3 \times a^{12}$

$= 8a^{12}$

d $2(x^2y)^0 \times (x^2y^3)^3 = 2 \times 1 \times x^6y^9$

$= 2x^6y^9$

The zero power

For all non-zero numbers a, we define $a^0 = 1$.

Exercise 8A

1 State the base and index of:

a 6^4 **b** 7^3 **c** 8^2 **d** 10^4 **e** 5 **f** 6^0

Example 2

2 Express as a power of a prime number.

a 8 **b** 27 **c** 64 **d** 243 **e** 125 **f** 81

3 Evaluate:

a 3^4 **b** 2^7 **c** 5^5 **d** 7^4 **e** $2^3 \times 3^5$ **f** $6^4 \times 3^2$

Example 3

4 Express as a product of powers of prime numbers.

a 18 **b** 24 **c** 144 **d** 90 **e** 700 **f** 84

Example 4

5 Simplify, leaving the answer as a power or a product of powers.

a $2^2 \times 2^3$ **b** $2^7 \times 2^3$ **c** $3^4 \times 3^5$ **d** $3^4 \times 3^7$

e $3^2 \times 3 \times 3^4$ **f** $3^3 \times 3^4 \times 3^5$ **g** $a^3 \times a^8$ **h** $b^7 \times b^{12}$

i $3a^2 \times a^3$ **j** $3x^2 \times 4x^3$ **k** $2y \times 3y^4$ **l** $4b^2 \times 3b^4$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

6 Simplify:

a $a^2b^3 \times b^2$ **b** $a^3b \times a^2b^3$ **c** $x^2y \times x^3y$

d $2xy^2 \times 3x^2y$ **e** $4a^3b^2 \times a^2b^4$ **f** $5a^4b \times 2ab^3$

Example 5

7 Simplify, leaving the answer as a power or a product of powers.

a $\frac{3^7}{3^2}$ **b** $\frac{2^6}{2^2}$ **c** $5^4 \div 5$ **d** $7^5 \div 7^3$

e $10^7 \div 10^2$ **f** $\frac{10^{12}}{10^4}$ **g** $\frac{a^4}{a}$ **h** $\frac{a^5}{a^3}$

i $\frac{2x^3}{x^2}$ **j** $\frac{6x^5}{2x^2}$ **k** $\frac{10y^{12}}{5y^3}$ **l** $\frac{27p^4}{9p}$

8 Simplify:

a $\frac{a^3b^2}{a^2}$ **b** $\frac{x^3y^2}{xy}$ **c** $\frac{a^5b^3}{a^4b}$ **d** $\frac{x^4y^7}{x^3y^2}$

e $\frac{12a^6b^2}{4a^2b}$ **f** $\frac{15xy^3}{3y^2}$ **g** $\frac{16a^4b^3}{12a^2b^2}$ **h** $\frac{27x^2y^3}{18xy^2}$

Example 6

9 Simplify:

a $\frac{a^3b^2}{ab} \times \frac{a^2b}{a}$ **b** $\frac{x^3y}{xy^2} \times \frac{x^4y^5}{x^2}$ **c** $\frac{2ab^2}{3a^2b^4} \times \frac{6a^4b^5}{ab}$ **d** $\frac{12x^4y^3}{3x^2y} \times \frac{x^2y^4}{x^3y^5}$

e $\frac{6ab^2}{5a^3b} \div \frac{12ab}{15a^5b}$ **f** $\frac{7x^3y^4}{2xy^2} \div \frac{21x^2y^3}{4x^3y^2}$ **g** $\frac{14a^4b^3}{3ab^2} \div \frac{7a^5b^4}{6a^3b^5}$ **h** $\frac{12x^2y}{x^3y^4} \div \frac{6xy^2}{x^6y^7}$

10 Copy and complete.

a $a^4 \times \ldots = a^{10}$ **b** $b^7 \times \ldots = b^{16}$ **c** $4a^3 \times \ldots = 12a^7$

d $9d^5 \times \ldots = 27d^6$ **e** $a^8 \div \ldots = a^4$ **f** $x^{10} \div \ldots = x^6$

g $15d^7 \div \ldots = 3d^2$ **h** $9d^6 \div \ldots = 3d$ **i** $m^4n^5 \times \ldots = m^{10}n^7$

j $8ab^4 \times \ldots = 24a^2b^6$ **k** $a^7b^4 \div \ldots = a^2b$ **l** $14x^5y^2 \times \ldots = 42x^{10}y^5$

m $\ell^6m^7 \div \ldots = \ell^2m^5$ **n** $9m^7n^4 \div \ldots = 3m^2$ **o** $18p^2q^6 \div \ldots = 3pq$

11 Simplify each expression. Check your answer for part **a** by substituting $x = 2$ into both the original expression and the simplified expression. Repeat for $x = 3$ in part **b** and $x = -2$ in part **c**.

a $\frac{6x^2}{x}$ **b** $\frac{6x^2}{3x}$ **c** $\frac{6x^2}{6x}$ **d** $\frac{8x^4}{4x}$

e $\frac{3x^5}{x^3}$ **f** $\frac{10x^4}{5x^3}$ **g** $\frac{10x^4}{2x^4}$ **h** $\frac{12x^4}{6x^2}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 8 **12** Simplify:

a a^0 **b** $2x^0$ **c** xy^0 **d** $7x^0y^0$

Example 9 **13** Simplify:

a $3a^0$ **b** $6a^0$ **c** $(4a)^0$ **d** $(3b)^0$

e $4a^0 + 3b^0$ **f** $6a^0 + 7m^0$ **g** $(2a+1)^0$ **h** $(4a+3b)^0$

i $(4b)^0 + 2b^0$ **j** $(3b)^0 - 5d^0$ **k** $(5m^0 + 7b)^0$ **l** $(6m - 2c^0)^0$

14 Simplify, leaving the answer as a power.

a $\left(2^3\right)^4$ **b** $\left(3^2\right)^3$ **c** $(a^2)^5$ **d** $(y^5)^6$

15 Simplify:

a $(a^3)^2 \times (a^3)^4$ **b** $(x^4)^2 \times (x^3)^3$ **c** $(b^4)^2 \div (b^3)^2$

d $\dfrac{(y^3)^4}{(y^4)^2}$ **e** $2ab^2 \times 3a(b^3)^2$ **f** $\dfrac{3ab}{(b^2)^3} \times \dfrac{4b^7}{3a}$

g $\dfrac{4(x^3)^2y^4}{3x^4y^3} \times \dfrac{3x^3(y^2)^2}{8xy^5}$ **h** $\dfrac{8a^2(b^3)^2}{3ab^2} \div \dfrac{16a^5b^3}{9(a^3)^2}$ **i** $\dfrac{3(x^3y)^2}{(x^2y)^2} \div \dfrac{12x^4y^2}{(2x^3y)^2}$

16 Copy and complete (using index law 3).

a $(a^6)^{\cdots} = a^{24}$ **b** $(b^3)^{\cdots} = b^{21}$ **c** $(m^5)^{\cdots} = m^{10}$

d $(m^6)^{\cdots} = m^{30}$ **e** $(\ldots)^6 = p^{36}$ **f** $(\ldots)^5 = p^{25}$

g $(\ldots)^4 = a^8$ **h** $(\ldots)^3 = m^{15}$ **i** $(\ldots)^{\cdots} = m^{20}$

17 **a** Is it true that $(a^2)^6 = (a^6)^2$? **b** Is it true that $(b^4)^7 = (b^7)^4$?

c Generalise your result.

Example 7 **18** Simplify by expanding the brackets.

a $(3a)^2$ **b** $(2x)^3$ **c** $(xy^3)^2$ **d** $(a^2b)^4$

e $\left(\dfrac{a}{5}\right)^2$ **f** $\left(\dfrac{2}{x}\right)^3$ **g** $\left(\dfrac{a}{b}\right)^5$ **h** $\left(\dfrac{x^2}{y}\right)^3$

19 Simplify:

a $(2a^2b)^2 \times 3ab^3$ **b** $(3xy^2)^3 \times (x^2y)^2$ **c** $(m^2n^3)^2 \times (mn)^3$

d $(5xy^2)^3 \times (x^2y^3)^2$ **e** $(2a^3b)^3 \times 3a^0$ **f** $(2xy^2)^0 \times (3x^2y)^3$

20 Simplify:

a $\left(\dfrac{x^2}{y}\right)^2 \times \left(\dfrac{y^2}{x}\right)^3$ **b** $\left(\dfrac{4a^2}{b}\right)^2 \times \left(\dfrac{b}{2a}\right)^3$ **c** $\left(\dfrac{x^3}{y^2}\right)^2 \div \left(\dfrac{x}{y^2}\right)^3$ **d** $\left(\dfrac{2x^4}{y}\right)^5 \div \left(\dfrac{4x^3}{y^3}\right)^2$

21 Simplify:

a $\dfrac{(3xy^2)^2 \times (2x^2y)^3}{(6x^2y)^2}$ **b** $\dfrac{3a^2b^4 \times (2ab^2)^3}{(4a^2b^3)^2}$ **c** $\dfrac{(2x^2y^3)^3 \times (5xy^2)^2}{(10x^2y)^2 \times (xy)^3}$ **d** $\dfrac{(6ab)^3 \times 2a^7b^4}{(2ab)^4 \times (3a^2b)^2}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

22 Copy and complete.

a $(\ldots)^4 = a^8b^{12}$ **b** $(\ldots)^6 = m^{30}n^{24}$ **c** $(p^3q)\ldots = p^9q^3$

d $(x^4y^7)\ldots = 1$ **e** $(\ldots)^4 = 16a^8$ **f** $(\ldots)^3 = 27q^9$

g $(\ldots)^2 = 49m^6$ **h** $(\ldots)^3 = 64\ell^9m^3$ **i** $(\ldots)^2 = 25m^{10}n^6$

8B Negative indices

In the last section, we defined a^n to be the product of n factors of a, where a is any number and n is a positive integer. We also defined $a^0 = 1$ for all non-zero numbers a.

We now give meaning to negative integer indices. For example, we want to give a meaning to 2^{-4}, 3^{-100} and so on. To work towards a useful definition, look at the following pattern:

$2^5 = 32,\quad 2^4 = 16,\quad 2^3 = 8,$

$2^2 = 4,\quad 2^1 = 2,\quad 2^0 = 1$

Each index is one less than the preceding one, and each number to the right of an equal sign is half the number in the previous expression.

This can be continued purely as a pattern:

$$2^{-1} = \frac{1}{2},\qquad 2^{-2} = \frac{1}{4} = \frac{1}{2^2},\qquad 2^{-3} = \frac{1}{8} = \frac{1}{2^3}\qquad \text{and so on.}$$

This suggests we define $2^{-n} = \dfrac{1}{2^n}$, for any positive integer n.

Negative indices

Define

$$a^{-n} = \frac{1}{a^n}$$

where a is a non-zero number and n is a positive integer.

Note that this tells us that $a^{-n} \times a^n = 1$. So $a^n = \dfrac{1}{a^{-n}}$

For example,

$$2^{10} = \frac{1}{2^{-10}}$$

So we can say that for all n, whether n is positive or negative, $a^{-n} = \dfrac{1}{a^n}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Example 10

Evaluate:

a 6^{-2} **b** 4^{-3} **c** 2^{-7} **d** 10^{-3}

Solution

a $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

b $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

c $2^{-7} = \frac{1}{2^7} = \frac{1}{128}$

d $10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$

Fractions and negative indices

The **reciprocal** of a fraction such as $\frac{4}{3}$ is $\frac{3}{4}$. The index -1 means 'the reciprocal of', so

$$\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$$

When raising a fraction to other negative indices take the reciprocal first.

$$\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^{3} = \frac{27}{8}$$

Example 11

Evaluate:

a $\left(\frac{1}{3}\right)^{-1}$ **b** $\left(\frac{2}{7}\right)^{-2}$ **c** $\left(4\frac{1}{4}\right)^{-2}$

Solution

a $\left(\frac{1}{3}\right)^{-1} = \frac{3}{1} = 3$

b $\left(\frac{2}{7}\right)^{-2} = \left(\frac{7}{2}\right)^{2} = \frac{49}{4}$

c $\left(4\frac{1}{4}\right)^{-2} = \left(\frac{17}{4}\right)^{-2} = \left(\frac{4}{17}\right)^{2} = \frac{16}{289}$

Note that in general $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.

The index laws we revised in Section 8A are also valid when using negative integer indices.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

For all integers m and n and non-zero numbers a and b the following are true.		
Zero index		$a^0 = 1$
Negative index		$a^{-n} = \frac{1}{a^n}$
Index law 1	Product of powers	$a^m a^n = a^{m+n}$
Index law 2	Quotient of powers	$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$
Index law 3	Power of a power	$(a^m)^n = a^{mn}$
Index law 4	Power of a product	$(ab)^n = a^n b^n$
Index law 5	Power of a quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

The results for negative indices m and n can be proved using the index laws for positive integer indices. An indication of the proof appears as a question in the Challenge exercise at the end of the chapter.

Example 12

Write as a single power and then evaluate.

a $3^4 \times 3^{-2}$ **b** $5^7 \times 5^{-8}$ **c** $13^{-8} \times 13^{15} \times 13^{-7}$ **d** $\left(\frac{2}{3}\right)^{-6} \times \left(\frac{2}{3}\right)^4$

Solution

a $3^4 \times 3^{-2} = 3^2$

$= 9$

b $5^7 \times 5^{-8} = 5^{-1}$

$= \frac{1}{5}$

c $13^{-8} \times 13^{15} \times 13^{-7} = 13^0$

$= 1$

d $\left(\frac{2}{3}\right)^{-6} \times \left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)^{-2}$

$= \left(\frac{3}{2}\right)^2$

$= \frac{9}{4}$

Example 13

Write as a single power and then evaluate.

a $\frac{2^4}{2^5}$ **b** $\frac{3^4}{3^7}$ **c** $\frac{5}{5^3}$ **d** $\frac{3^4}{3^6}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Solution

a $\frac{2^4}{2^5} = 2^{-1}$
$= \frac{1}{2}$

b $\frac{3^4}{3^7} = 3^{-3}$
$= \frac{1}{3^3}$
$= \frac{1}{27}$

c $\frac{5}{5^3} = 5^{-2}$
$= \frac{1}{5^2}$
$= \frac{1}{25}$

d $\frac{3^4}{3^6} = 3^{-2}$
$= \frac{1}{3^2}$
$= \frac{1}{9}$

Example 14

Simplify, expressing the answers with positive indices.

a $a^2b^{-3} \times a^{-4}b^5$

b $\frac{x^2y^3}{x^3y^2}$

c $(2a^{-2}b^3)^{-2}$

d $\left(\frac{3m^2}{n}\right)^{-4}$

Solution

a $a^2b^{-3} \times a^{-4}b^5 = a^{2-4} \times b^{-3+5}$
$= \frac{1}{a^2} \times b^2$
$= \frac{b^2}{a^2}$

b $\frac{x^2y^3}{x^3y^2} = x^{-1}y^1$
$= \frac{1}{x} \times y$
$= \frac{y}{x}$

c $(2a^{-2}b^3)^{-2} = 2^{-2} \times a^4 \times b^{-6}$
$= \frac{1}{2^2} \times a^4 \times \frac{1}{b^6}$
$= \frac{a^4}{4b^6}$

d $\left(\frac{3m^2}{n}\right)^{-4} = \left(\frac{n}{3m^2}\right)^4$
$= \frac{n^4}{81m^8}$

Exercise 8B

Example 10

1 Express with a positive index and then evaluate.

a 2^{-1} **b** 5^{-1} **c** 3^{-2} **d** 6^{-2} **e** 9^{-2} **f** 10^{-2}

g 2^{-4} **h** 3^{-3} **i** 5^{-3} **j** 3^{-4} **k** 10^{-5} **l** 2^{-7}

2 Write each fraction as a power of a prime with a negative index.

a $\frac{1}{8}$ **b** $\frac{1}{9}$ **c** $\frac{1}{27}$ **d** $\frac{1}{49}$ **e** $\frac{1}{121}$ **f** $\frac{1}{125}$

g $\frac{1}{16}$ **h** $\frac{1}{64}$ **i** $\frac{1}{169}$ **j** $\frac{1}{81}$ **k** $\frac{1}{31}$ **l** $\frac{1}{32}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

3 Express with positive indices, evaluating where possible.

a a^{-3} **b** x^{-7} **c** $3a^{-4}$ **d** $5x^{-7}$ **e** $4a^{-5}$ **f** $\frac{1}{x^{-3}}$

g $\frac{3}{a^{-4}}$ **h** $\frac{5}{x^{-5}}$ **i** $3^{-2}a^{-2}$ **j** $4^{-2}x^{-2}$ **k** $\frac{13^{-2}}{n^{-6}}$ **l** $\frac{y^{-4}}{x^{-3}}$

Example 11, 12

4 Simplify where possible and then evaluate.

a $\left(\frac{1}{4}\right)^{-1}$ **b** $\left(\frac{2}{5}\right)^{-2}$ **c** $\left(3\frac{1}{3}\right)^{-2}$ **d** $\left(\frac{2}{3}\right)^{-3}$

e $3^5 \times 3^{-2}$ **f** $5^{11} \times 5^{-8}$ **g** $7^3 \times 7^{-5}$ **h** $4^3 \times 4^{-5}$

Example 13

5 Write as a single power and then evaluate.

a $\frac{2^3}{2^6}$ **b** $\frac{4^2}{4^4}$ **c** $\frac{3^8}{3^9}$ **d** $\frac{6^5}{6^8}$ **e** $\frac{7^1}{7^3}$ **f** $\frac{5^7}{5^{10}}$

g $\frac{8^6}{8^7}$ **h** $\frac{20^4}{20^6}$ **i** $\frac{3^5}{3^9}$ **j** $\frac{2^7}{2^{13}}$ **k** $\frac{10^2}{10^6}$ **l** $\frac{12^{12}}{12^{14}}$

6 Express with negative index.

a $\frac{3}{x}$ **b** $\frac{5}{x^2}$ **c** $\frac{8}{x^4}$ **d** $\frac{3}{2x^4}$ **e** $\frac{4}{3x^7}$ **f** $\frac{2}{3x^5}$

7 Evaluate.

a $\left(\frac{1}{2}\right)^{-1}$ **b** $\left(\frac{2}{3}\right)^{-1}$ **c** $\left(\frac{1}{2}\right)^{-2}$ **d** $\left(\frac{4}{5}\right)^{-2}$ **e** $\left(2\frac{1}{4}\right)^{-2}$ **f** $\left(1\frac{1}{5}\right)^{-3}$

Example 14

8 Simplify, expressing the answer with positive indices.

a $x^{-6}y^4 \times x^2y^{-2}$ **b** $a^{-3}b^{-5} \times a^5b^{-3}$ **c** $3x^{-2}y^5 \times 5x^{-7}y^{-2}$

d $2a^{-1}b^5 \times 7ab^{-3}$ **e** $7a^3m^{-4} \times 8a^{-5}m^{-3}$ **f** $3r^2s^3 \times 4r^{-3}s^{-5}$

g $\frac{8a^{-4}}{2a^6}$ **h** $\frac{16a^{-4}}{8a^5}$ **i** $\frac{18a^{-4}}{4a^5}$ **j** $\frac{27m^{-3}}{9m^{-2}}$ **k** $\frac{56t^{-7}}{8t^{-2}}$

l $\frac{36h^{-9}}{9h^{-4}}$ **m** $\frac{144x^7y^5}{12x^{-3}y^4}$ **n** $\frac{72a^4b^{-3}}{36ab^{-2}}$ **o** $\frac{7a^2b^{-3}c^{-4}}{21a^5b^{-7}c^{-9}}$ **p** $\frac{9m^3n^4p^{-5}}{21m^{-3}n^4p^2}$

9 Copy and complete.

a $6^4 \times \ldots = 6^2$ **b** $9^5 \times \ldots = 9^4$ **c** $b^9 \times \ldots = b^7$

d $m^5 \times \ldots = m^{-6}$ **e** $a^{11} \div \ldots = a^{14}$ **f** $b^7 \div \ldots = b^{15}$

g $d^{-7} \div \ldots = d^{15}$ **h** $e^{-7} \div \ldots = e^{-5}$ **i** $(m^{-2})^{\cdots} = m^{10}$

j $(a^5)^{\cdots} = a^{-15}$ **k** $(\ldots)^{-4} = \frac{m^8}{16}$ **l** $(\ldots)^{-3} = \frac{1}{27a^9}$

m $(\ldots)^{-2} = \frac{m^6}{25}$ **n** $(\ldots)^{-3} = \frac{a^6}{b^9}$ **o** $(\ldots)^{-6} = \frac{m^{12}n^{18}}{p^6}$

p $(\ldots)^{-2} = p^4q^{-6}$ **q** $(\ldots)^{\cdots} = a^6b^{-4}$ **r** $(\ldots)^{\cdots} = \frac{m^6}{n^9}$

Write two possible alternatives for part **q** and part **r**.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

10 Simplify, expressing the answers with positive indices. Evaluate powers where possible.

a $(3a^2b^{-2})^3 \times (2a^4)^{-2}$ **b** $(5x^4y^6)^{-3} \times (5^2xy^{-1})^3$ **c** $(5m^2n^{-3})^{-2} \times 2(m^{-2}n^3)^2$

d $(6a^5b^{-4})^{-3} \times 2(a^3b^{-3})^2$ **e** $\dfrac{(x^2)^2}{y} \times \dfrac{(y^2)^{-3}}{x^3}$ **f** $\dfrac{(2x^3)^{-2}}{y^4} \times \dfrac{(2x^7)^2}{3y^5}$

g $\dfrac{(2a^4b^{-2})^3}{c^2} \times \dfrac{(2^2a^{-3}b^2)^{-1}}{c}$ **h** $\dfrac{(m^2n^3)^2}{p^{-3}} \times (mnp^{-2})^{-3}$ **i** $\dfrac{(a^2)^3}{b^3} \div \left(\dfrac{a}{b^2}\right)^{-2}$

j $\dfrac{(2a^4)^2}{b^7} \div \dfrac{(a^2)^{-3}}{2b}$ **k** $\dfrac{(4c^4d^{-3})^2}{9} \div \dfrac{3c^{-2}}{d}$ **l** $\dfrac{(3m^2n^3)^{-2}}{p^4} \div \dfrac{p^{-3}}{m}$

8C Fractional indices

Fractional indices with numerator one

We are now going to extend our study of indices by looking at fractional indices. We begin by considering what we mean by powers such as $3^{\frac{1}{2}}$, $2^{\frac{1}{3}}$ and $7^{\frac{1}{10}}$, in which the index is the reciprocal of a positive integer.

If a is a positive number, then $\sqrt{a}$ is the positive square root and $(\sqrt{a})^2 = a = a^1$.

We now introduce the alternative notation $a^{\frac{1}{2}}$ for $\sqrt{a}$. We do this because the third index law then continues to hold, that is:

$$\left(a^{\frac{1}{2}}\right)^2 = a^{2\times\frac{1}{2}} = a^1$$

Keep in mind that $a^{\frac{1}{2}}$ is nothing more than an alternative notation for $\sqrt{a}$.

For example, $49^{\frac{1}{2}} = 7$, $64^{\frac{1}{2}} = 8$, $100^{\frac{1}{2}} = 10$, and so on.

Every positive number a has a cube root $\sqrt[3]{a}$. It is the positive number whose cube is a.

For example:

$\sqrt[3]{8} = 2$ because $2^3 = 8$ $\sqrt[3]{27} = 3$ because $3^3 = 27$

We define $a^{\frac{1}{3}}$ to be $\sqrt[3]{a}$. The third index laws continues to hold.

$$\left(a^{\frac{1}{3}}\right)^3 = a^{3\times\frac{1}{3}} = a^1$$

Similarly we define

$a^{\frac{1}{4}} = \sqrt[4]{a}$, $a^{\frac{1}{5}} = \sqrt[5]{a}$, and so on.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Fractional indices

Let a be positive or zero and let n be a positive integer.

Define $a^{\frac{1}{n}}$ to be the n^{th} root of a. That is, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

For example, $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$

Square roots, cube roots, fourth roots and so on are usually irrational numbers, and a calculator can be used to obtain approximations. Using a calculator, we obtain:

$$10^{\frac{1}{2}} = \sqrt{10} \approx 3.1623, \qquad 10^{\frac{1}{3}} = \sqrt[3]{10} \approx 2.1544, \qquad 10^{\frac{1}{4}} = \sqrt[4]{10} \approx 1.7783, \ldots$$

Using our new notation, here are some other numerical approximations, all recorded correct to 4 decimal places. You can use your calculator to check these.

$$2^{\frac{1}{5}} \approx 1.1487, \qquad 10^{\frac{1}{8}} \approx 1.3335, \qquad 0.2^{\frac{1}{4}} \approx 0.6687, \qquad 3.2^{\frac{1}{6}} \approx 1.2139$$

Example 15

Evaluate:

a $\sqrt[3]{27}$ **b** $\sqrt[4]{16}$ **c** $\sqrt[5]{243}$ **d** $\sqrt[3]{125}$

Solution

a $\sqrt[3]{27}$ is the number that when cubed is 27.
Thus $3^3 = 27$, because $\sqrt[3]{27} = 3$

b $\sqrt[4]{16} = 2$ because $2^4 = 16$.

c $\sqrt[5]{243} = \sqrt[5]{3^5}$
$= 3$

d $\sqrt[3]{125} = \sqrt[3]{5^3}$
$= 5$

Example 16

Evaluate:

a $9^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $16^{\frac{1}{4}}$ **d** $1000000^{\frac{1}{6}}$

Solution

a $9^{\frac{1}{2}} = 3$ (since $3^2 = 9$)

b $27^{\frac{1}{3}} = 3$ (since $3^3 = 27$)

c $16^{\frac{1}{4}} = 2$ (since $2^4 = 16$)

d $1000000^{\frac{1}{6}} = 10$ (since $10^6 = 1000000$)

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Positive fractional indices

In the previous section, we defined $8^{\frac{1}{3}} = \sqrt[3]{8}$. For consistency with the index laws we now define $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$.

Positive fractional indices

If a is a positive number or zero and, p and q are positive integers, define:

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{a}\right)^p$$

Example 17

Evaluate:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{2}{3}}$ **c** $81^{\frac{3}{4}}$

Solution

a $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3$

$= 3^3$ (since $2^2 = 4$)

$= 8$

b $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$

$= 2^2$ (since $2^3 = 8$)

$= 4$

c $81^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)^3$

$= 3^3$ (since $3^4 = 81$)

$= 27$

The index laws also hold for fractional indices. A proof of these more general results is given as a question in the Challenge exercise.

Example 18

Simplify:

a $(d^{21})^{\frac{1}{7}}$ **b** $p^{\frac{2}{3}} \times p^{\frac{4}{5}}$ **c** $q^{\frac{4}{5}} \div q^{\frac{2}{3}}$

Solution

a $(d^{21})^{\frac{1}{7}} = d^{21\times\frac{1}{7}}$

$= d^3$

b $p^{\frac{2}{3}} \times p^{\frac{4}{5}} = p^{\frac{2}{3}+\frac{4}{5}}$

$= p^{\frac{10}{15}+\frac{12}{15}}$

$= p^{\frac{22}{15}}$

c $q^{\frac{4}{5}} \div q^{\frac{2}{3}} = q^{\frac{4}{5}-\frac{2}{3}}$

$= q^{\frac{12-10}{15}}$

$= q^{\frac{2}{15}}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Negative fractional indices

Since $4^{-1} = \frac{1}{4}$, we define $4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \left(\frac{1}{4}\right)^{\frac{3}{2}}$.

We can now combine the definitions of negative indices and fractional indices.

Negative fractional indices

Let a be a positive number or zero, and let p and q be positive integers. Define

$$a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \left(\frac{1}{a}\right)^{\frac{p}{q}}$$

Note: If $a > 0$ and $b > 0$, $\left(\frac{a}{b}\right)^{-\frac{p}{q}} = \left(\frac{b}{a}\right)^{\frac{p}{q}}$.

Example 19

Evaluate:

a $4^{-\frac{3}{2}}$

b $\left(\frac{8}{27}\right)^{-\frac{1}{3}}$

Solution

a $4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}}$

$= \frac{1}{8}$

b $\left(\frac{8}{27}\right)^{-\frac{1}{3}} = \left(\frac{27}{8}\right)^{\frac{1}{3}}$

$= \frac{27^{\frac{1}{3}}}{8^{\frac{1}{3}}}$

$= \frac{3}{2}$

Example 20

Simplify, writing each answer with positive indices.

a $p^{-\frac{1}{4}} \times p^{\frac{2}{3}}$

b $x^{\frac{2}{3}} \div x^{-\frac{1}{2}}$

c $(125n^{-6})^{\frac{1}{3}}$

d $x^{\frac{1}{5}} \div x^{\frac{1}{3}}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Solution

a $p^{-\frac{1}{4}} \times p^{\frac{2}{3}} = p^{-\frac{1}{4}+\frac{2}{3}}$

$= p^{-\frac{3}{12}+\frac{8}{12}}$

$= p^{\frac{5}{12}}$

b $x^{\frac{2}{3}} \div x^{-\frac{1}{2}} = x^{\frac{2}{3}-\left(-\frac{1}{2}\right)}$

$= x^{\frac{4}{6}+\frac{3}{6}}$

$= x^{\frac{7}{6}}$

c $(125n^{-6})^{\frac{1}{3}} = 125^{\frac{1}{3}} \times n^{-6\times\frac{1}{3}}$

$= 5 \times n^{-2}$

$= \dfrac{5}{n^2}$

d $x^{\frac{1}{5}} \div x^{\frac{1}{3}} = x^{\frac{3}{15}-\frac{5}{15}}$

$= x^{-\frac{2}{15}}$

$= \dfrac{1}{x^{\frac{2}{15}}}$

Exercise 8C

Example 15

1 Evaluate:

a $\sqrt[3]{8}$ **b** $\sqrt[5]{32}$ **c** $\sqrt[3]{216}$ **d** $\sqrt[4]{81}$ **e** $\sqrt[3]{64}$ **f** $\sqrt[5]{2^{10}}$

2 Write using fractional indices. Evaluate, correct to 4 decimal places.

a $\sqrt{14}$ **b** $\sqrt[4]{64}$ **c** $\sqrt[5]{7}$ **d** $\sqrt[7]{11}$ **e** $\sqrt[3]{2^7}$

Example 16

3 Evaluate:

a $4^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $243^{\frac{1}{5}}$ **d** $81^{\frac{1}{4}}$

e $64^{\frac{1}{2}}$ **f** $25^{\frac{1}{2}}$ **g** $125^{\frac{1}{3}}$ **h** $64^{\frac{1}{3}}$

i $32^{\frac{1}{5}}$ **j** $625^{\frac{1}{4}}$ **k** $216^{\frac{1}{3}}$ **l** $49^{\frac{1}{2}}$

Example 17

4 Evaluate:

a $4^{\frac{5}{2}}$ **b** $25^{\frac{3}{2}}$ **c** $125^{\frac{2}{3}}$ **d** $64^{\frac{5}{6}}$

e $32^{\frac{2}{5}}$ **f** $81^{\frac{3}{4}}$ **g** $216^{\frac{2}{3}}$ **h** $243^{\frac{3}{5}}$

i $\left(\sqrt[4]{16}\right)^3$ **j** $\left(\sqrt[3]{27}\right)^2$ **k** $\sqrt[5]{32^4}$ **l** $\sqrt[3]{2^6}$

Example 18

5 Simplify:

a $\left(a^{\frac{1}{2}}\right)^2$ **b** $\left(b^{\frac{1}{3}}\right)^6$ **c** $(c^{12})^{\frac{1}{4}}$ **d** $(c^{10})^{\frac{1}{5}}$

e $x^{\frac{1}{2}} \times x^{\frac{3}{2}}$ **f** $y^{\frac{1}{3}} \times y^{\frac{2}{3}}$ **g** $p^{\frac{3}{4}} \times p^{\frac{2}{5}}$ **h** $q^{\frac{3}{2}} \times q^{\frac{2}{3}}$

i $x^{\frac{3}{2}} \div x^{\frac{1}{2}}$ **j** $y^{\frac{2}{3}} \div y^{\frac{1}{3}}$ **k** $p^{\frac{3}{4}} \div p^{\frac{2}{5}}$ **l** $q^{\frac{3}{2}} \div q^{\frac{2}{3}}$

m $(4m^6)^{\frac{1}{2}}$ **n** $(27n^{12})^{\frac{1}{3}}$ **o** $\left(2x^{\frac{2}{3}}\right)^3$ **p** $\left(3y^{\frac{1}{2}}\right)^4$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 19

6 Evaluate:

a $4^{-\frac{1}{2}}$ **b** $25^{-\frac{1}{2}}$ **c** $\left(\frac{8}{125}\right)^{-\frac{1}{3}}$ **d** $\left(\frac{64}{27}\right)^{-\frac{1}{3}}$ **e** $32^{-\frac{2}{5}}$

f $\left(\frac{1}{81}\right)^{-\frac{1}{4}}$ **g** $81^{-\frac{1}{4}}$ **h** $\left(\frac{1}{25}\right)^{-\frac{1}{2}}$ **i** $\left(\frac{16}{81}\right)^{-\frac{1}{4}}$ **j** $\left(\frac{32}{243}\right)^{-\frac{1}{5}}$

Example 20

7 Simplify, expressing the answer with positive indices.

a $\left(a^{\frac{1}{2}}\right)^{-2}$ **b** $\left(b^{-\frac{2}{3}}\right)^{6}$ **c** $\left(2x^{\frac{2}{3}}\right)^{-3}$ **d** $\left(3y^{\frac{1}{2}}\right)^{-4}$ **e** $x^{\frac{1}{2}} \times x^{-\frac{3}{2}}$

f $y^{\frac{1}{3}} \times y^{-\frac{2}{3}}$ **g** $p^{\frac{3}{4}} \times p^{-\frac{2}{5}}$ **h** $q^{\frac{3}{2}} \times q^{-\frac{2}{3}}$ **i** $x^{\frac{3}{2}} \div x^{-\frac{1}{2}}$ **j** $y^{\frac{2}{3}} \div y^{-\frac{1}{3}}$

k $p^{\frac{3}{4}} \div p^{-\frac{2}{5}}$ **l** $q^{\frac{3}{2}} \div q^{-\frac{2}{3}}$ **m** $(4m^{-6})^{\frac{1}{2}}$ **n** $(27n^{-12})^{\frac{1}{3}}$ **o** $\left(2x^{-\frac{2}{5}}\right)^{5}$

8D Scientific notation

Scientific notation, or **standard form**, is a convenient way to represent very large or very small numbers. It allows such numbers to be easily read.

Here is an example with a very large number. The star Sirius A is approximately 81362000000000 km from the Sun. This is about 81 trillion kilometres. This distance can be written neatly in scientific notation as 8.1362×10^{13} km. You can verify that if we move the decimal point 13 places to the right, inserting the necessary zeros, we arrive back at the number we started with.

We can also use this notation for very small numbers. For example, an angstrom (Å) is a unit of length equal to 0.0000000001 m, which is the approximate diameter of a small atom. In scientific notation this is written as 1.0×10^{-10} m or 1×10^{-10} m. When we move the decimal point 10 places to the left, inserting the zeros, we arrive back at the original number. To further this example, the approximate diameter of a uranium atom is 0.00000000038 m, or 3.8×10^{-10} m, or 3.8 Å.

By definition, a positive number is in scientific notation (or standard form) if it is written as $a \times 10^{b}$, where $1 \le a < 10$ and b is an integer.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

We convert a number into scientific notation by placing a decimal point after the first non-zero digit and multiplying by the appropriate power of 10.

Note: If the number to be written in scientific notation is greater than 1, then the index is positive or zero. If the number is positive and less than 1, then the index is negative.

Example 21

Write in scientific notation.

a 610 **b** 21000 **c** 46 000 000 **d** 81

e 0.0067 **f** 0.000 02 **g** 0.07 **h** 8.17

Solution

a $610 = 6.1 \times 100$
$= 6.1 \times 10^2$

b $21000 = 2.1 \times 10\,000$
$= 2.1 \times 10^4$

c $46\,000\,000 = 4.6 \times 10\,000\,000$
$= 4.6 \times 10^7$

d $81 = 8.1 \times 10$
$= 8.1 \times 10^1$

e $0.0067 = 6.7 \div 1000$
$= 6.7 \times \frac{1}{1000}$
$= 6.7 \times 10^{-3}$

f $0.000\,02 = 2 \div 100\,000$
$= 2 \times \frac{1}{100\,000}$
$= 2 \times 10^{-5}$

g $0.07 = 7 \div 100$
$= 7 \times 10^{-2}$

h $8.17 = 8.17 \times 10^0$

Example 22

Write in decimal form.

a 2.1×10^3 **b** 6.3×10^5 **c** 5×10^{-4} **d** 8.12×10^{-2}

Solution

a $2.1 \times 10^3 = 2.1 \times 1000$
$= 2100$

b $6.3 \times 10^5 = 6.3 \times 100\,000$
$= 630\,000$

c $5 \times 10^{-4} = 5 \times \frac{1}{10\,000}$
$= 5 \div 10\,000$
$= 0.0005$

d $8.12 \times 10^{-2} = 8.12 \times \frac{1}{100}$
$= 8.12 \div 100$
$= 0.0812$

Since numbers written in scientific notation involve powers, when these numbers are multiplied, divided or raised to a power, the index laws come into play.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 23

Simplify and write in scientific notation.

a $(3 \times 10^4) \times (2 \times 10^6)$

b $(9 \times 10^7) \div (3 \times 10^4)$

c $(4.1 \times 10^4)^2$

d $(2 \times 10^5)^{-2}$

Solution

a $$\begin{aligned}(3 \times 10^4) \times (2 \times 10^6) &= 3 \times 10^4 \times 2 \times 10^6\\ &= 3 \times 2 \times 10^4 \times 10^6\\ &= 6 \times 10^{10}\end{aligned}$$

b $$\begin{aligned}(9 \times 10^7) \div (3 \times 10^4) &= \frac{9 \times 10^7}{3 \times 10^4}\\ &= \frac{9}{3} \times \frac{10^7}{10^4}\\ &= 3 \times 10^3\end{aligned}$$

c $$\begin{aligned}(4.1 \times 10^4)^2 &= 4.1^2 \times \left(10^4\right)^2\\ &= 16.81 \times 10^8\\ &= 1.681 \times 10^9\end{aligned}$$

d $$\begin{aligned}(2 \times 10^5)^{-2} &= 2^{-2} \times \left(10^5\right)^{-2}\\ &= \frac{1}{2^2} \times 10^{-10}\\ &= 0.25 \times 10^{-10}\\ &= 2.5 \times 10^{-11}\end{aligned}$$

Scientific notation

- **Scientific notation**, or **standard form**, is a convenient way to represent very large and very small numbers.
- To represent a number in scientific notation, insert a decimal point after the first non-zero digit and multiply by an appropriate power of 10.

  For example:

  $81\,362\,000\,000\,000 = 8.1362 \times 10^{13}$ and $0.000\,000\,000\,38 = 3.8 \times 10^{-10}$

Some of the exercises in the rest of this chapter are best done using a calculator.

Exercise 8D

1 Write as a power of 10.

a 10 **b** 100 **c** 1000

d 10 000 **e** 1 000 000 **f** 1 000 000 000

g a googol – which is 1 followed by 100 zeros.

Note: 10^6 is a million, 10^9 is a billion and 10^{12} is a trillion.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

2 Write as a power of 10.

a $\frac{1}{10}$ **b** $\frac{1}{100}$ **c** $\frac{1}{1000}$

d 1 trillionth **e** $\frac{1}{100\,000}$ **f** 1 millionth

Example 21

3 Write in scientific notation.

a 510 **b** 5300 **c** 26 000

d 796 000 000 **e** 576 000 000 000 **f** 4 000 000 000 000

g 0.008 **h** 0.06 **i** 0.000 72

j 0.000 041 **k** 0.000 000 006 **l** 0.000 000 206

Example 22

4 Write in decimal form:

a 3.24×10^4 **b** 7.2×10^3 **c** 8.6×10^2

d 2.7×10^6 **e** 5.1×10^0 **f** 7.2×10^1

g 5.6×10^{-2} **h** 1.7×10^{-3} **i** 8.72×10^{-4}

j 2.01×10^{-3} **k** 9.7×10^{-1} **l** 2.6×10^{-7}

5 The mass of the Earth is approximately 6 000 000 000 000 000 000 000 000 kg. Write this value in scientific notation.

6 Light travels approximately 299 000 km in a second. Express this in scientific notation.

7 The mass of a copper sample is 0.0089 kg. Express this in scientific notation.

8 The distance between interconnecting lines on a silicon chip for a computer is approximately 0.000 000 04 m. Express this in scientific notation.

Example 23

9 Simplify, expressing the answer in scientific notation.

a $(4 \times 10^5) \times (2 \times 10^6)$ **b** $(2.1 \times 10^6) \times (3 \times 10^7)$

c $(4 \times 10^2) \times (5 \times 10^{-7})$ **d** $(3 \times 10^6) \times (8 \times 10^{-3})$

e $(5 \times 10^4) \div (2 \times 10^3)$ **f** $(8 \times 10^9) \div (4 \times 10^3)$

g $(6 \times 10^{-4}) \div (8 \times 10^{-5})$ **h** $(1.2 \times 10^6) \div (4 \times 10^7)$

i $(2.1 \times 10^2)^4$ **j** $(3 \times 10^{-2})^3$

k $\frac{(2 \times 10^5) \times (4 \times 10^4)}{1.6 \times 10^3}$ **l** $\frac{(8 \times 10^6) \times (4 \times 10^3)}{5 \times 10^7}$

m $(4 \times 10^{-2})^2 \times (5 \times 10^7)$ **n** $(6 \times 10^{-3}) \times (4 \times 10^7)^2$

o $\frac{(4 \times 10^5)^3}{(8 \times 10^4)^2}$ **p** $\frac{(2 \times 10^{-1})^5}{(4 \times 10^{-2})^3}$

10 If light travels at 3×10^5 km/s and our galaxy is approximately 80 000 light years across, how many kilometres is it across? (A light year is the distance light travels in a year.)

11 The mass of a hydrogen atom is approximately 1.674×10^{-27} kg and the mass of an electron is approximately 9.1×10^{-31} kg. How many electrons, correct to the nearest whole number, will have the same mass as a single hydrogen atom?

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

12 If the average distance from the Earth to the Sun is 1.4951×10^8 km and light travels at 3×10^5 km/s, how long does it take light to travel from the Sun to the Earth?

13 The furthest galaxy detected by optical telescopes is approximately 4.6×10^9 light years from us. How far is this in kilometres? (Light travels at 3×10^5 km/s.)

14 In a lottery there are $\frac{45 \times 44 \times 43 \times 42 \times 41 \times 40}{720}$ different possible outcomes. If I mark each outcome on an entry form one at a time, and it takes me an average of 1 minute to mark each outcome, how long will it take me to cover all different possible outcomes?

8E Significant figures

Suppose that we are using a ruler marked in centimetres and millimetres to measure the length of a sheet of paper. We find that the length of the edge of the paper falls between 15.2 cm and 15.3 cm but is closer to 15.3 cm. The normal procedure is to use rounding and write the measurement as 15.3 cm.

In scientific notation, our measurement is 1.53×10^1 cm. We say that the length is 1.53×10^1 cm **correct to 3 significant figures**. This means that the length is between 1.525 cm and 1.535 cm.

All measurements involve rounding to one level of accuracy or another. For example, you may read that the mass of an electron is about 0.000 000 000 000 000 000 000 000 000 910 938 26 g. This usually means that a measurement was made and the last digit 6 is a rounding digit and is therefore not completely accurate. The other digits are accurate. Writing this in scientific notation, we say that the mass of an electron is $9.109\,3826 \times 10^{-28}$ g **correct to 8 significant figures** because there are 8 digits in the factor 9.109 3826 before the power of 10.

The mass of the Earth is about 5.9736×10^{24} kg. How many significant digits are there in this measurement?

There are 5 digits in 5.9736, so the measurement is correct to 5 significant figures.

Notice that, to use the idea of significant figures, we must first express the number in scientific notation.

People sometimes apply the same kind of ideas to shorten a given decimal number containing many digits. This need not have anything to do with measurement. It is a way of abbreviating the information you are given. The procedure is called **writing the number to a certain number of significant figures**.

To write a number to a specified number of significant figures, first write the number in scientific notation and then round to the required number of significant figures.

For example, $0.000\,340\,61 = 3.4061 \times 10^{-4}$

$$\approx 3 \times 10^{-4} \quad \text{(correct to 1 significant figure)}$$

$$\approx 3.4 \times 10^{-4} \quad \text{(correct to 2 significant figures)}$$

$$\approx 3.41 \times 10^{-4} \quad \text{(correct to 3 significant figures)}$$

$$\approx 3.406 \times 10^{-4} \quad \text{(correct to 4 significant figures)}$$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Example 24

Write in scientific notation and then round correct to 3 significant figures.

a 235.674 **b** 0.007 245 46

Solution

a $235.674 = 2.356\,74 \times 10^2$

$\approx 2.36 \times 10^2$

b $0.007\,245\,46 = 7.245\,46 \times 10^{-3}$

$\approx 7.25 \times 10^{-3}$

Example 25

Write in scientific notation and then round correct to 2 significant figures.

a 276 000 000 **b** 0.000 000 654

Solution

a $276\,000\,000 = 2.76 \times 10^8$

$\approx 2.8 \times 10^8$

b $0.000\,000\,654 = 6.54 \times 10^{-7}$

$\approx 6.5 \times 10^{-7}$

Significant figures

- A number may be expressed with different numbers of **significant figures**.

  For example: π is 3.1 to 2 significant figures, 3.14 to 3 significant figures, 3.142 to 4 significant figures and so on.
- To write a number correct to a specified number of significant figures, first write the number in scientific notation and then round to the required number of significant figures.

Exercise 8E

Example 24

1 Write in scientific notation, correct to 3 significant figures.

a 2.7043 **b** 634.96 **c** 8764.37

d 256 412 **e** 0.003 612 **f** 0.024 186

Example 25

2 Write in scientific notation, correct to 2 significant figures.

a 368.2 **b** 278 000 **c** 0.004 321 **d** 0.000 021 906

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

3 Along each row, write the respective number in scientific notation, correct to the indicated number of significant figures.

	4 sig. figs	3 sig. figs	2 sig. figs	1 sig. fig.
274.62				
0.041236				
1704.28				
1.9925×10^{27}				

4 Use a calculator to evaluate the following, giving the answer in scientific notation correct to 3 significant figures.

a 3.24×0.067

b $6.24 \div 0.026$

c $4.736 \times 10^{13} \times 2.34 \times 10^{-6}$

d $(5.43 \times 10^{-6}) \div (6.24 \times 10^{-4})$

e $0.0276^2 \times \sqrt{0.723}$

f $\dfrac{17.364 \times 24.32 \times 5.4^2}{3.6 \times 7.31^2}$

g $\dfrac{6.54(5.26^2 + 3.24)}{5.4 + \sqrt{6.34}}$

h $\dfrac{6.283 \times 10^8 \times 5.24 \times 10^6}{(4.37 \times 10^7)^2}$

5 Use a calculator to evaluate, giving the answer in scientific notation correct to 4 significant figures.

a 1.234×0.1988

b $1.234 \div 0.1988$

c 1.9346^3

d $(7.919 \times 10^{21})^2$

e $\sqrt{4.863 \times 10^{-12}}$

f $\dfrac{177.41 \times 0.048}{16.23}$

g $\dfrac{7.932 \times 10^{12} \times 9.4 \times 10^{-10}}{0.000\,000\,000\,416}$

h $\dfrac{579.2 \times 0.6231}{79.05 \times 115.4}$

i $\dfrac{74\,510\,000\,000}{6.4 \times 10^{-18} \times 4.4 \times 10^{23}}$

j $\dfrac{79.99}{\sqrt{48.92} + 11.68^2}$

k $\dfrac{15.62^2(79.1 + 111.7)}{12.46 + 4.48^3}$

l $56.21 \times 12 + \dfrac{1}{2} \times 9.8 \times 12^2$

6 Estimate each of the following, correct to 1 significant figure, using appropriate units. In each case explain how you obtained your answer.

a The thickness of a sheet of paper

b The volume of your classroom

c The height of a six-storey building

d The area required for a car park for 500 cars

e The total printed area of a 600-page novel

f The volume of a warehouse that can store 300 000 pairs of shoes (still in their boxes)

g The length of a queue if every student in your school is standing in it

Discuss your answers with others in your class and with your teacher.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Review exercise

1 Evaluate:

a 4^3 **b** 2^6 **c** 8^2 **d** 10^6

2 Express as a product of powers of prime numbers.

a 120^2 **b** 900^3 **c** 315^4 **d** 490^5

3 Simplify and evaluate where possible.

a $a^6 \times a^7$ **b** $b^4 \times b^9$ **c** $3a^4 \times 5a^5$ **d** $2x^3 \times 5x^6$

e $a^7 \div a^4$ **f** $m^{12} \times m^6$ **g** $\dfrac{12b^7}{6b^2}$ **h** $\dfrac{18p^{10}}{9p}$

i $(a^4)^3$ **j** $(b^6)^5$ **k** $(2a^7)^3$ **l** $(3m^2)^4$

m a^0 **n** $3b^0$ **o** $5m^0$ **p** $(3q)^0$

4 Simplify and evaluate where possible.

a $4a^2b^3 \times 5ab^4$ **b** $2m^4n^3 \times 5m^6n^7$ **c** $\dfrac{20a^4b^2}{5a^2b}$

d $\dfrac{24m^9n^4}{18m^6n^2}$ **e** $(3a^3b)^4$ **f** $(5a^2b)^2 \times 4a^4b^3$

g $\dfrac{5a^6b^7}{4a^3b^2} \times \dfrac{12a^{10}b^9}{a^6b^7}$ **h** $\dfrac{8m^4n^2}{7m^3n} \div \dfrac{3m^3n^5}{14m^9n^{16}}$ **i** $\dfrac{(2x^2y)^3}{5x^6y^2} \times \left(\dfrac{x^3}{2y^2}\right)^3$

j $\dfrac{(4ab)^3 \times 5a^2b}{(10ab^2)^2}$ **k** $\dfrac{(3x^2y)^3 \times 2(xy^2)^3}{(3xy)^4}$ **l** $\dfrac{(4a^2b^3)^2}{3a^6b^4} \div \dfrac{ab^5}{(3a^3b^2)^3}$

5 Evaluate:

a 6^{-2} **b** 8^{-3} **c** 2^{-7} **d** 4^{-3}

e $\left(\dfrac{4}{5}\right)^{-2}$ **f** $\left(\dfrac{2}{3}\right)^{-4}$ **g** $\left(2^{\frac{1}{2}}\right)^{-3}$ **h** $\left(16^{\frac{1}{4}}\right)^{-3}$

6 Simplify, expressing the answer with positive indices.

a $\dfrac{4m^2n^5p^{-6}}{16m^{-2}n^5p^3}$ **b** $(4y)^{-3}$ **c** $\left(2^2y^3\right)^{-5}$ **d** $\left(5^{-2}x^3\right)^{-5}$

e $\left(3^{-3}a^2b^{-1}\right)^{-4}$ **f** $\left(\dfrac{a^3}{b^2}\right)^{-2}$ **g** $\left(\dfrac{5g^2}{h^{-3}}\right)^{-2}$ **h** $\left(\dfrac{m^{-3}}{(2n)^{-4}}\right)^{-2}$

7 Simplify, expressing the answer with positive indices.

a $4a^2 \times 5a^{-3}$ **b** $8m^2n^{-3} \times 5m^{-4}n^6$ **c** $14a^{-4} \div 7a^{-5}$

d $\dfrac{18m^2n^{-3}}{9m^4n^{-1}}$ **e** $(4m^2n^{-2})^{-3}$ **f** $\dfrac{(5m^2n)^{-2}}{10m^4n^3}$

g $(3a^4b^{-3})^{-3} \div 6a^2b^{-7}$ **h** $\dfrac{2m^3n^4}{(5m)^2} \times \dfrac{10m}{3n^{-4}}$ **i** $\dfrac{(5a^4b^{-3})^2}{a^{-2}b} \div \dfrac{5(a^{-1}b)^{-3}}{ab^{-4}}$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

8 Evaluate:

a $49^{\frac{1}{2}}$ **b** $8^{\frac{1}{3}}$ **c** $125^{\frac{2}{3}}$ **d** $64^{\frac{2}{3}}$

e $9^{\frac{3}{2}}$ **f** $16^{-\frac{3}{2}}$ **g** $\left(\frac{1}{8}\right)^{-\frac{2}{3}}$ **h** $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$

9 Simplify, expressing the answer with positive indices.

a $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$ **b** $3b^{\frac{2}{3}} \times 4b$ **c** $p^{\frac{2}{3}} \div p^{\frac{1}{2}}$ **d** $m^{-\frac{1}{2}} \div m^{-2}$

e $\left(4x^{\frac{1}{3}}\right)^4$ **f** $\left(2x^{-\frac{1}{3}}\right)^{-2}$ **g** $\left(9a^{\frac{1}{3}}\right)^{\frac{1}{2}}$ **h** $(8p^{-2}q^3)^{\frac{1}{2}}$

10 Write in scientific notation.

a 47 000 **b** 164 000 000 **c** 0.0047

d 0.0035 **e** 840 **f** 0.840

11 Write in decimal form.

a 6.8×10^4 **b** 7.5×10^3 **c** 2.6×10^{-3}

d 9.4×10^{-2} **e** 6.7×10^0 **f** 3.2×10^{-4}

12 Simplify, writing each answer in scientific notation.

a $(3.1 \times 10^4) \times (2 \times 10^{-2})$ **b** $(5.2 \times 10^{-7}) \div (2 \times 10^3)$

c $(3 \times 10^4)^3$ **d** $(5 \times 10^{-4})^2$

e $(4 \times 10^2)^2 \times (5 \times 10^{-6})$ **f** $\dfrac{(3 \times 10^4)^3}{9 \times 10^{-2}}$

13 Write in scientific notation correct to the number of significant figures indicated in the brackets.

a 18.62 (2) **b** 18.62 (3) **c** 18.62 (1)

d 0.004 276 (2) **e** 5973.4 (2) **f** 0.473 952 (4)

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Challenge exercise

1 The following table gives information regarding the population, approximate rate of population growth, and land area of several countries in 1996. Use the information to answer the questions below.

Country	Population (in millions)	Rate of population growth p.a.	Area of country (in km^2)
Australia	18.2	1.4%	7.6×10^6
China	1200	1.1%	9.6×10^6
Indonesia	201	1.6%	1.9×10^6
Singapore	2.9	1.1%	633
New Zealand	3.4	0.6%	2.7×10^5
United States	261	1.0%	9.2×10^6

a How many times larger is Australia than Singapore, correct to the nearest whole number?

b How many times larger is China than New Zealand, correct to 1 decimal place?

c Express Indonesia's population as a percentage of the United States' population. (Calculate your answer correct to the nearest per cent.)

d The population density of a country is defined to be the average number of people for each square kilometre of land. Calculate the population density for each country and find:

i how many times larger the population density of China is compared to Australia

ii the country with the lowest population density

iii the country with the highest population density

e If each country maintains its present rate of population growth, what will the population of each country be (assume compounding growth):

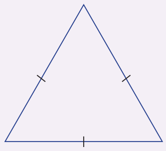
i 5 years from 1996?

ii 10 years from 1996?

iii 100 years from 1996?

f Find the population density of each country 100 years from 1996, assuming each country maintains its present growth rate.

2 **a** Consider the equilateral triangle shown opposite. If each side of the triangle is of length 3 cm, what is the perimeter of the triangle?



b Suppose that the following procedure is performed on the triangle above: 'On the outside of each side of the triangle, draw a triangle with side lengths one-third of the side lengths of the original triangle'.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7