CHAPTER

# 9

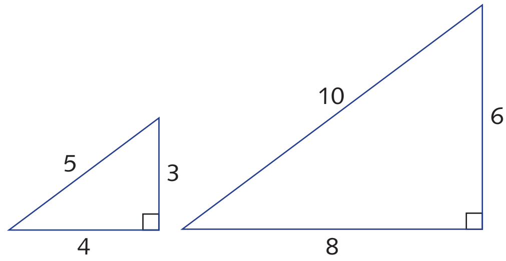
Measurement and Geometry

# Enlargements and similarity

This chapter introduces a transformation called an enlargement. This allows us to develop the idea of similar figures, which is a more general idea than congruence.

For example, the two right-angled triangles shown here are similar because they have exactly the same shape, but they are not congruent because they have different side lengths.

Similarity and proportion are closely related, as we shall see. This idea has been used in art and architecture for centuries. It is now used extensively in photography and in the zoom function of a computer application.



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# 9A Enlargements

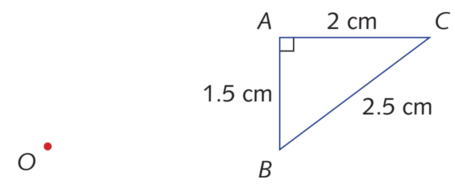
You have already met three types of transformations: translations, rotations and reflections.

This section introduces a fourth type of transformation called an **enlargement**, in which the lengths of the sides are increased or decreased by the same factor. In *ICE-EM Mathematics Year 8*, we looked at these ideas under the topic heading of **scale drawings**. Now we are going to examine these ideas a little more closely.

The figure to the right below is a right-angled triangle $ABC$, whose sides have length 1.5 cm, 2 cm and 2.5 cm.

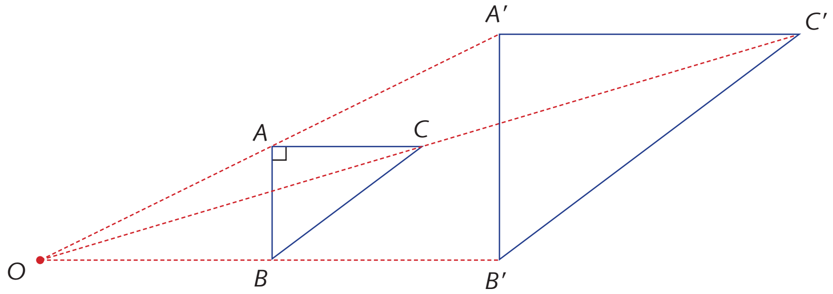
To define an enlargement of this figure, two things need to be specified.

First, we have to specify a **centre of enlargement** – in our example, it is the point $O$ to the left of the triangle. The centre of enlargement stays fixed in the one place while the enlargement expands or shrinks everything else around it.




Secondly, we have to specify an **enlargement ratio** or **enlargement factor** – we will use an enlargement ratio of 2 here. This means that every point will double its distance from $O$. As a consequence, the side lengths of the triangle will double, as we saw in scale drawings.

The second diagram shows how this is done.




- The interval $OA$ is joined, then produced to $A'$ so that $OA' = 2 \times OA$.
- The interval $OB$ is joined, then produced to $B'$ so that $OB' = 2 \times OB$.
- The interval $OC$ is joined, then produced to $C'$ so that $OC' = 2 \times OC$.

The triangle $A'B'C'$ is then joined up, and is called the **image** of triangle $ABC$ with enlargement factor 2 and centre of enlargement $O$.

The image triangle $A'B'C'$ is related to the original triangle in three important ways.

First, each side of the image triangle $A'B'C'$ is twice the length of the matching sides of the original triangle. We will usually write this using ratios:

$$\frac{A'B'}{AB} = 2 \qquad \frac{B'C'}{BC} = 2 \qquad \frac{A'C'}{AC} = 2$$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Secondly, each angle of the image is equal to the matching angle of the original:

$\angle B'A'C' = \angle BAC \qquad \angle A'C'B' = \angle ACB \qquad \angle C'B'A' = \angle CBA$

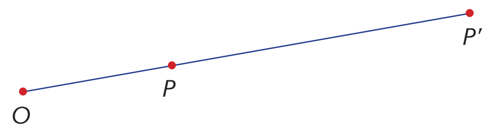
Thus the image $A'B'C'$ is a scale drawing of the original figure $ABC$, with ratio 2.

Thirdly, since matching angles are equal we can see that $A'B' \parallel AB$, $B'C' \parallel BC$ and $C'A' \parallel CA$.

The same kinds of results hold if we perform an enlargement by any positive factor.

## Enlargement transformations

- Each enlargement has a centre of enlargement $O$ and an enlargement factor $k > 0$.
- The enlargement moves each point $P$ to a new point $P'$ on the ray with vertex $O$ passing through $P$.
- The distance of $P'$ from $O$ is $k$ times the distance of $P$ from $O$. That is, $OP' = k \times OP$.



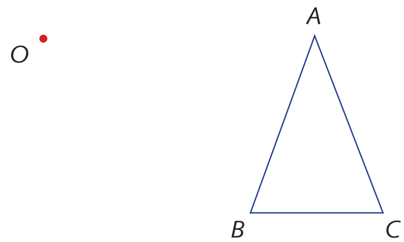

When a figure is enlarged by a factor $k$:

- The image of each interval has $k$ times the length of the original interval.
- The image of an angle has the same size as the original angle.
- The image of an interval is parallel to the original interval.

The first three dot points follow from the definition of an enlargement. If $k > 1$, then the image is larger than the original figure. If $k < 1$, then the image is smaller. If $k = 1$, then the original figure is unchanged by the transformation.

## Exercise 9A

1 The following diagram shows an isosceles triangle $ABC$ and a point $O$. Copy or trace the diagram into your exercise book, leaving plenty of room to the right and below your diagram.




a Use ruler and compasses to apply an enlargement transformation with centre $O$ and enlargement factor 3. Label your image $A'B'C'$.

b Verify that each side length of $\Delta A'B'C'$ is three times the matching side length of $\Delta ABC$.

c Use a protractor to verify that each angle of $\Delta A'B'C'$ is equal to the matching angle of $\Delta ABC$.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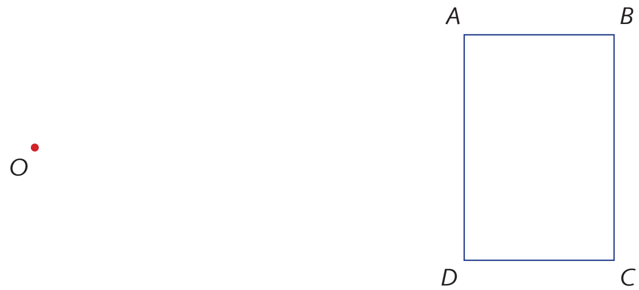
**2** The diagram below shows a rectangle $ABCD$ and a point $O$. Copy or trace the diagram into your exercise book.




**a** Now apply an enlargement with centre $O$ and factor $\frac{1}{2}$ using these steps:

- **i** Join $OA$ and mark the midpoint $A'$ of $OA$.
- **ii** Join $OB$ and mark the midpoint $B'$ of $OB$.
- **iii** Join $OC$ and mark the midpoint $C'$ of $OC$.
- **iv** Join $OD$ and mark the midpoint $D'$ of $OD$.

**b** Use compasses to verify that each side length of $A'B'C'D'$ length of $ABCD$.

**c** Use a protractor to verify that each angle of $A'B'C'D'$ is a right angle, matching the angles of $ABCD$.

**3** I have built a primitive slide projector by shining light from a point source through a transparent slide held up 10 cm away. The light then falls onto a wall that is parallel to the slide and distant 1 metre from it.

**a** Draw a labelled diagram of a triangle on the slide and its image on the wall.

**b** What are the centre of enlargement and the enlargement factor of the transformation?

**4** A ginger cat is sitting on a thin railing 2 metres horizontally from a desk lamp, producing a shadow of the cat on a wall 3 metres behind the cat. What is the enlargement factor of the shadow with respect to the profile of the cat?

**5** Ernest is making scale drawings of atoms, in which his drawing of a hydrogen atom is a circle of radius 1 cm. Here are the actual approximate radii of some of the atoms in his drawings, given in picometres (1 pm = $10^{-12}$ metres).

Hydrogen: 25 pm   Carbon: 70 pm   Gold: 135 pm   Radium: 215 pm

Calculate the radii of Ernest's drawings of carbon, gold and radium atoms.

**6** A map of a country is drawn to a scale of 1 : 12 500 000. Find the actual distance in kilometres between two points whose separation on the map is:

**a** 1 cm   **b** 4 cm   **c** 0.6 cm   **d** 1 mm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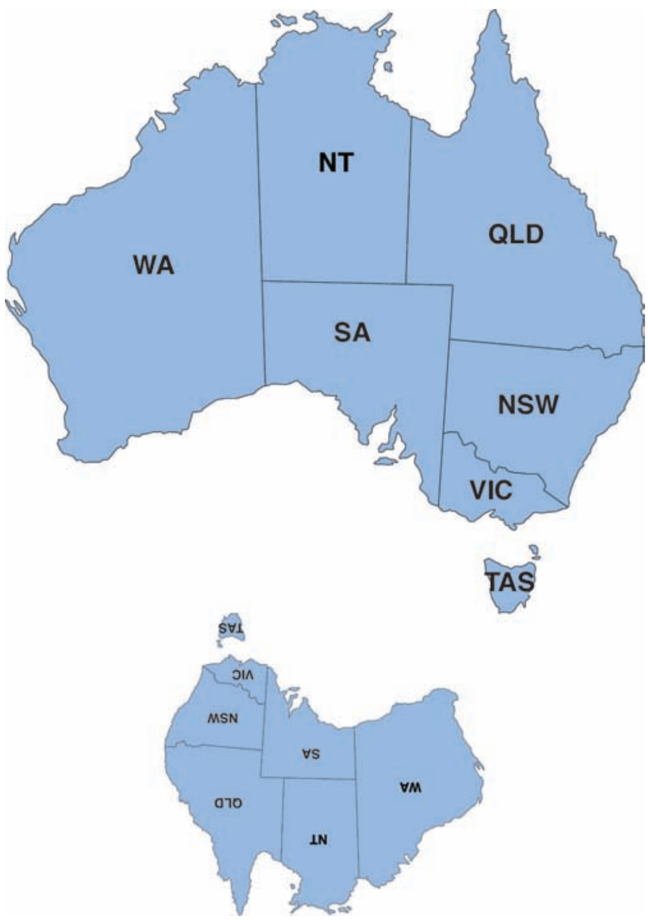
# 9B Using ratios in similar figures

Two figures are called **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.

In simple terms this means that by enlarging or shrinking one figure, we get the other figure perhaps translated, rotated or reflected.

Thus similar figures have the same shape, but not necessarily the same size, just as a scale drawing has the same shape as the original, but has a different size.

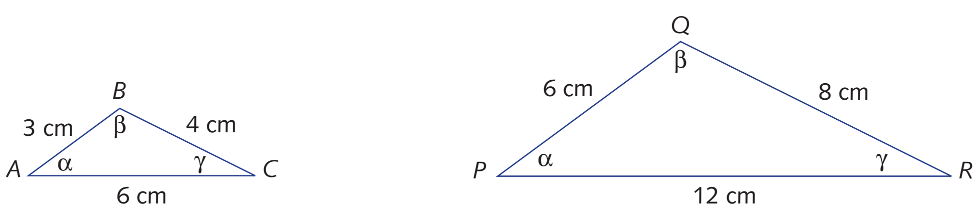
- Matching lengths of similar figures are therefore always in the same ratio, called the **similarity ratio**.
- Matching angles of similar figures are equal.




We will use the same convention for matching angles as we did for congruence.

If $\Delta ABC$ is similar to $\Delta PQR$ then, $\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$

The two triangles in the diagram below are similar.




$$\frac{PQ}{AB} = \frac{QR}{BC} = \frac{PR}{AC} = 2$$

Hence $\Delta ABC$ is similar to $\Delta PQR$ and the similarity ratio is 2.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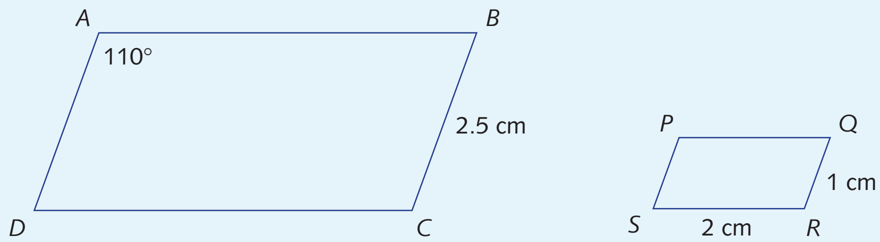
### Example 1

These two parallelograms are known to be similar.




**a** What is the size of $\angle P$?

**b** Find the base length of the large parallelogram.

**c** What is the similarity ratio of the transformation taking *PQRS* to *ABCD*?

### Solution

**a** $\angle P = 110°$ (matching angles of similar figures)

**b** $\frac{DC}{SR} = \frac{BC}{QR}$ (ratio of matching sides)

$\frac{DC}{2} = \frac{2.5}{1}$

$DC = 5\text{ cm}$

**c** The similarity ratio is 2.5.

*Note 1*: In part **b**, we could have written the alternative ratio statement

$\frac{DC}{BC} = \frac{SR}{QR}$ (ratios within figures, 'larger' figure on the left, 'smaller' figure on the right)

It does not matter at all which ratio statement you choose to use. The subsequent algebra will be easier, however, if you always start by placing the unknown length on the top of the left-hand side of the equation.

*Note 2*: An alternative solution to part **b** would be to calculate $\frac{BC}{QR} = 2.5$ as the similarity ratio.

Thus $DC = 2.5 \times SR$

### Similar figures

Two figures are called **similar** if an enlargement of one figure is congruent to the other figure.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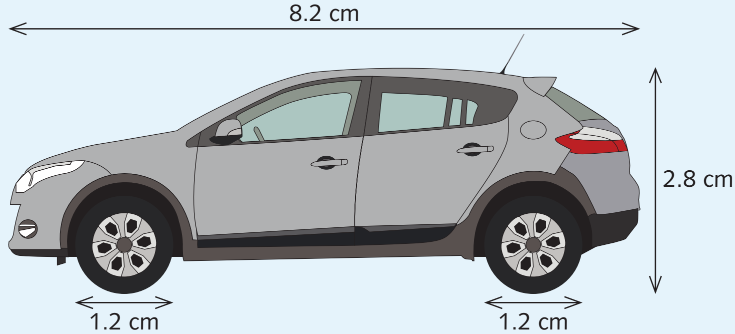
## Matching lengths and angles of similar figures

- Matching lengths in similar figures are in the same ratio, called the **similarity ratio**.
- Matching angles in similar figures are equal.

## Example 2

The car shown in the photograph below is 3.8 metres long.




**a** Use your ruler and your calculator to find approximate ratios for the following, each correct to 1 decimal place. Your measurements are for the diagram, but the ratio is for both the diagram and the real car.

**i** $\dfrac{\text{height of the car}}{\text{diameter of wheel}}$ **ii** $\dfrac{\text{length of the car}}{\text{height of the car}}$

**b** Find the height of the real car.

### Solution

**a** In the picture, height of the car ≈ 2.8 cm

diameter of a wheel ≈ 1.2 cm

length of the car ≈ 8.2 cm

**i** $\dfrac{\text{height of the car}}{\text{diameter of wheel}} \approx \dfrac{2.8}{1.2}$

$\approx 2.3$

**ii** $\dfrac{\text{length of the car}}{\text{height of the car}} \approx \dfrac{8.2}{2.8}$

$\approx 2.9$

**b** Enlargement factor $\approx \dfrac{380}{8.2} \approx 46$

The height of the car $\approx 2.8 \times 46 \approx 128$ cm $= 1.28$ m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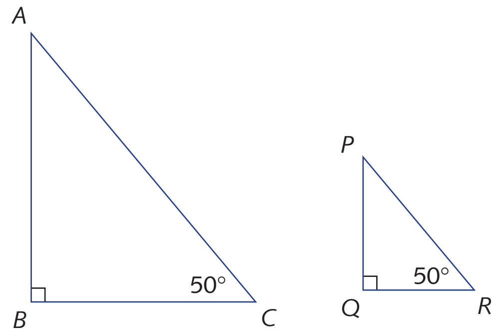
## Exercise 9B

Example 1

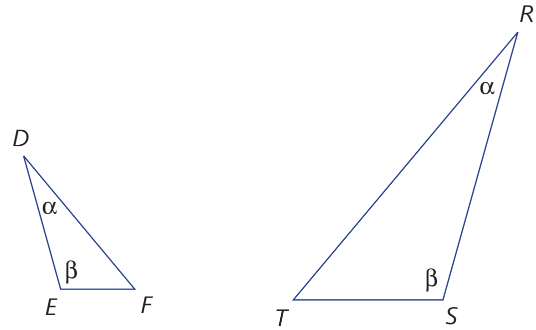
**1** The diagram in each part shows two figures that are known to be similar. Copy and complete the statements below each diagram.

**a**



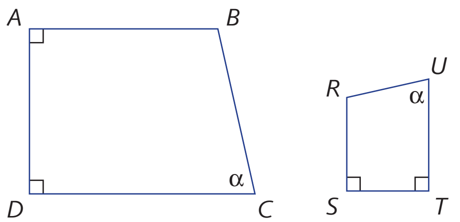

**i** $\frac{PR}{\dots} = \frac{PQ}{\dots}$ **ii** $\frac{QR}{\dots} = \frac{RP}{\dots}$

**b**



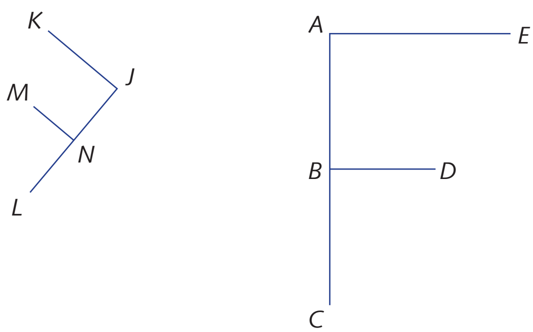

**i** $\frac{DE}{\dots} = \frac{DF}{\dots}$ **ii** $\frac{FE}{\dots} = \frac{FD}{\dots}$

**c**




**i** $\frac{AB}{\dots} = \frac{AD}{\dots}$ **ii** $\frac{CB}{\dots} = \frac{CD}{\dots}$

**d**

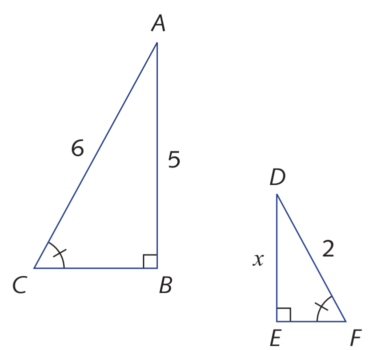



**i** $\frac{JK}{\dots} = \frac{MN}{\dots}$ **ii** $\frac{JL}{\dots} = \frac{MN}{\dots}$

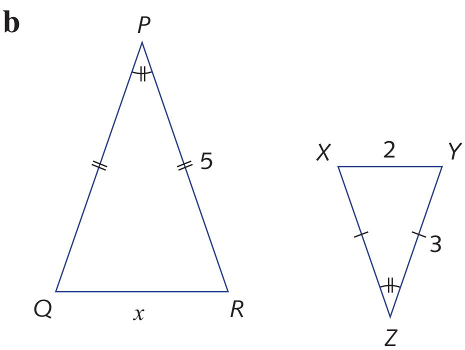
Example 1c

**2** The diagram in each part shows two figures that are known to be similar. Write a ratio statement. Hence find the value of the pronumeral.

**a**




**b**





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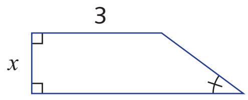
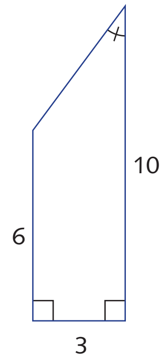
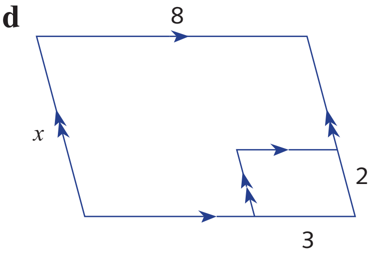
**c**




**d**




**3** The diagram below shows some of the keys of a piano.

By measuring lengths with your ruler, find approximately:

**a** the ratio of the lengths of the black and the white keys

**b** the ratio of the widths of the black and white keys at their front edges

**c** the ratio of the length of a white key and the width of its front edge

**d** the width of its front edge if the length of a white key is 13.3 cm

*In questions **4** to **6**, assume that the triangles are named in matching order.*

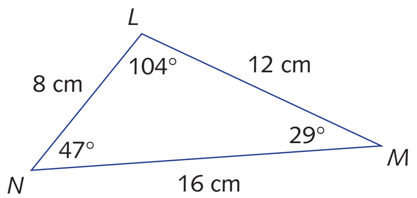
**4** $\Delta LMN$ is similar to $\Delta TUV$, where $\Delta LMN$ is shown below, and $TU = 24$ cm, find:

**a** $\angle TUV$

**b** $UV$

**c** $\angle VTU$

**d** $VT$



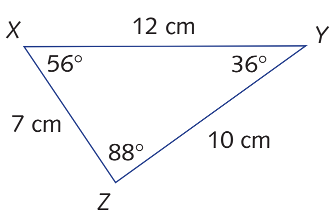

**5** $\Delta XYZ$ is similar to $\Delta GHK$, where $\Delta XYZ$ is shown below, and $HK = 18$ cm, find:

**a** $\angle GHK$

**b** $\angle GKH$

**c** $GH$

**d** $GK$




**6** In two triangles $ABC$ and $LMN$, $AB = 6$ cm, $LM = 12$ cm, $LN = 8$ cm and $MN = 16$ cm. What is the similarity ratio between the two triangles if:

**a** $\Delta ABC$ is similar to $\Delta LMN$?

**b** $\Delta ABC$ is similar to $\Delta LNM$?

**c** $\Delta ABC$ is similar to $\Delta MNL$?

**d** $\Delta ABC$ is similar to $\Delta MLN$?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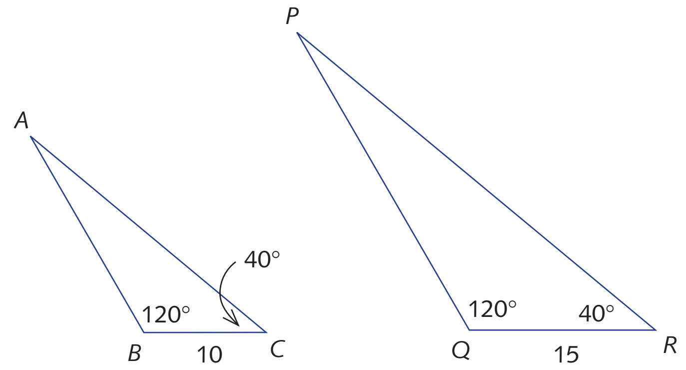
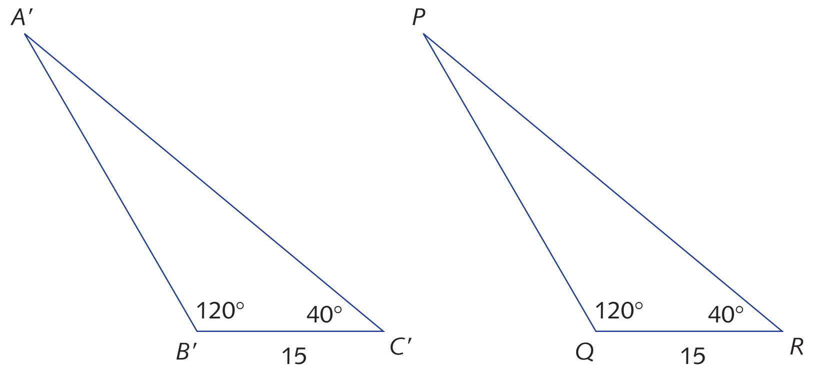
# 9C The AAA similarity test for triangles

As with congruence, most of the problems involving similarity come down to establishing that two triangles are similar. In this section and the next two sections, we shall establish four **similarity tests** for triangles and use them in problems.

The two triangles below have the same interior angles, but have different sizes.




Let us now enlarge the left-hand triangle by an enlargement factor of $\frac{3}{2}$ so that $B'C' = QR$, as in the diagram following. We have seen that the angles do not change, and that all three side lengths increase by the same factor.




By the AAS congruence test, the new triangle $\Delta A'B'C$ is congruent to $\Delta PQR$.

This is because the two triangles have two pairs of angles equal, and the matching sides $B'C'$ and $QR$ are also equal.

Hence the original triangle $\Delta ABC$ is similar to $\Delta PQR$, because its enlargement $\Delta A'B'C$ is congruent to $\Delta PQR$.

### The AAA similarity test

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

*Note*: When using this test, it is sufficient to prove the equality of just two pairs of angles – the third pair must then also be equal since the angle sum of any triangle is 180°. Thus the test is sometimes called 'the AA similarity test'.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## Similarity statements

Arguments using similar triangles depend on a similarity statement that two triangles are similar. As with congruence, always give the similarity test in brackets afterwards, and be particularly careful to name the vertices in matching order.

Thus the conclusion of our argument above is written as:

$\triangle ABC$ is similar to $\triangle PQR$ (AAA)

There are two common notations for similarity. These are:

$\triangle ABC \sim \triangle PQR$ and $\triangle ABC \parallel\!\mid \triangle PQR$

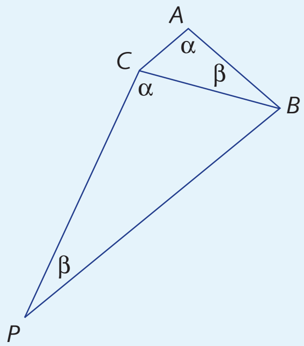
These are both read as 'triangle $ABC$ is similar to triangle $PQR$'.

### Example 3

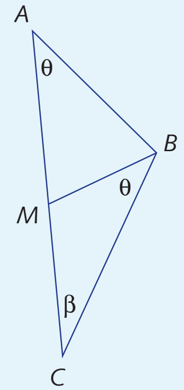
In each diagram below, write a similarity statement beginning with '$\triangle ABC$ is similar to . . .'

Be careful to name the vertices in matching order.

**a**




**b**




### Solution

**a** $\triangle ABC$ is similar $\triangle CPB$. (AAA)

**b** $\triangle ABC$ is similar $\triangle BMC$. (AAA)

Notice that in part **b** the angle $\angle C$ is **common** to both triangles.

## Proving similarity

In many geometric problems, we need to prove that two triangles are similar. The proofs are set out just like congruence proofs.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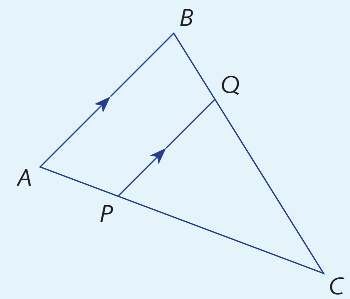
### Example 4

Prove that the two triangles in the diagram to the right are similar.




### Solution

In the triangles $ABC$ and $PQC$:

| | |
|---|---|
| $\angle BAC = \angle QPC$ | (corresponding angles, $AB \parallel PQ$) |
| $\angle C = \angle C$ | (common) |
| so $\Delta ABC$ is similar to $\Delta PQC$. | (AAA) |

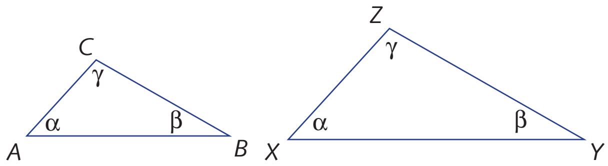
## Using similar triangles to find lengths

Once we have established that two triangles are similar, we know that matching sides have the same ratio.

If $\Delta ABC$ is similar to $\Delta XYZ$, then:

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$$

This ratio is the **similarity ratio**.

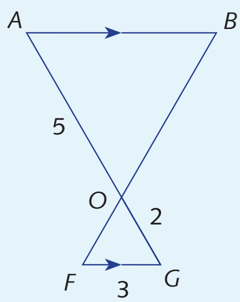



### Example 5

In the diagram to the right:

**a** Prove that $\Delta ABO$ is similar to $\Delta GFO$.

**b** Hence find the length $AB$.





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### Solution

**a** In the triangles $ABO$ and $GFO$:

$\angle AOB = \angle GOF$ (vertically opposite at $O$)

$\angle A = \angle G$ (alternate angles, $AB \parallel FG$)

so $\Delta ABO$ is similar to $\Delta GFO$. (AAA)

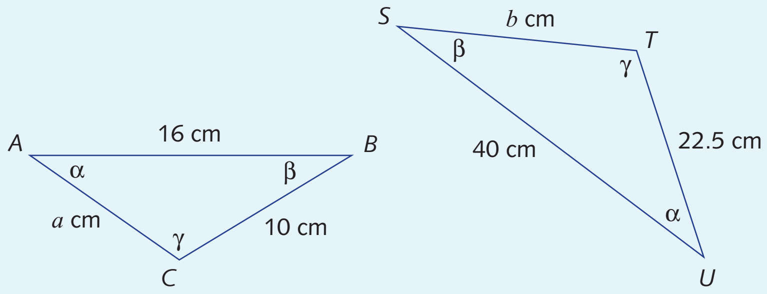
**b** Hence $\dfrac{AB}{3} = \dfrac{5}{2}$ (matching sides of similar triangles)

$$AB = 3 \times \frac{5}{2}$$

$$= 7\tfrac{1}{2}$$

### Example 6

Find the values of $a$ and $b$ in the following diagrams.




### Solution

$\Delta ABC$ is similar to $\Delta UST$. (AAA)

Hence $\dfrac{ST}{BC} = \dfrac{SU}{BA}$ (matching sides of similar triangles)

$$\frac{b}{10} = \frac{40}{16}$$

$$b = 25$$

Also $\dfrac{AC}{UT} = \dfrac{BA}{SU}$ (matching sides of similar triangles)

$$\frac{a}{22.5} = \frac{16}{40}$$

So $a = 9$

Alternatively, since $\dfrac{SU}{BA} = \dfrac{40}{16} = 2.5$, the similarity ratio from $\Delta ABC$ to $\Delta UST$ is 2.5; that is, every length in $\Delta UST$ will be 2.5 times the matching length in $\Delta ABC$.

Thus $ST = 2.5 \times BC$ and $UT = 2.5 \times AC$

$b = 2.5 \times 10$ $\quad$ $22.5 = 2.5 \times a$

$= 25$ $\quad$ $a = 9$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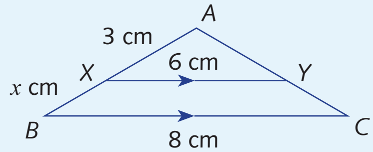
## Example 7

Find the value of $x$.




### Solution

In the triangles $ABC$ and $AXY$:

$\angle A = \angle A$ (common)

$\angle ABC = \angle AXY$ (corresponding angles, $XY \parallel BC$)

so $\Delta ABC$ is similar to $\Delta AXY$ (AAA).

Hence $\frac{AB}{AX} = \frac{BC}{XY}$ (matching sides of similar triangles)

$$\frac{x+3}{3} = \frac{8}{6} = \frac{4}{3}$$

$$x + 3 = 4$$

$$x = 1$$

## Example 8

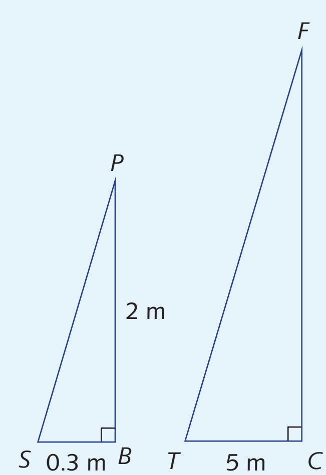
A tall vertical flagpole throws a 5 metre shadow. At the same time of day, a fencepost of height 2 metres throws a 30 cm shadow.

**a** Draw a diagram, identifying two similar triangles and proving that they are similar.

**b** Hence find the height of the flagpole.

### Solution

**a** In the diagram, $FC$ is the flagpole, $PB$ is the fencepost, and $T$ and $S$ mark the end of the shadows.




In the triangles $FCT$ and $PBS$:

$\angle C = \angle B = 90°$ (pole and post are vertical)

$\angle T = \angle S$ (both are the angle of elevation of the Sun)

so $\Delta FCT$ is similar to $\Delta PBS$. (AAA)

**b** Hence $\frac{FC}{2} = \frac{5}{0.3}$ (matching sides of similar triangles)

$FC = 33\frac{1}{3}$ metres

Thus the height of the flagpole is $33\frac{1}{3}$ metres.



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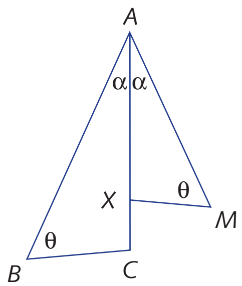
## Exercise 9C

Example 3

**1** Copy and complete each similarity statement, giving the similarity test and naming the vertices in matching order. Then copy and complete the ratio statement.

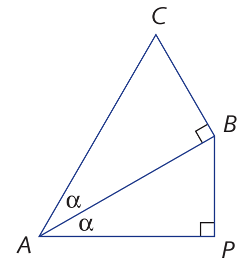
**a**




$\Delta ABC$ is similar to … (…)

$$\frac{AB}{\ldots} = \frac{BC}{\ldots}$$

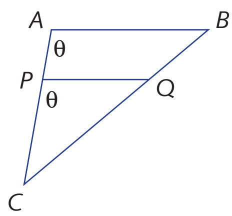
**b**




$\Delta ABC$ is similar to … (…)

$$\frac{AB}{\ldots} = \frac{BC}{\ldots}$$

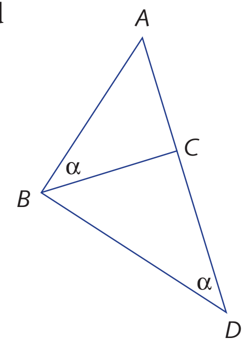
**c**




$\Delta ABC$ is similar to … (…)

$$\frac{AB}{\ldots} = \frac{BC}{\ldots}$$

**d**



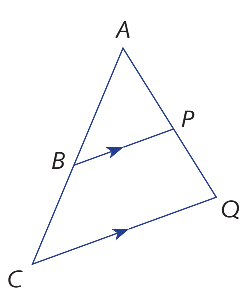

$\Delta ABC$ is similar to … (…)

$$\frac{AB}{\ldots} = \frac{BC}{\ldots}$$

Example 4

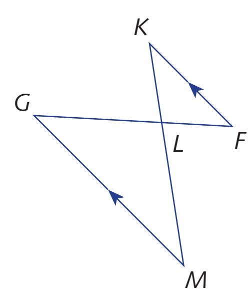
**2** Prove that the two triangles in each diagram are similar, naming the vertices in matching order. Then copy and complete the ratio statement.

**a**




$$\frac{AB}{\ldots} = \frac{AP}{\ldots}$$

**b**




$$\frac{GL}{\ldots} = \frac{GM}{\ldots}$$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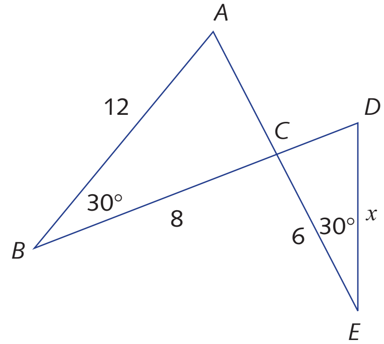
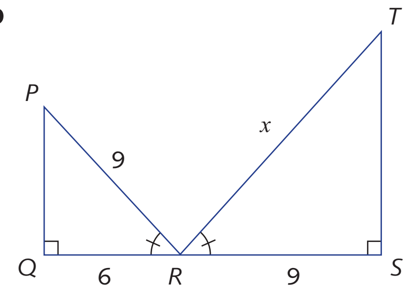
Example 5, 6, 7

**3** Prove that the two triangles in each diagram are similar. Then write a ratio statement and find the value of the pronumeral.

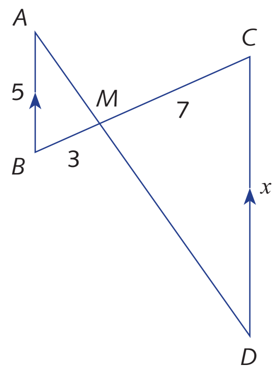
**a**



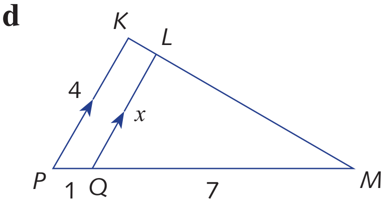

**b**



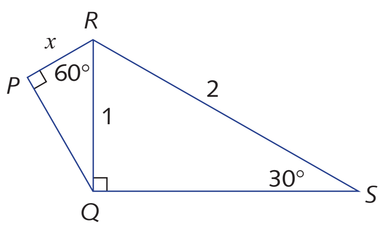

**c**



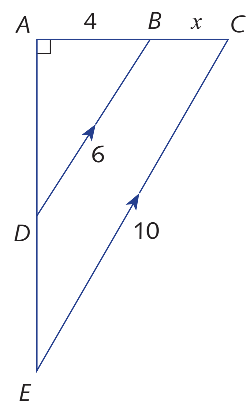

**d**




**e**




**f**




Example 8

**4** In each part, you will need to draw a diagram and prove that two triangles are similar.

**a** A stick 1.5 metres long leans against a wall and reaches 1 metre up the wall. How far up the wall will a ladder 10 metres long reach if it is parallel to the stick?

**b** A 3 metre high road-sign casts a shadow 1.4 metres long. How high is a telegraph pole that casts a shadow 3.5 metres long at the same time of day?

**c** A steep road rises 1.2 metres for every 6 metres travelled along the road surface. How far will a car have to travel to rise 600 metres?

**d** The front of an A-frame hut is an isosceles triangle of height 5 metres and base 8 metres. How wide will a model of the hut be if its height is 0.8 metres?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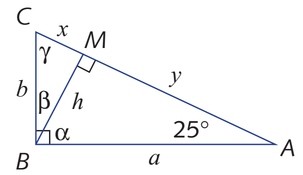
**5** **a** Find the sizes of $\alpha$, $\beta$ and $\gamma$ in the diagram on the right.




**b** The three triangles in the diagram are similar.

Copy and complete the statement:

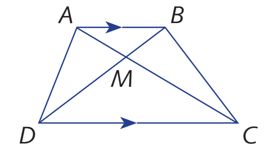
$\Delta ABC$ is similar to … which is similar to … (AAA).

**c** Hence copy and complete:

**i** $\frac{a}{b} = \frac{\dots}{\dots} = \frac{\dots}{\dots}$ **ii** $\frac{h}{a} = \frac{\dots}{\dots} = \frac{\dots}{\dots}$

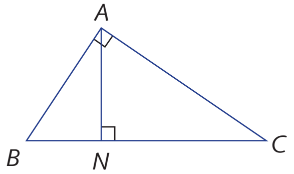
**6** Prove that: *The diagonals of a trapezium dissect the trapezium into four triangles, two of which are similar*.

Let the diagonals of the trapezium $ABCD$ meet at $M$. Identify the two similar triangles and prove that they are similar.




**7** Prove that: *The altitude to the hypotenuse of a right-angled triangle divides the hypotenuse into two intervals whose product equals the square of the altitude*.

Let $AN$ be the altitude to the hypotenuse $BC$ of the right-angled triangle $\Delta ABC$.

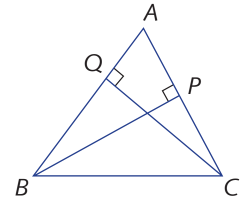



**a** Prove that $\Delta ABN$ is similar to $\Delta CAN$.

**b** Hence prove that $AN^2 = BN \times CN$.

**8** Let $ABC$ be a triangle, and let $BP$ and $CQ$ be altitudes.

**a** Prove that $\Delta ABP$ is similar to $\Delta ACQ$.




**b** Hence prove that $\frac{BP}{CQ} = \frac{AB}{AC}$.

**c** As an alternative approach, explain why $\frac{1}{2} \times AC \times BP$ and $\frac{1}{2} \times AB \times CQ$ are both equal to the area of $\Delta ABC$.

**d** Hence prove that $\frac{BP}{CQ} = \frac{AB}{AC}$.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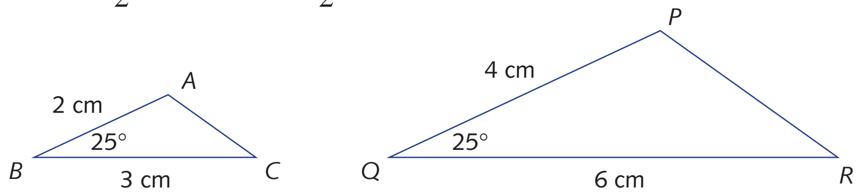
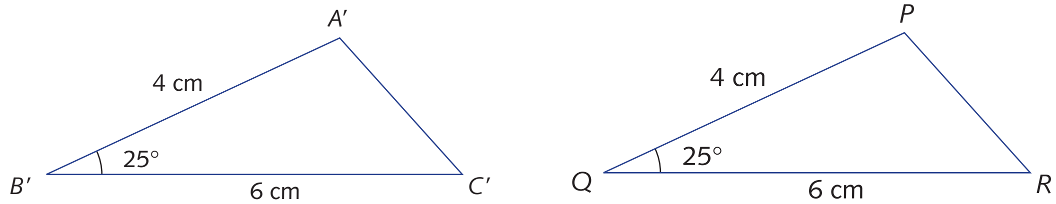
# 9D The SAS similarity test for triangles

The two triangles below have a pair of equal angles, and the sides including this angle are in the same ratio 1 : 2.

$$\angle B = \angle Q, AB = \frac{1}{2}PQ \text{ and } BC = \frac{1}{2}QR.$$




Using the same procedure as before, we can enlarge the first triangle by a factor of 2 to produce a new triangle $\Delta A'B'C'$, and $\Delta A'B'C'$ is congruent to $\Delta PQR$ by the SAS congruence test.




Hence the original triangle $\Delta ABC$ is similar to $\Delta PQR$.

This gives us our second similarity test.

**The SAS similarity test**

If the ratio of the lengths of two pairs of matching sides are equal and the included angles are equal then the two triangles are similar.

The statement that the two triangles above are similar is thus written as

$\Delta ABC$ is similar to $\Delta PQR$ (SAS).

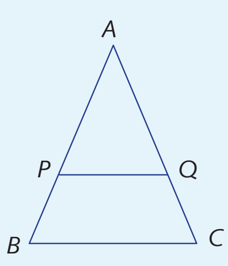
**Example 9**

In triangle $ABC$,

$AB = AC = 10$ and $AP = AQ = 7$.

**a** Prove that $\Delta ABC$ is similar to $\Delta APQ$.

**b** Prove that $PQ$ is parallel to $BC$.



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**Solution**

**a** In the triangles $ABC$ and $APQ$:

$\frac{AB}{AP} = \frac{10}{7}$ (given)

$\frac{AC}{AQ} = \frac{10}{7}$ (given)

$\angle A = \angle A$ (common)

so $\Delta ABC$ is similar to $\Delta APQ$ (SAS).

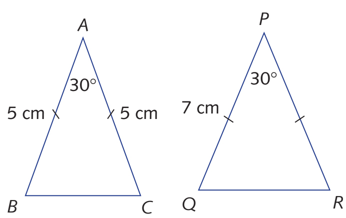
**b** $\angle ABC = \angle APQ$ (matching angles of similar triangles)

So $BC \parallel PQ$. (corresponding angles are equal)

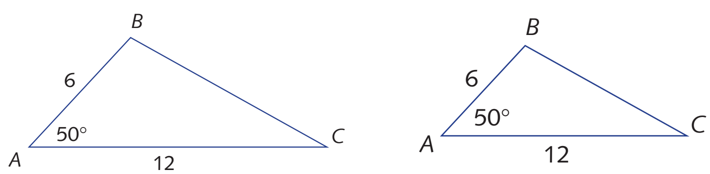
## Exercise 9D

**1** Prove that the two triangles in each diagram are similar, and write the similarity statement and the similarity factor (left to right).

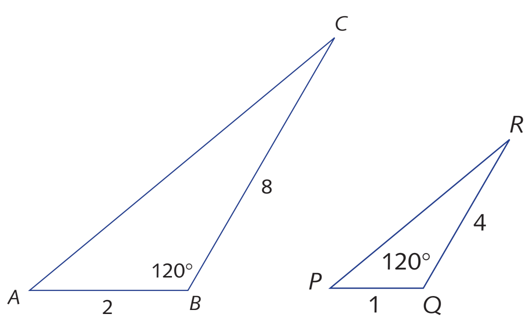
**a**



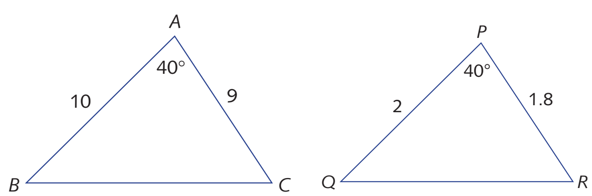

**b**




**c**



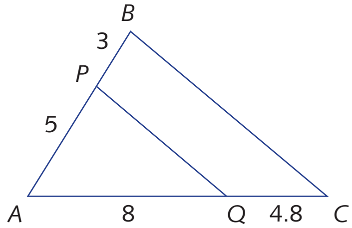

**d**



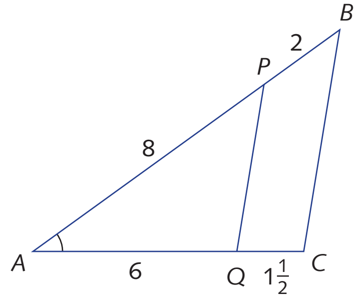

Example 9

**2** In each part below, prove that $\Delta ABC$ is similar to $\Delta APQ$. Then prove that $PQ$ is parallel to $BC$.

**a**




**b**





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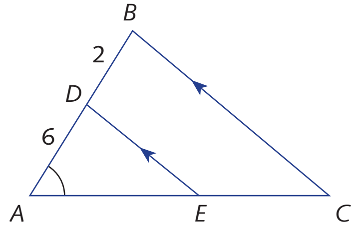
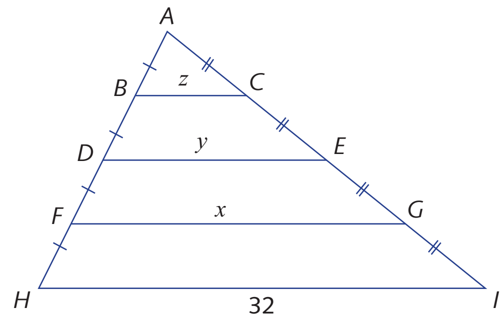
**3** Prove that $\Delta ABC$ is similar to $\Delta ADE$ and copy and complete the ratio statement.




$$\frac{AB}{\ldots} = \frac{AC}{\ldots} = \ldots$$

**4** Prove that triangles $ABC$, $ADE$, $AFG$, and $AHI$ are similar and find $x$, $y$ and $z$.




**5** In triangles $PQR$ and $XYZ$, $\angle P = \angle Z$ and $PQ \times YZ = PR \times XZ$. Name the other pairs of equal angles in the triangles.

**6** Prove that if $ABCD$ is a quadrilateral such that $\angle A = \angle C$ and $\frac{DA}{AB} = \frac{BC}{CD}$, then $ABCD$ is a parallelogram.

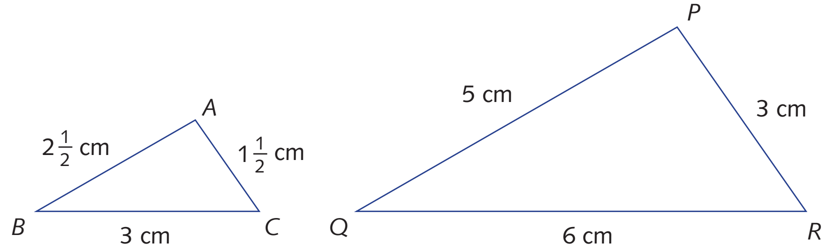
# 9E The SSS and RHS similarity tests

We now introduce two further similarity tests, giving four tests altogether.

## The SSS similarity test

The two triangles below have different sizes, but their sides can be paired up so that matching sides have the same ratio 1 : 2.

$$AB = \frac{1}{2}PQ,\ BC = \frac{1}{2}QR \text{ and } CA = \frac{1}{2}RP$$





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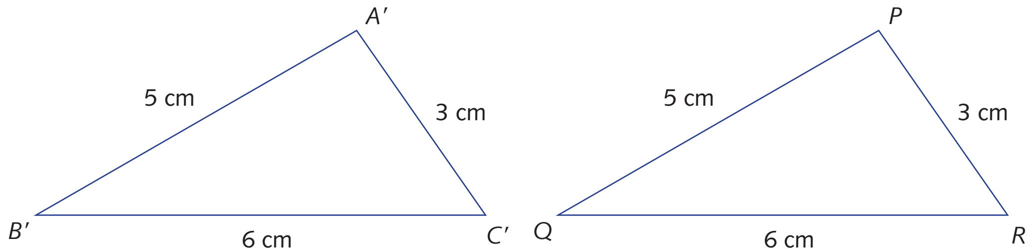
Let us now enlarge the first triangle by a factor 2, so that the side lengths all double.




By the SSS congruence test, the new triangle $\Delta A'B'C'$ is congruent to $\Delta PQR$.

Hence the original triangle $\Delta ABC$ is similar to $\Delta PQR$.

The statement that the two triangles above are similar is thus written as

$\Delta ABC$ is similar to $\Delta PQR$ (SSS).

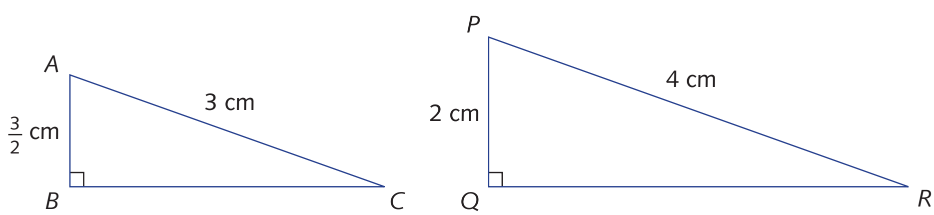
As an interesting example, the Pythagorean triples $3, 4, 5; 6, 8, 10; 9, 12, 15; \ldots$ give an infinite family of similar right-angled triangles.

**The SSS similarity test**

If we can match up the sides of one triangle with the sides of the other so that the ratio of matching lengths is constant, then the triangles are similar.

## The RHS similarity test

The two triangles below are both right-angled, and the hypotenuse and one side are in the same ratio $3 : 4$.




Once again, we can enlarge the first triangle by a factor of $\frac{4}{3}$ to produce a triangle $\Delta A'B'C'$, and $\Delta A'B'C'$ is congruent to $\Delta PQR$ by the RHS congruence test.

Hence the original triangle $\Delta ABC$ is similar to $\Delta PQR$.

This gives us our fourth and last similarity test.

**The RHS similarity test**

If the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides, then the two triangles are similar.

The statement that the two triangles above are similar is thus written as

$\Delta ABC$ is similar to $\Delta PQR$ (RHS).

Recall that given the hypotenuse and one side of a right-angled triangle, the third side can be found by Pythagoras' theorem.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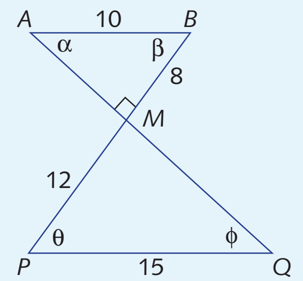
**Example 10**

**a** Prove that the two triangles in the diagram to the right are similar.

**b** Identify the equal angles in two triangles.

**c** Prove that $AB \parallel PQ$.




**Solution**

**a** In the triangles $ABM$ and $QPM$

$\angle AMB = \angle QMP = 90°$ (vertically opposite)

$\frac{AB}{QP} = \frac{2}{3} = \frac{BM}{PM}$ (given)

so $\Delta ABM$ is similar to $\Delta QPM$ (RHS similarity).

**b** Hence $\alpha = \phi$ and $\beta = \theta$ (matching angles of similar triangles).

**c** Hence $AB \parallel PQ$ (alternate angles are equal).

## Using the four similarity tests in problems

Here are examples using the AAA, SSS, SAS and RHS similarity tests in problems.

**Example 11**

**a** Prove that the two triangles in the diagram are similar.

**b** Which of the marked angles are equal?

**Solution**

**a** In the triangles $\Delta ABC$ and $\Delta CBD$:

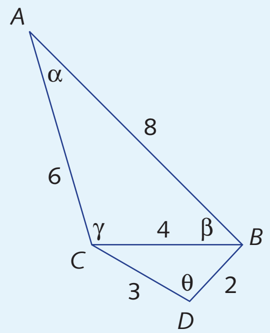
$AB = 2 \times CB$ (given)

$BC = 2 \times BD$ (given)

$CA = 2 \times DC$ (given)

so $\Delta ABC$ is similar to $\Delta CBD$ (SSS).

**b** Hence $\gamma = \theta$ (matching angles of similar triangles).



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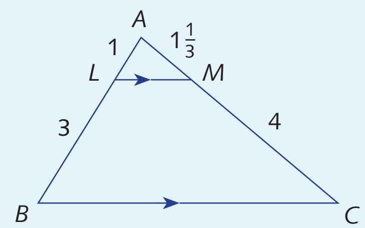
## Example 12

**a** Prove that the two triangles in the diagram are similar.

**b** Hence prove that $LM \parallel BC$.

### Solution




**a** In the triangles $\Delta ABC$ and $\Delta ALM$:

$AB = 4 \times AL$

$AC = 4 \times AM$

$\angle BAC = \angle LAM$ (common)

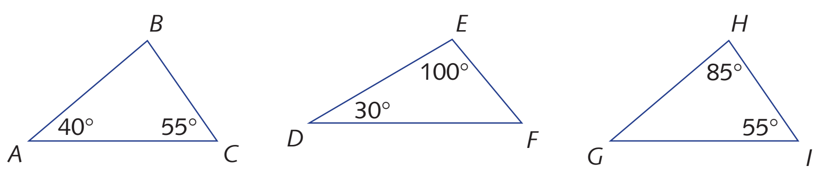
so $\Delta BAC$ is similar to $\Delta LAM$ (SAS).

**b** Hence $\angle ABC = \angle ALM$ (matching angles of similar triangles) and so $LM \parallel BC$ (corresponding angles are equal).

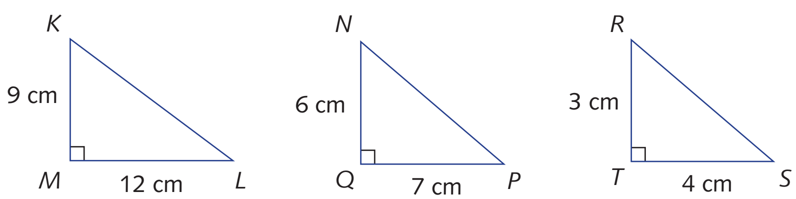
## Exercise 9E

**1** Name the pair of similar triangles in each part, naming the vertices in matching order and stating the similarity test.

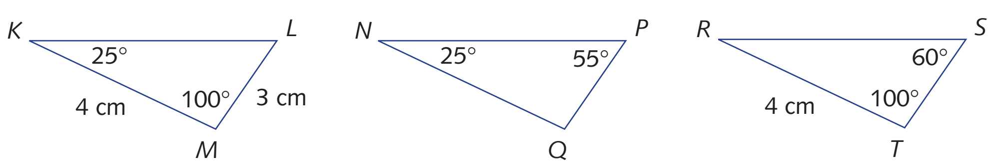
**a**



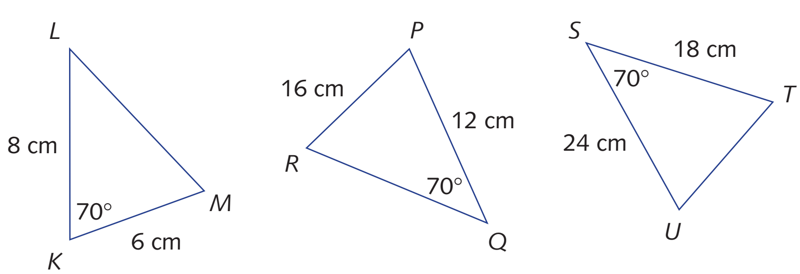

**b**




**c**




**d**





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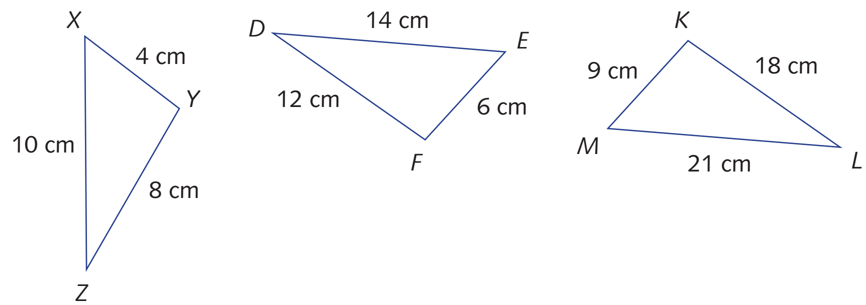
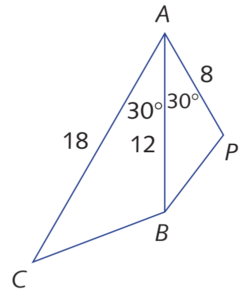
**e**




**2** Copy and complete each similarity statement, stating the similarity test, and naming the vertices in matching order. They copy and complete the ratio statement.

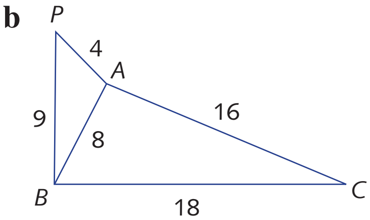
**a**




$\Delta ABC$ is similar to … (…)

$\frac{AB}{\ldots} = \frac{BC}{\ldots}$

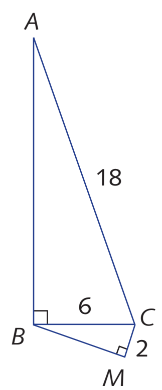
**b**




$\Delta ABC$ is similar to … (…)

$\frac{AB}{\ldots} = \frac{BC}{\ldots}$

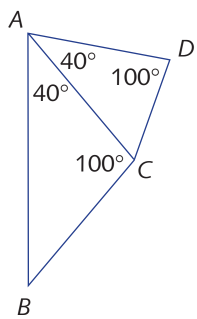
**c**




$\Delta ABC$ is similar to … (…)

$\frac{AB}{\ldots} = \frac{BC}{\ldots}$

**d**

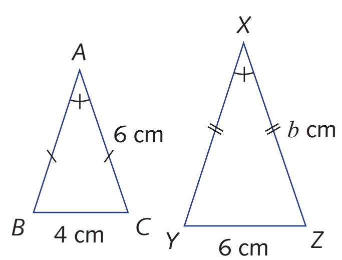



$\Delta ABC$ is similar to … (…)

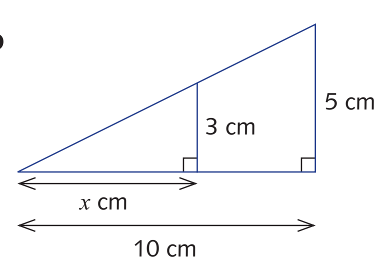
$\frac{AB}{\ldots} = \frac{BC}{\ldots}$

**3** Find the values of the pronumerals in each part.

**a**




**b**



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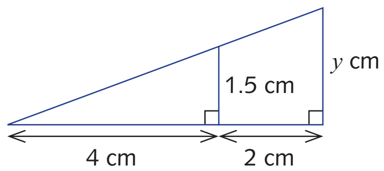
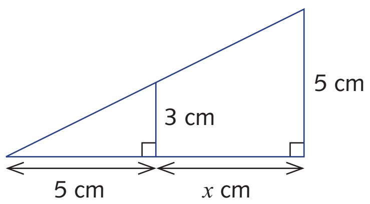
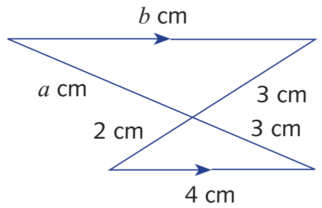
**c**




**d**




**e**

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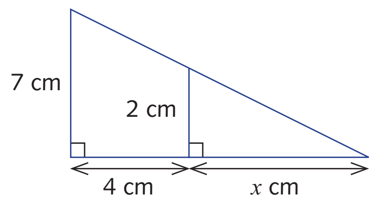



**4** **a** Find the size of:

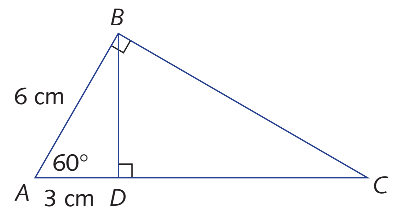
**i** $\angle ABD$ **ii** $\angle ACB$

**b** Justify the statement that $\Delta ABC$ is similar to $\angle ADB$.

**c** Find $AC$.

**d** Show that $BD = 3\sqrt{3}$ cm.

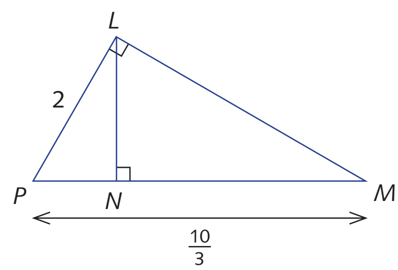
**e** Find $BC$.



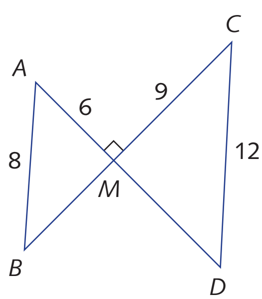

**5** **a** Find $LM$.

**b** Find $LN$.

**c** Find $MN$.




**6** **a**

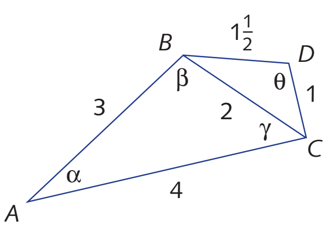



**i** Prove that $\Delta ABM$ is similar to $\angle CDM$.

**ii** Hence prove that $\angle B = \angle D$.

**iii** Is it true that $AB \parallel CD$?

**b**




**i** Prove that $\Delta ABC$ is similar to $\Delta\ldots$

**ii** Which of $\alpha$, $\beta$ and $\gamma$ is equal to $\theta$?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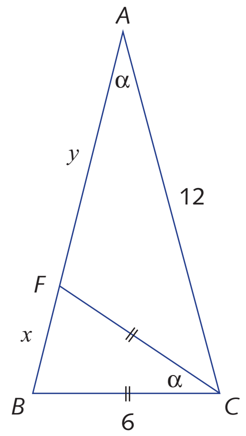
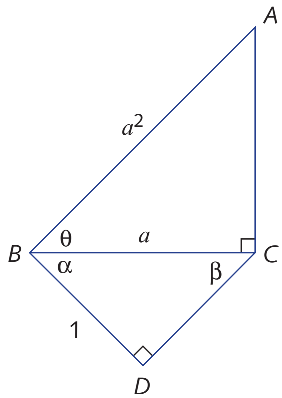
**c**




**i** Prove that $\Delta ABC$ is similar to $\Delta CBF$.

**ii** Hence find $x$ and $y$.

**d**




**i** Prove that $\Delta ABC$ is similar to $\Delta$…

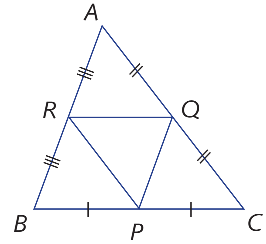
**ii** Is α, or β equal to θ?

**7** **a** Prove that: *The line interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.*

**b** Prove that: *The line through the midpoint of one side of a triangle parallel to another side of the triangle meets the third side at its midpoint.*

**c** Prove that: *The midpoints of the sides of a quadrilateral form the vertices of a parallelogram.* Let $ABCD$ be a quadrilateral, and let $P$, $Q$, $R$ and $S$ be the midpoints of the sides $AB$, $BC$, $CD$ and $DA$ respectively. Construct diagonals $AC$ and $BD$.

**8** Prove that: *The intervals joining the midpoints of the sides of a triangle dissect the triangle into four congruent triangles each similar to the original triangle.*




**9** If you think the statement is true, justify your answer using a similarity test. If you think it is false, draw two triangles that provide a counter-example.

**a** Any two equilateral triangles are similar.

**b** Any two isosceles triangles are similar.

**c** Any two isosceles triangles with equal apex angles are similar.

**d** Any two isosceles triangles with equal length bases are similar.

**e** Any two right-angled triangles are similar.

**f** Any two right-angled triangles with equal length hypotenuses are similar.

**g** Any two isosceles right-angled triangles are similar.

**h** Any two right-angled triangles in which one other angle is 40° are similar.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**10** If you think the statement is true, justify your answer. If you think it is false, draw two figures that provide a counterexample.

**a** Any two squares are similar.

**b** Any two rectangles are similar.

**c** Any two rectangles in which one side is three times another are similar.

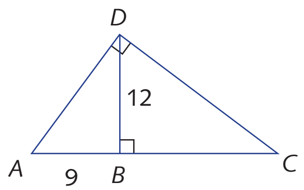
**d** Any two rectangles whose diagonals meet at 30° are similar.

**e** Any two rectangles whose diagonals have length 6 cm are smilar.

**f** Any two rhombuses are similar.

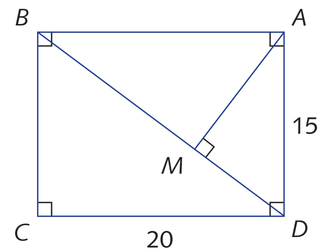
**g** Any two rhombuses with vertex angles 50° and 130° are similar.

**11** **a**




Find $AD$, $DC$ and $BC$.

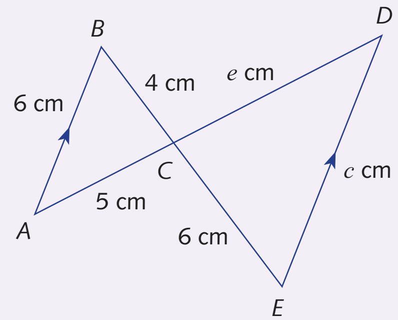
**b**



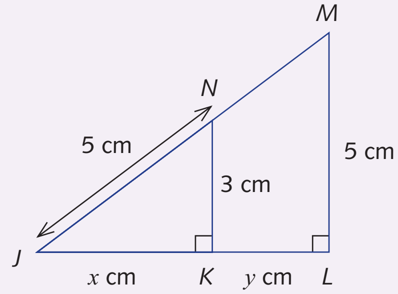

Find $AM$, $BM$ and $DM$.

# Review exercise

**1** **a** Prove that triangle $ABC$ is similar to triangle $DEC$ and find the value of $e$ and $c$.




**b** Prove that triangle $JNK$ is similar to triangle $JML$ and find the values of $x$ and $y$.



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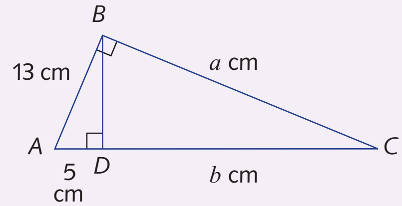
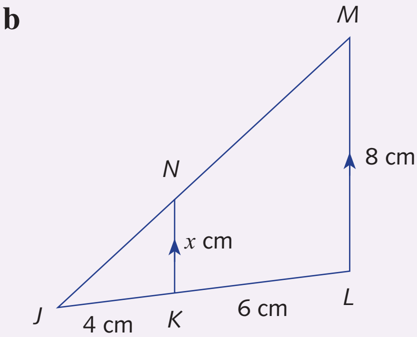
**2** Write down a pair of similar triangles and find the value of each pronumeral.

**a**

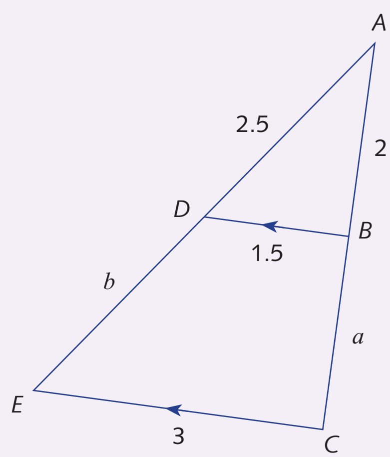



**b**

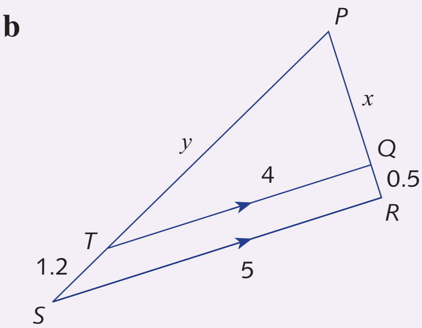



**3** Write down a pair of similar triangles and find the value of each pronumeral.

**a**

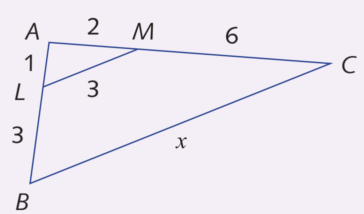



**b**




**4** A vertical stick of length 30 cm casts a shadow of length 5 cm at 12 p.m. Find the length, in centimetres, of the shadow cast by a 1 metre ruler placed in the same position at the same time of the day.

**5** **a**

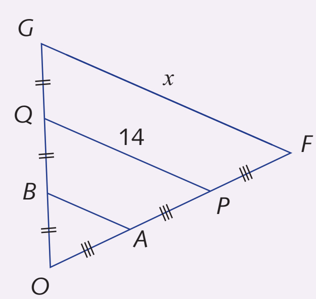



**i** Prove that $\Delta ALM$ is similar to $\Delta ABC$.

**ii** Hence prove that $LM \parallel BC$.

**iii** Find $x$.

**b**




**i** Prove that $\Delta OPQ$ is similar to $\Delta OFG$.

**ii** Hence prove that $PQ \parallel FG$.

**iii** Find $x$.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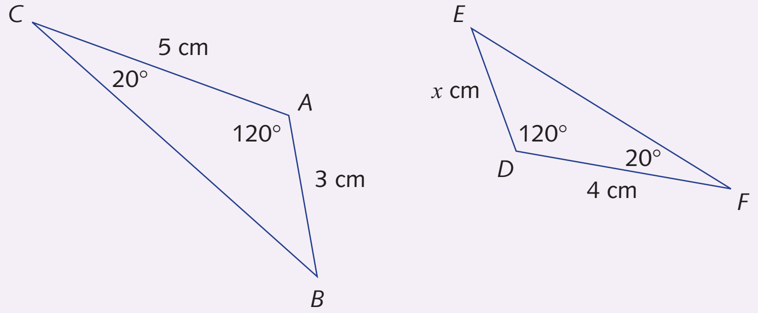
**6** Prove that $\Delta ABC$ is similar to $\Delta DEF$.

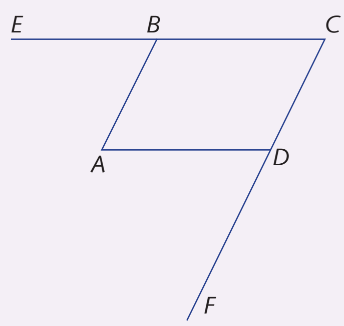
Find $x$.




**7** A tower standing on level ground casts a shadow 40 m long. A vertical stick 3 m high is placed at the tip of the shadow. The stick is found to cast a shadow 6 m long. Find the height of the tower.

**8** A line from the top of a church steeple to the ground just passes over the top of a pole 2.5 m high and meets the ground at a point $A$, 1.5 m from the base of the pole. If the distance from $A$ to a point directly below the church steeple is 30 m, find the height of the steeple.

**9** $ABCD$ is a parallelogram with $\angle BAD$ acute. The point $E$ lies on the ray $CB$ such that $\Delta ABE$ is isosceles with $AB = AE$. The point $F$ lies on the ray $CD$ such that $\Delta ADF$ is isosceles with $AD = AF$.




**a** Prove that $\Delta ABE$ is similar to $\Delta ADF$.

**b** Prove that $DE = BF$.

**10** $ABC$ is a triangle, $M$ is a point in the interval $AB$ such that $AB = 3AM$ and $N$ is a point in the line segment $AC$ such that $AC = 3AN$.

**a** Prove that $BC \parallel MN$.

**b** If $BN$ and $CM$ intersect at $P$, prove that $BP = 3PN$.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# Challenge exercise

**1** You proved in Exercise 9E, Question **8** that the midpoints of the sides of a quadrilateral form the vertices of a parallelogram. Now prove that this parallelogram has area half the area of the original quadrilateral.

**2** Let $ABCD$ be a trapezium with $AB \parallel CD$. Let the diagonals $AC$ and $BD$ meet at $O$.

Let the line through $O$ parallel to $AB$ meet $AD$ at $P$ and $BC$ at $Q$.

By considering the pairs of triangles $ADB$ and $POD$, $BDC$ and $QOB$, and $AOB$ and $COD$, prove that $O$ is the midpoint of $PQ$.

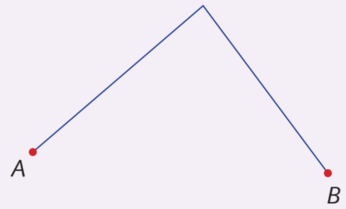
**3** Let $ABCD$ be a rectangle, and let $K$ and $L$ be the midpoints of $AB$ and $CD$ respectively. Let $AC$ meet $KL$ at $M$.

**a** Show that $M$ is the midpoint of $KL$.

**b** Let $DK$ meet $AC$ at $X$. Find the ratio $AX : XC$.

**c** Find the ratio of the area of $\Delta KXA$ to the area of the rectangle $ABCD$.

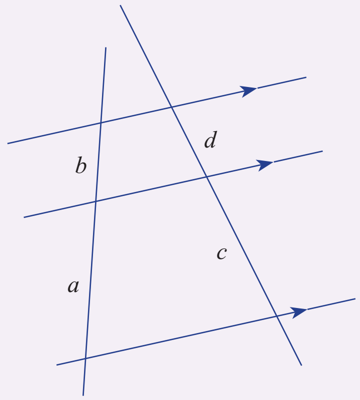
**4** Take any two points $A$ and $B$ in the plane and a piece of string longer than the distance $AB$. Imagine fixing one end of the string at $A$ and the other at $B$, and pulling the string taut by using a pencil.




Now move the pencil around keeping the string taut and letting the string slide around the point of the pencil. In this way we trace out a curve called an **ellipse**.

Prove that an enlargement transformation takes an ellipse to another ellipse.

**5** Take three parallel lines and any two transversals that are not parallel.

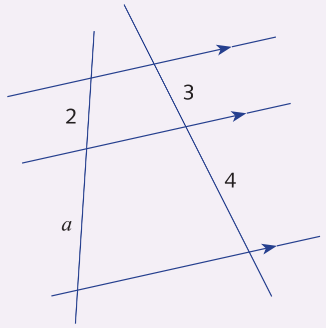



**a** Prove that the ratio of the lengths cut off by the parallel lines is the same for both transversals.

That is, prove that $\frac{a}{b} = \frac{c}{d}$ in the diagram opposite.

**b** Is the result still true if the transversals are parallel?

**c** In the diagram below, find $a$.



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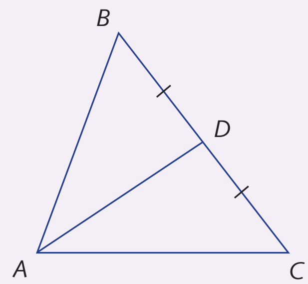
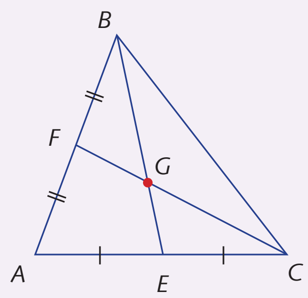
**6** An **altitude** of a triangle is an interval from a vertex meeting the opposite side at right angles. Prove that if two altitudes of a triangle are equal, then the triangle is isosceles.

**7** In this question we will prove that: *The medians of a triangle are concurrent and trisect each other*.




A **median** of a triangle is the line which joins a vertex to the middle of the opposite side.

**a** Let the medians $BE$ and $CF$ meet at $G$. Show that $BG = 2GE$ and $CG = 2GF$.




**b** Deduce that the third median $AD$ also passes through $G$. The point $G$ is called the **centroid** of the triangle $ABC$.

**c** Prove that if two of the medians of a triangle have equal length, then the triangle is isosceles.

**8** **Theorem**

Let $ABC$ and $PQR$ be equiangular triangles, with $\angle A = \angle P = \alpha$, $\angle B = \angle Q = \beta$, $\angle C = \angle R = \gamma$. Let $BC = a$, $CA = b$ and $AB = c$, and let $QR = ka$, where $k$ is a positive rational number. Then $RP = kb$ and $PQ = kc$.

Use congruence (not similarity) to prove the result for $k = 2$ and then $k = 3$.

This is a beginning of the proof that the four similarity tests can be proven from the congruence tests without enlargements when the similarity factor is a rational number.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