CHAPTER

11

Measurement and Geometry

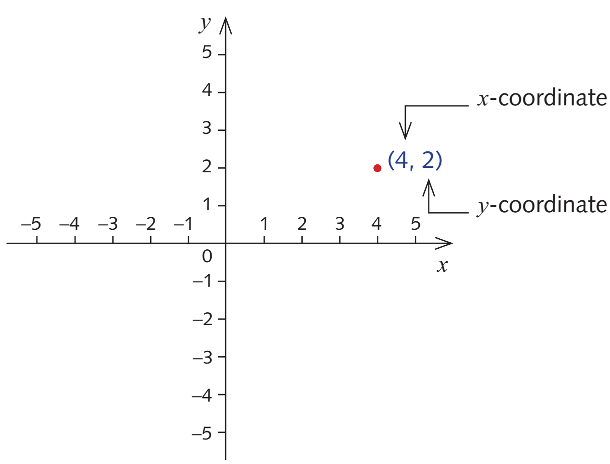
Coordinate geometry

The Cartesian or number plane is divided into four **quadrants** by two perpendicular axes called the ***x*-axis** (a horizontal line) and the ***y*-axis** (a vertical line). These axes intersect at a point called the **origin**. The position of any point in the plane can be represented by an ordered pair of numbers (x, y). These numbers are the **coordinates** of the point.

The point with coordinates (4, 2) has been plotted on the Cartesian plane shown. The coordinates of the origin are (0, 0).

René Descartes (1596–1650) introduced coordinates to show how algebra could be used to solve geometric problems and we therefore use the adjective 'Cartesian' for the number plane.

In coordinate geometry a point is represented by a pair of numbers, and a line is represented by a linear equation. In this chapter we graph points and lines and find out how to determine whether lines are perpendicular or parallel. We will learn how to calculate the distance between two points and the coordinates of the midpoint of an interval.



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11A Distance between two points

Once the coordinates of two points are known, the distance between them can easily be found.

Example 1

Find the distance between each pair of points.

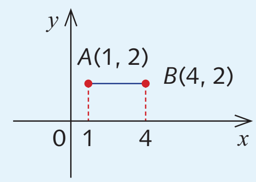
a $A(1, 2)$ and $B(4, 2)$

b $A(1, -2)$ and $B(1, 3)$

Solution

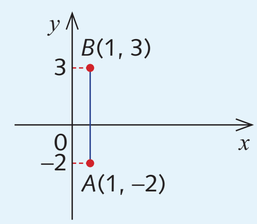
a The distance $AB = 4 - 1 = 3$

Note: The distance AB is the positive difference of the x-coordinates of the two points.



b The distance $AB = 3 - (-2) = 5$

Note: The distance AB is the positive difference of the y-coordinates of the two points.



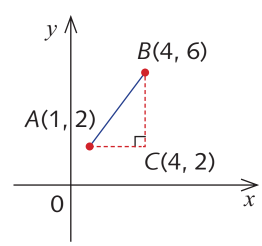
The above example involves the special cases when the interval AB is horizontal or vertical. To calculate the distance between two points when the interval between them is neither vertical nor horizontal, we use Pythagoras' theorem.

The distance between the points $A(1, 2)$ and $B(4, 6)$ is calculated below.

$AC = 4 - 1 = 3$ and $BC = 6 - 2 = 4$

By Pythagoras' theorem

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= 3^2 + 4^2 \\ &= 9 + 16 \\ &= 25 \end{aligned}$$



so $AB = \sqrt{25} = 5$

The general case

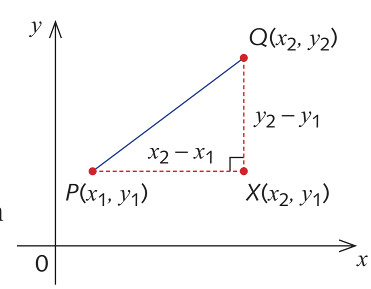
We can obtain a formula for the length of any interval.

Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points, as shown opposite.

Form the right-angled triangle PQX, where X is the point (x_2, y_1). Then

$PX = x_2 - x_1$

$QX = y_2 - y_1$



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By Pythagoras' theorem

$$PQ^2 = PX^2 + QX^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

so $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In practice, we sometimes work out the square of PQ and then take the square root.

Note: You will notice that our diagram assumes $x_2 - x_1$ and $y_2 - y_1$ are positive. If either or both of these are negative, it is not necessary to change the formula since they are squared.

Example 2

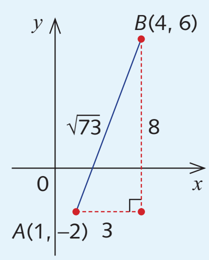
Find the distance between each pair of points.

a $A(1, -2)$ and $B(4, 6)$

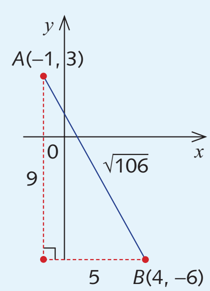
b $A(-1, 3)$ and $B(4, -6)$

Solution

a
$$AB^2 = (4-1)^2 + (6-(-2))^2$$
$$= 3^2 + 8^2$$
$$= 9 + 64$$
$$= 73$$
$$AB = \sqrt{73}$$



b
$$AB^2 = (-1-4)^2 + (3-(-6))$$
$$= (-5)^2 + 9^2$$
$$= 25 + 81$$
$$= 106$$
$$AB = \sqrt{106}$$



Example 3

Use the distance formula to find the distance between the points.

a $A(-4, -3)$ and $B(5, 7)$

b $A(7, -3)$ and $B(0, -7)$

Solution

a Let $x_1 = -4$, $x_2 = 5$, $y_1 = -3$ and $y_2 = 7$

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$= (5-(-4))^2 + (7-(-3))^2$$
$$= 9^2 + 10^2$$
$$= 181$$
$$AB = \sqrt{181}$$

b
$$AB^2 = (0-7)^2 + (-7-(-3))^2$$
$$= 49 + 16$$
$$= 65$$
$$AB = \sqrt{65}$$

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Distance between two points

The distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by:

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

or

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is just Pythagoras' theorem.

Exercise 11A

Example 1

1 Find the distance between each pair of points.

a (2, 3) and (8, 3) **b** (5, 7) and (–1, 7)

c (–1, 2) and (–1, 12) **d** (–2, –7) and (–2, 8)

Example 2, 3

2 Find the distance between each pair of points.

a (2, 3) and (8, 11) **b** (5, 7) and (8, 3)

c (–1, 0) and (4, 12) **d** (–2, –7) and (6, 8)

e (–1, –3) and (1, –1) **f** (–3, 3) and (3, 0)

g (2, –3) and (6, –5) **h** (–6, –3) and (1, –1)

3 How far are the following points from the origin?

a (2, 3) **b** (–5, 7) **c** (–1, 4) **d** (–4, –5)

4 Which of the two points $M(3, 6)$ and $N(6, -4)$ is closer to $P(-2, -1)$?

5 Show that the point $A(8, 4)$ is equidistant (that is, the same distance) from points $B(-4, -1)$ and $C(13, 16)$.

6 Given the three points $A(0, 0)$, $B(3, 4)$ and $C(6, 0)$:

a calculate the distance AB **b** calculate the distance BC

c calculate the distance AC **d** identify the type of triangle ABC

7 The points $A(5, 3)$, $B(-17, -8)$ and $C(-6, -19)$ are joined to form a triangle. Prove that the triangle is isosceles.

8 The points $A(-2, 1)$, $B(1, 3)$ and $C(7, -6)$ are joined to form a triangle. Prove that the triangle is right-angled.

9 Calculater the perimeter of ΔPQR with vertices $P(3, 1)$, $Q(8, 6)$ and $R(10, 0)$.

10 Show that the points $A(0, -5)$, $B(5, 0)$, $C(6, 7)$ and $D(1, 2)$ are the vertices of a rhombus.

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11B The midpoint of an interval

The coordinates of the midpoint of an interval can be found by averaging the coordinates of its endpoints.

Example 4

Find the coordinates of the midpoint of the interval AB, given:

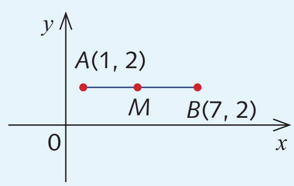
a $A(1,\ 2)$ and $B(7,\ 2)$ **b** $A(1, -2)$ and $B(1,\ 3)$

Solution

a AB is a horizontal interval since the y-coordinates of A and B are equal.

The coordinates of the midpoint of AB are (4, 2).

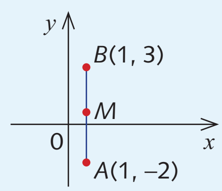
Note: 4 is the *average* of 1 and 7; that is, $4 = \dfrac{1+7}{2}$



b The midpoint of AB has coordinates $\left(1, \dfrac{1}{2}\right)$.

Note: $\dfrac{1}{2}$ is the *average* of 3 and –2;

that is, $\dfrac{1}{2} = \dfrac{3+(-2)}{2}$



What happens when the interval is not parallel to one of the axes?

The general case

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points and let $M(x, y)$ be the midpoint of the interval PQ.

The triangles PMS and MQT are congruent (AAS), so $PS = MT$ and $MS = QT$.

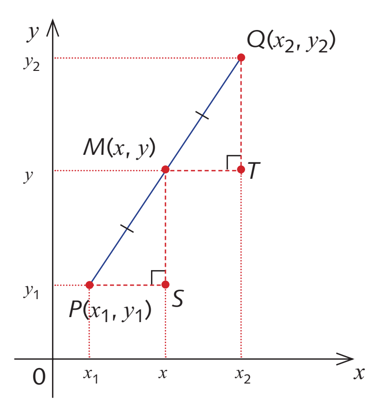
Hence the x-coordinate of M is the average of the x-value of the two points, P and Q.

$$x = \frac{x_1 + x_2}{2}$$

And the y-coordinate of M is the average of the y-values of the two points, P and Q.

$$y = \frac{y_1 + y_2}{2}$$

This provides a formula for the midpoint of any interval.



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Midpoint of an interval

The midpoint of the interval with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$ has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The coordinates are found by calculating the average of x_1 and x_2 and the average of y_1 and y_2.

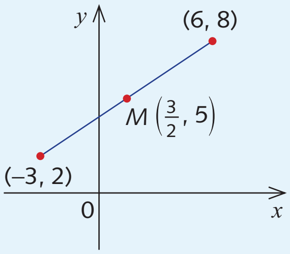
Example 5

Find the coordinates of the midpoint of the interval joining the points (6, 8) and (–3, 2).

Solution

The midpoint M has coordinates

$$\left(\frac{6+(-3)}{2}, \frac{8+2}{2}\right) = \left(\frac{3}{2}, 5\right)$$



Finding an endpoint given the midpoint and the other endpoint

Example 6

If $M(3, 6)$ is the midpoint of the interval AB and A has coordinates $(-1, 1)$, find the coordinates of B.

Solution

The coordinates of A are $(-1, 1)$.

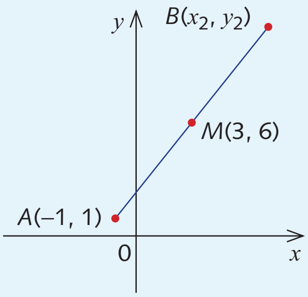
Let the coordinates of B be (x_2, y_2).

$$\frac{x_2 + (-1)}{2} = 3 \quad \text{and} \quad \frac{y_2 + 1}{2} = 6$$

$$x_2 - 1 = 6 \quad \text{and} \quad y_2 + 1 = 12$$

$$x_2 = 7 \quad \text{and} \quad y_2 = 11$$

Thus B has coordinates $(7, 11)$.



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Exercise 11B

Example 4, 5

1 Find the coordinates of the midpoint of the interval AB with endpoints:

a (2, 3) and (8, 3) **b** (5, 7) and (−1, 7)

c (−1, 0) and (4, 12) **d** (−2, −7) and (6, 8)

e (−1, −3) and (1, −1) **f** (−3, 3) and (3, 0)

g (2, −3) and (6, −5) **h** (−6, −3) and (0, −1)

Example 6

2 The point M is the midpoint of the interval AB. Find the coordinates of B given that the coordinates of A and M are:

a $A(1, 6)$ and $M(10, 6)$ **b** $A(1, 6)$ and $M(-10, 6)$

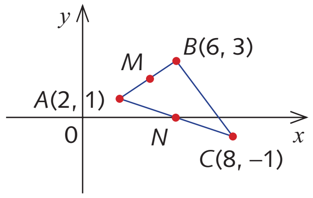
c $A(-1, 4)$ and $M(-3, 2)$ **d** $A(2, -6)$ and $M(1, 8)$

e $A(3, -1)$ and $M(-2, 10)$ **f** $A(5, -6)$ and $M(-2, -4)$

3 In each of the following, M is the midpoint of the interval AB. Fill in the missing entries.

	A	B	M
a	(1, 6)	(5, 10)	
b		(3, 6)	(9, 12)
c	(−3, −4)		(1, −1)
d	(2.3, 6.1)	(8.5, 3.2)	
e		(−1.6, −2.4)	(0.4, −1.9)
f	(−3.6, 4)		(0, 3)

4 Consider the three points $A(2, 1)$, $B(6, 3)$ and $C(8, -1)$. Let M and N be the midpoints of AB and AC respectively.



a Find the coordinates of M.

b Find the coordinates of N.

c Calculate the distance BC.

d Calculate the distance MN.

e Compare the distance BC with the distance MN.

f What is the relation between ΔABC and ΔAMN?

5 The parallelogram $PQRS$ has vertices $P(1, 6)$, $Q(5, 12)$, $R(3, 3)$ and $S(-1, -3)$.

a Find the coordinates of the midpoint of the diagonal PR.

b Find the coordinates of the midpoint of the diagonal SQ.

c What well-known property of a parallelogram does this demonstrate?

6 Use the midpoint formula to find the coordinates of three more points that lie on the line passing through the points (0, 0) and (2, 3).

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7 The triangle ABC has vertices $A(-2,\ 1)$, $B(-1,\ 3)$ and $C(7,\ -1)$.

a Find the coordinates of M, the midpoint of AC.

b Find the distance BM.

c Show that M is equidistant from A, B and C.

8 The triangle ABC has vertices $A(2,\ 1)$, $B(4,\ 5)$ and $C(6,\ 1)$.

a Find the coordinates of the midpoints M and N of sides AB and BC respectively.

b $P(4,\ 1)$ is the midpoint of side AC. Show that $BC = 2MP$ and $AB = 2PN$.

9 The triangle ABC has vertices $A(1,\ 6)$, $B(1,\ 10)$ and $C(a,\ 8)$. The area of ΔABC is 16. Find the possible values of a.

10 The quadrilateral $ABCD$ has vertices $A(1, 0)$, $B(4, 8)$, $C(11, 12)$ and $D(-1, 10)$. M, N, O and P are the midpoints of the sides AB, BC, CD and DA respectively. Find the lengths of the sides of quadrilateral $MNOP$ and describe this quadrilateral.

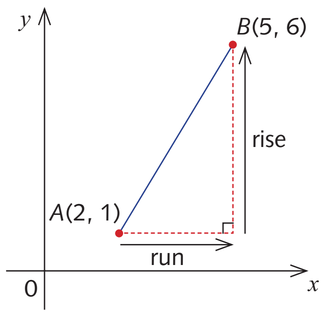
11C The gradient of a line

Gradient of an interval

The **gradient** of an interval AB is defined as $\frac{\text{rise}}{\text{run}}$, where the **rise** is the change in the y values as you move from A to B and the **run** is the change in the x values as you move from A to B.

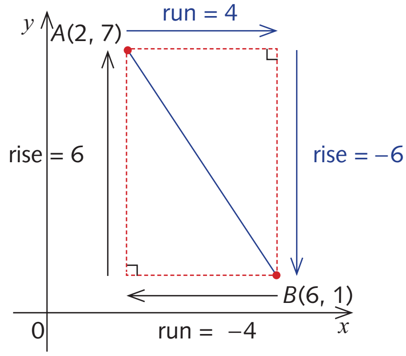
For the points $A(2,\ 1)$ and $B(5,\ 6)$,

$$\begin{aligned}\text{gradient of interval } AB &= \frac{\text{rise}}{\text{run}}\\ &= \frac{6-1}{5-2}\\ &= \frac{5}{3}\end{aligned}$$



Notice that as you move from A to B along the interval, the y value increases as the x value increases. This means the gradient is **positive**. In the diagram below,

$$\begin{aligned}\text{gradient of interval } AB &= \frac{\text{rise}}{\text{run}}\\ &= \frac{1-7}{6-2}\\ &= \frac{-6}{4}\\ &= -\frac{3}{2}\end{aligned}$$



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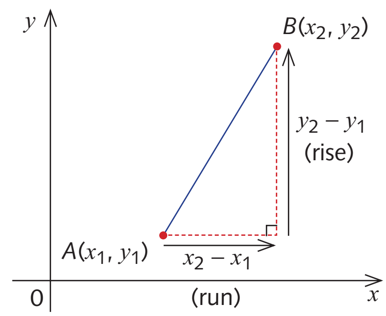
Notice that in this case the y value decreases as the x value increases. This means the gradient is **negative**.

Notice also that the gradient of AB = the gradient of BA ($\frac{\text{rise}}{\text{run}} = \frac{6}{-4} = -\frac{3}{2}$ for the gradient of BA).

In general, provided $x_2 \neq x_1$,

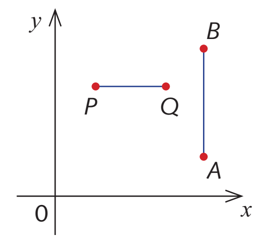
$$\text{gradient of interval } AB = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$



Since $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$, it does not matter which point we take as the first and which point we take as the second.

If the rise is zero, the interval is horizontal, as shown by the interval PQ. The gradient of the interval is zero.

If the run is zero, the interval is vertical, as shown by the interval AB. This interval does not have a gradient.



The gradient of PQ is zero.
The gradient of AB is not defined.

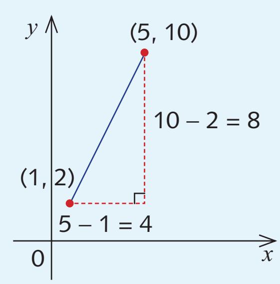
Example 7

Find the gradient of the interval joining the points (1, 2) and (5, 10).

Solution

For the points (1, 2) and (5, 10),

$$\begin{aligned}\text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{10 - 2}{5 - 1} \\ &= \frac{8}{4} \\ &= 2\end{aligned}$$



Gradient of a line

The **gradient of a line** is defined as the gradient of any interval within the line.

For this to make sense, we must show that any two intervals on a line have the same gradient.

Suppose that AB and PQ are two intervals on the same straight line. Draw right-angled triangles ABX and PQY, with sides AX and PY parallel to the x-axis and sides BX and QY parallel to the y-axis.

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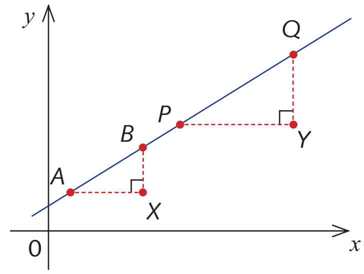


Triangle ABX is similar to triangle PQY since the corresponding angles are equal. Therefore,

$$\frac{QY}{PY} = \frac{BX}{AX} \qquad \text{(ratios of sides in similar triangles)}$$

That is, the intervals have the same gradient.

Example 8

A line passes through the point (5, 7) and has gradient $\frac{2}{3}$. Find:

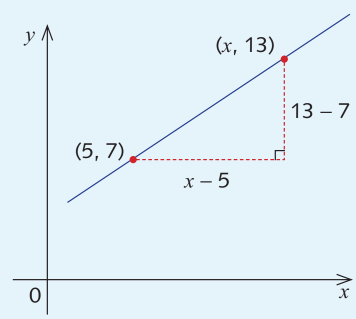
a the x-coordinate of the point on the line with y-coordinate 13

b the y-coordinate of the point on the line with x-coordinate -1

Solution

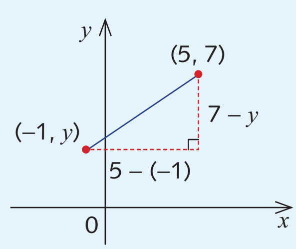
a

$$\frac{13-7}{x-5} = \frac{2}{5}$$
$$\frac{6}{x-5} = \frac{2}{3}$$
$$18 = 2(x-5)$$
$$9 = x-5$$
$$x = 14$$



b

$$\frac{7-y}{5-(-1)} = \frac{2}{3}$$
$$3(7-y) = 12$$
$$7-y = 4$$
$$y = 3$$



The ***x*-intercept** of a line is the x value of the point at which the line cuts the x-axis.

The ***y*-intercept** of a line is the y value of the point at which the line cuts the y-axis.

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Example 9

a Find the gradient of the line with:

i x-intercept 3 and y-intercept 6

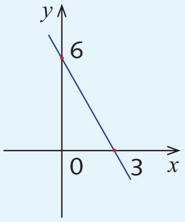
ii x-intercept -4 and y-intercept 8

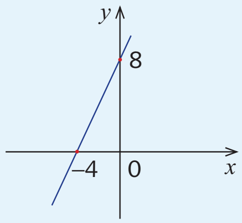
b Find the y-intercept of the line with gradient -2 and x-intercept -4.

Solution

a i Since (0, 6) and (3, 0) lie on the line,

$$\text{gradient} = \frac{6-0}{0-3} = -2$$





ii Since (0, 8) and (−4, 0) lie on the line,

$$\text{gradient} = \frac{8-0}{0-(-4)} = 2$$

b Gradient is −2,

so $\frac{\text{rise}}{\text{run}} = -2$

Let the y-intercept be b

$$\frac{b-0}{0-(-4)} = -2$$

So $b = -8$

Gradient of a line

- The **gradient of an interval** AB joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$\frac{y_2 - y_1}{x_2 - x_1}, \qquad \text{provided that } x_2 \neq x_1$$

- The **gradient of a line** is defined as the gradient of any interval within the line.
- A vertical line does not have a gradient.
- A horizontal line has gradient zero.

Exercise 11C

Example 7

1 Find the gradient of each interval AB.

a $A(6, 3), B(2, 0)$

b $A(-2, 6), B(0, 10)$

c $A(-1, 10), B(6, -4)$

d $A(2, 3), B(-4, 5)$

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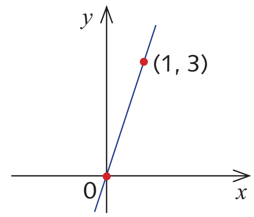
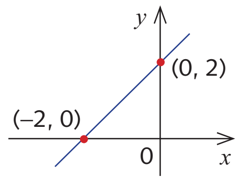
2 Find the gradient of each line.

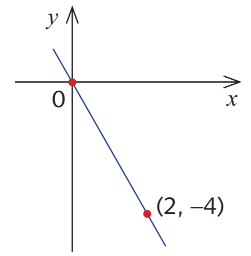
a



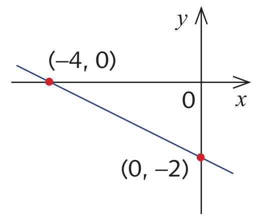
b



c



d



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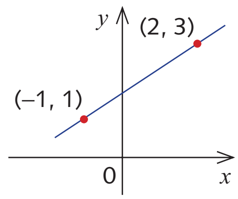

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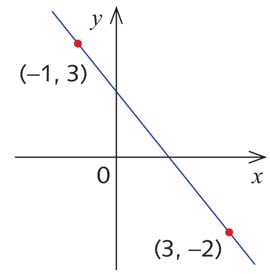

Example 7

3 Find the gradient of the line passing through each pair of points.

a $(0, 0)$ and $(6, -2)$ **b** $(0,\ 0)$ and $(-2,\ -3)$

c $(-2, -5)$ and $(-4,\ -1)$ **d** $(-2,\ 9)$ and $(3,\ -1)$

Example 8

4 **a** A line passes through the point $(4,\ 8)$ and has gradient $\frac{1}{2}$. Find the y-coordinate of the point on the line when $x = 8$.

b A line passes through the point $(-1,\ 6)$ and has gradient -1. Find the y-coordinate of the point on the line when $x = 4$.

5 A line passes through the point $(1,\ 3)$ and has gradient 3. Find:

a the x-coordinate of the point on the line when $y = 12$

b the y-coordinate of the point on the line when $x = 3$

c the x-coordinate of the point on the line when $y = 0$

d the y-coordinate of the point on the line when $x = -2$

6 **a** A line passes through $(0, 0)$ and has gradient 2. Copy and complete the table of values.

x	−2		1	
y		−2		6

b A line passes through the point (2, 1) and has gradient $\frac{3}{4}$. Copy and complete the following table of values.

x		−2	2	6	
y	−5		1		10

c A line passes through the point $(2, -6)$ and has gradient -2. Copy and complete the following table of values.

x		−4		0	2
y	10		0		−6

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Example 9

7 Copy and complete the following table. (Each part refers to a straight line. You may need to draw a diagram for each part.)

	x-intercept	y-intercept	Gradient
a	–1	2	
b	–2		3
c		4	$\frac{1}{2}$
d		–2	$\frac{2}{3}$
e	–5		–1
f		8	–2
g	10	5	

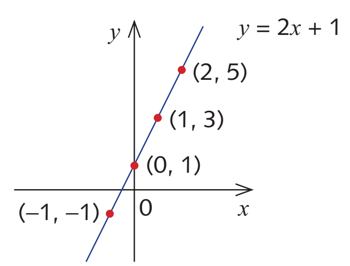
8 A line passes through the point $(3,\ 6)$ and has gradient 2. Find where the line crosses the x-axis and the y-axis.

9 A line passes through the point $(2,\ 6)$ and crosses the y-axis at the point $(0, 4)$. At what point does it cross the x-axis?

10 A line passes through $(0,\ b)$ and $(a,\ 0)$. Find the gradient of the line.

11D The equation of a straight line

When we plot points that satisfy the equation $y = 2x + 1$, we find that they lie in a straight line. When we deal with lines in coordinate geometry, we will be dealing with their equations.

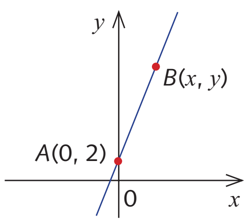


Can we find the equation of the line given suitable geometric information about the line? The following shows that this can be done given the gradient of the line and the y-intercept.

The line $y = 3x + 2$

Consider the line with gradient 3 and y-intercept 2. This passes through the point $A(0,\ 2)$. Let $B(x, y)$ be any point on this line.

$$\text{Gradient of interval } AB = \frac{\text{rise}}{\text{run}} = \frac{y-2}{x-0} = \frac{y-2}{x}$$



We know the gradient of the line is 3.

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Therefore $\dfrac{y-2}{x} = 3$

$y - 2 = 3x$

$y = 3x + 2$

So the coordinates (x, y) of B satisfy the equation $y = 3x + 2$.
Conversely, suppose that the point $B(x, y)$ in the plane satisfies $y = 3x + 2$. Then

$y - 2 = 3x$

$\dfrac{y-2}{x-0} = 3$

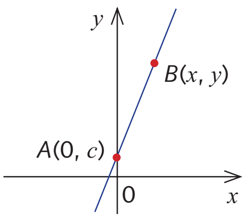
Thus we have shown that the interval joining (x, y) with $(0,\ 2)$ has gradient 3.

So B lies on the line with gradient 3 and y-intercept 2.

We summarise this by saying that the **equation of the line** is $y = 3x + 2$.

The line $y = mx + c$

Consider the line with gradient m and y-intercept c. It passes through the point $A(0, c)$. Let $B(x, y)$ be any point on this line.



Gradient of interval $AB = \dfrac{y-c}{x-0}$

$= \dfrac{y-c}{x}$

We know the gradient of the line is m.

Therefore $\dfrac{y-c}{x} = m$

$y - c = mx$

$y = mx + c$

That is, the line in the Cartesian plane with gradient m and y-intercept c has equation $y = mx + c$.
Conversely, the points whose coordinates satisfy the equation $y = mx + c$ always lie on the line with gradient m and y-intercept c.

Example 10

The gradient of a line is -6 and the y-intercept is 2. Find the equation of the line.

Solution

The equation of a straight line can be written as $y = mx + c$.

Thus the equation of the line is $y = -6x + 2$.

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Example 11

Write down the gradient and y-intercept of the line with equation $y = 3x - 4$.

Solution

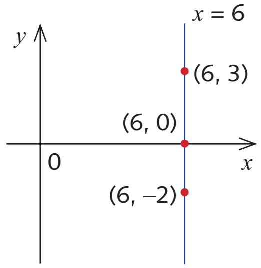
The gradient of the line is 3 and the y-intercept is -4.

Horizontal lines

All points on a horizontal line have the same y-coordinate, but the x-coordinate can take any value. In general, the equation of the horizontal line through $P(a, b)$ is $y = b$.

Vertical lines

All points on a vertical line have the same x-coordinate, but the y-coordinate can take any value. For example, the equation of the vertical line through the point $(6, 0)$ is $x = 6$.



In general, the equation of the vertical line through $P(a, b)$ is $x = a$.

Because a vertical line does not have a gradient, its equation does not fit the form $y = mx + c$.

Checking whether a point lies on the graph

We check whether the coordinates of the point satisfy the equation of the line.

Example 12

Check whether or not each of the following points lie on the line with equation $y = 2x + 3$.

a $(3, 9)$ **b** $(-2, 7)$ **c** $(4, 11)$

Solution

a Substitute $(3, 9)$ into the equation.

$\text{LHS} = 9, \text{RHS} = 2(3) + 3 = 9$

The point $(3, 9)$ lies on the line $y = 2x + 3$.

b Substitute $(-2, 7)$ into the equation.

$\text{LHS} = 7, \text{RHS} = 2(-2) + 3 = -1$

The point $(-2, 7)$ does not lie on the line $y = 2x + 3$.

c Substitute $(4, 11)$ into the equation.

$\text{LHS} = 1, \text{RHS} = 2(4) + 3 = 11$

The point $(4, 11)$ lies on the line $y = 2x + 3$.

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Finding the coordinates of a point on a line by substitution

We can find the unknown coordinate of a point on a line by substituting into the equation of the line.

Example 13

a Find the y-coordinate for the point on the line $y = 6x - 7$ with the x-coordinate:

i -1 **ii** 0 **iii** 20

b Find the x-coordinate for the point on the line $y = 6x - 7$ with the y-coordinate:

i 11 **ii** 0 **iii** 23

Solution

a i For $x = -1$,
$$y = 6 \times (-1) - 7 = -6 - 7 = -13$$
The y-coordinate is -13.

ii For $x = 0$,
$$y = 6 \times 0 - 7 = -7$$
The y-coordinate is -7.

iii For $x = 20$,
$$y = 6 \times 20 - 7 = 113$$
The y-coordinate is 113.

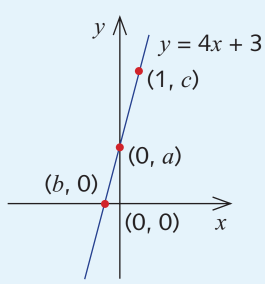
b i For $y = 11$,
$$6x - 7 = 11$$
$$6x = 18$$
$$x = 3$$
The x-coordinate is 3.

ii For $y = 0$,
$$6x - 7 = 0$$
$$6x = 7$$
$$x = \frac{7}{6}$$
The x-coordinate is $\frac{7}{6}$.

iii For $y = 23$,
$$6x - 7 = 23$$
$$6x = 30$$
$$x = 5$$
The x-coordinate is 5.

Example 14

The graph of $y = 4x + 3$ is shown opposite.
Find the values of a, b and c.



Solution

When the x-coordinate is 1, the y-coordinate is c.
$$c = 4 \times 1 + 3 = 7$$
When the x-coordinate is 0, the y-coordinate is a.
$$a = 4 \times 0 + 3 = 3$$

(*continued over page*)

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When the y-coordinate is 0, the x-coordinate is b.

$$4b + 3 = 0$$
$$4b = -3$$
$$b = -\frac{3}{4}$$

That is, $a = 3$, $b = -\frac{3}{4}$ and $c = 7$

Equation of a straight line

- The equation of the vertical line through $P(a, b)$ is $x = a$.
- Every non-vertical line in the Cartesian plane has equation $y = mx + c$, where m is the gradient of the line and c is the y-intercept.
- Conversely, the points whose coordinates satisfy the equation $y = mx + c$ all lie on the line with gradient m and y-intercept c.

Example 15

Rewrite each equation in the form $y = mx + c$. Hence find the value of the gradient and y-intercept of the line.

a $2x + 3y = 6$

b $-2x + 8y = 15$

Solution

a We rearrange the equation to make y the subject.

$$2x + 3y = 6$$

so
$$3y = 6 - 2x$$
$$y = 2 - \frac{2x}{3}$$
$$y = -\frac{2x}{3} + 2$$

The gradient of the line is $-\frac{2}{3}$ and the y-intercept is 2.

b $-2x + 8y = 15$

$$8y = 15 + 2x$$
$$y = \frac{2x}{8} + \frac{15}{8}$$
$$y = \frac{x}{4} + \frac{15}{8}$$

The gradient of the line is $\frac{1}{4}$ and the y-intercept is $\frac{15}{8}$.

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Exercise 11D

Example 10

1 Write down the equation of the line that has:

a gradient 2 and y-intercept 3

b gradient 3 and y-intercept 4

c gradient –2 and y-intercept 1

d gradient –1 and y-intercept 3

e gradient $\frac{2}{3}$ and y-intercept 1

f gradient $-\frac{3}{4}$ and y-intercept 0

Example 11

2 Write down the gradient and y-intercept of each line. Draw a graph of each line by plotting two points.

a $y = 2x + 1$

b $y = 3x + 4$

c $y = -2x + 5$

d $y = -2x - 6$

e $y = \frac{2}{3}x + 1$

f $y = -2x$

g $y = -4x$

h $y = 1 - 3x$

i $y = 2 - 5x$

Example 12

3 Check whether or not each of these points lies on the line with equation $y = -2x + 3$.

a (3, 9)

b (–2, 7)

c (–1, 5)

d (4, –5)

4 Check whether or not each of these points lies on the line with equation $y = -6x$.

a (0, 0)

b (1, 6)

c (–1, 6)

d (4, –10)

Example 13a

5 Find the y-coordinate of the point on the line $y = 3x - 4$ with x-coordinate:

a 2

b 0

c –2

6 Find the y-coordinate of the point on the line $y = -3x + 4$ with x-coordinate:

a 5

b –2

c 0

Example 13b

7 Find the x-coordinate of the point on the line $y = 2x + 6$ with y-coordinate:

a 10

b 0

c –4

8 Find the x-coordinate of the point on the line $y = -2x - 8$ with y-coordinate:

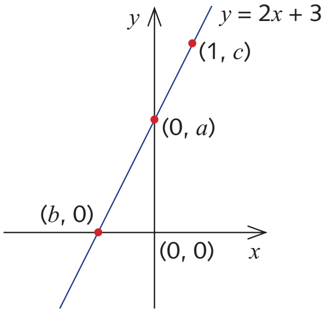
a 10

b 0

c –3

Example 14

9 The graph of $y = 2x + 3$ is shown below. Find the values of a, b and c.



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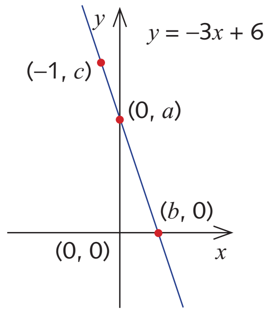
10 The graph of $y = -3x + 6$ is shown below. Find the values of a, b and c.



Example 15

11 Rewrite in the form $y = mx + c$ and then write down the gradient and y-intercept.

a $2x + y = 10$ **b** $10x + 2y = 4$ **c** $3x - 2y = 6$

d $4x - 3y = 12$ **e** $5y - 2x = 9$ **f** $3x - 4y = 6$

g $x = 2y - 4$ **h** $x = 3y + 1$ **i** $x = -2y$

j $x = -4y$ **k** $y + 3x = 0$ **l** $x - 2y = 0$

12 **a** Express the equation $ax + by = d$, where a, b and d are constants $(b \neq 0)$, in the form $y = mx + c$.

b Write down the gradient and y-intercept of the line whose equation is $ax + by = d$.

13 Show that the line passing through $(a, 0)$ and $(0, b)$ has equation $\frac{x}{a} + \frac{y}{b} = 1$.

11E Graphing straight lines

Two-point method

Two points determine a straight line. To draw a line we use the equation to find the coordinates of two points on the line.

Example 16

Draw the graph of

a $y = 2x + 3$ **b** $y = 3x$

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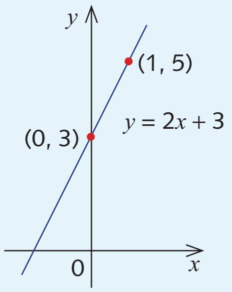
Solution

a Substitute $x = 0$, so $y = 2 \times 0 + 3$
$= 3$

Substitute $x = 1$, so $y = 2 \times 1 + 3$
$= 5$

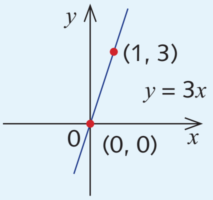
Two points on the line are (0, 3) and (1, 5).



b When $x = 0$, $y = 0$

When $x = 1$, $y = 3$

Two points on the line are (0, 0) and (1, 3).



Two-intercept method

The points where the line crosses the axes are usually of particular interest. The ***x*-intercept** is found by substituting $y = 0$ and the ***y*-intercept** is found by substituting $x = 0$.

This method does not work if the line is parallel to an axis, or passes through the origin, since such lines have only one intercept.

Example 17

Using the two-intercept method, sketch the graph of:

a $y = 3x - 4$ **b** $2x + 3y + 6 = 0$

Solution

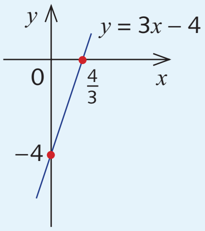
a When $x = 0, y = -4$

When $y = 0$, $3x - 4 = 0$

$3x = 4$

$x = \frac{4}{3}$

The two points $(0, -4)$ and $\left(\frac{4}{3}, 0\right)$ lie on the line.



b When $x = 0$, $3y + 6 = 0$

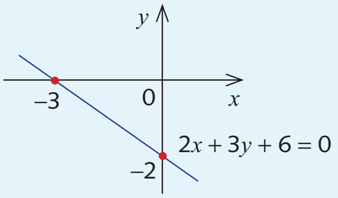
$3y = -6$

$y = -2$

When $y = 0$, $2x + 6 = 0$

$2x = -6$

$x = -3$



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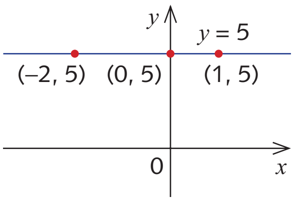
Horizontal lines

On a horizontal line, all points have the same y-coordinate, but the x-coordinate can take any value. For example, the equation of the horizontal line through the point $(0, 5)$ is $y = 5$.

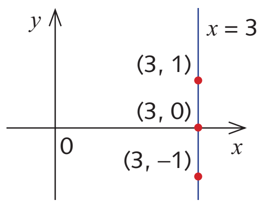
The equation of the horizontal line through the point (a, b) is $y = b$.



Vertical lines

Recall that, on a vertical line, all points have the same x-coordinate, but the y-coordinate can take any value. For example, the equation of the vertical line through the point $(3, 0)$ is $x = 3$.

The equation of the vertical line through the point (a, b) is $x = a$.



Graphing straight lines

- Two points determine a straight line. To draw a line we use the equation of the line to find the coordinates of two points on the line.
- If the line is not parallel to one of the axes and does not pass through the origin, the x-intercept and y-intercept can be found and the graph drawn.
- The equation of the horizontal line through the point (a, b) is $y = b$.
- The equation of the vertical line through the point (a, b) is $x = a$.

Exercise 11E

Example 16a

1 Sketch the graph of each line by calculating the y-intercept and finding one other point.

a $y = 2x + 1$ **b** $y = 3x + 2$ **c** $y = 3x - 2$

d $y = 4x + 5$ **e** $y = 4 - 2x$ **f** $y = 3 - x$

g $y = \frac{1}{2}x + 1$ **h** $y = \frac{2}{3}x + 2$ **i** $y = -2x + 3$

Example 16b

2 Sketch the graph of:

a $y = 4x$ **b** $y = 2x$ **c** $y = \frac{1}{2}x$

d $y = -2x$ **e** $y = -x$ **f** $x + 2y = 0$

g $2y - 3x = 0$ **h** $3x + y = 0$ **i** $\frac{x}{3} - \frac{y}{2} = 0$

Example 17

3 Sketch the graph of each line by calculating the coordinates of the x- and y-intercepts.

a $2x + y = 4$ **b** $x + 3y = 6$ **c** $2x + 3y = 6$

d $3x + y = 2$ **e** $x - y = 4$ **f** $3x - y = 3$

g $x - 2y = 4$ **h** $3y - x = 4$ **i** $2y - 3x = 4$

j $\frac{x}{2} + \frac{y}{3} = 1$ **k** $\frac{x}{3} - \frac{y}{4} = 1$ **l** $\frac{2x}{3} - \frac{3y}{4} = 2$

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4 Sketch the graph of:

a $y = 3$ **b** $x = -1$ **c** $x + 2 = 0$

d $y - 5 = 0$ **e** $4 - y = 0$ **f** $7 + x = 0$

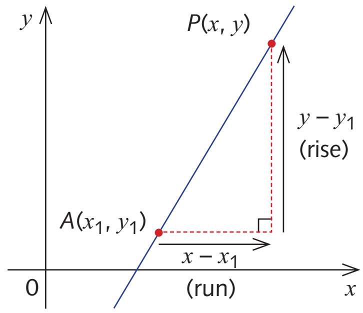
11F The equation of a line using the gradient and a point

Let $A(x_1, y_1)$ be a given point. Consider the line through A with gradient m. Let $P(x, y)$ be a general point on the line $x \neq x_1$.

Then $$m = \frac{y - y_1}{x - x_1}$$

and $$y - y_1 = m(x - x_1)$$

This is the equation of the line that passes through the point $A(x_1, y_1)$ and has gradient m.



Example 18

Find the equation of the line that passes through the point (–2, 3) and has gradient –4.

Solution

We use $y - y_1 = m(x - x_1)$.

The equation of this line is

$$\begin{aligned} y - 3 &= -4(x - (-2)) \\ y - 3 &= -4x - 8 \\ y &= -4x - 5 \end{aligned}$$

The equation of a line through two given points

The equation of a line can be found if the coordinates of two points on the line are known by first calculating the gradient and then using the method given above.

Example 19

Find the equation of the line that passes through the point $A(1, 3)$ and $B(4, 8)$.

Solution

$$\text{Gradient of interval } AB = \frac{8 - 3}{4 - 1} = \frac{5}{3}$$

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Now, using $y - y_1 = m(x - x_1)$, where $(x_1, y_1) = (1, 3)$ and $m = \frac{5}{3}$:

$$y - 3 = \frac{5}{3}(x - 1)$$
$$y - 3 = \frac{5}{3}x - \frac{5}{3}$$
$$y = \frac{5}{3}x + \frac{4}{3}$$

Note: The point $B(4, 8)$ could be used instead of $A(1, 3)$.

Equation of a line using the gradient and a point

- The equation of the line that passes through a point $A(x_1, y_1)$ and has gradient m is
$$y - y_1 = m(x - x_1)$$
- The equation of the line is determined if the coordinates of two points on the line are known. For the points $A(x_1, y_1)$ and $B(x_2, y_2)$, $x_2 \neq x_1$
  - find the gradient, $m = \frac{y_2 - y_1}{x_2 - x_1}$
  - the equation is then $y - y_1 = m(x - x_1)$.
- If $x_1 = x_2$ then the equation is $x = x_1$.

Exercise 11F

Example 18

1 Find the equation of the line that passes through:

a the point $(1, 3)$ and has gradient 1

b the point $(2, 4)$ and has gradient 3

c the point $(-1, 1)$ and has gradient 4

d the point $(2, -2)$ and has gradient $\frac{2}{3}$

e the point $(-1, 0)$ and has gradient $-\frac{1}{3}$

f the point $(-1, -4)$ and has gradient $-\frac{2}{5}$

2 The line ℓ has equation $y = 3x + 5$.

a What is the y-intercept of ℓ?

b What is the gradient of ℓ?

c Find the equation of the line that has the same y-intercept as ℓ but has a gradient of $\frac{1}{2}$.

d Find the equation of the line that has the same gradient as ℓ but has a y-intercept of -2.

3 A line passes through the points $A(1, 3)$ and $B(4, 12)$.

a Find the gradient of AB.

b Using the gradient and the point A, find the equation of line AB.

c Using the gradient and the point B, find the equation of line AB.

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Example 19

4 Find the equation of the line through each pair of points A and B.

a $A(5, 3)$ and $B(2, -1)$

b $A(4, 1)$ and $B(6, 7)$

c $A(1, 2)$ and $B(2, 4)$

d $A(-1, 6)$ and $B(2, -3)$

e $A(-2, 4)$ and $B(1, -6)$

f $A(-1, -2)$ and $B(3, 4)$

In the following questions, draw a sketch showing all relevant points and lines.

5 Consider the interval AB with endpoints $A(-1, -4)$ and $B(3, 8)$.

a Find the gradient of AB.

b Find the coordinates of C, the midpoint of interval AB.

c Find the equation of the line that passes through C with gradient $-\frac{1}{2}$.

d What are the coordinates of the point D, where the line intersects the y-axis?

e How far is the point C from the point D?

6 The quadrilateral $ABCD$ has vertices $A(1, 2)$, $B(5, 6)$, $C(8, 0)$ and $D(6, -2)$.

a Find the coordinates of M, the midpoint of AB.

b Find the coordinates of N, the midpoint of CD.

c Find the gradient of MN.

d Find the equation of the line MN.

e Find the coordinates of P, the midpoint of AC.

f Find the equation of the line that passes through P with gradient 3.

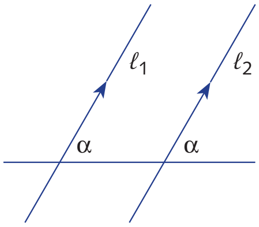
11G Parallel and perpendicular straight lines

Parallel lines

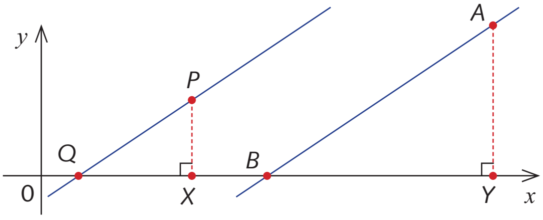
If two lines ℓ_1 and ℓ_2 are parallel, then the corresponding angles are equal.

Conversely, if the corresponding angles are equal, then the lines are parallel.

We are now going to show that two lines are parallel if they have the same gradient and, conversely, if they have the same gradient, they are parallel.



In the diagram below, the line through P meets the x-axis at Q. The line through A meets the x-axis at B. XP and AY are perpendicular to the x-axis.



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Proof that lines are parallel implies equal gradients

If the lines are parallel, $\angle PQX = \angle ABY$ (corresponding angles).

Triangles QPX and BAY are similar by the AAA test.

Therefore $\frac{PX}{QX} = \frac{AY}{BY}$ (matching sides in similar triangles).

So, the gradients are equal.

Proof that equal gradients implies lines are parallel

If the gradients are equal, $\frac{PX}{QX} = \frac{AY}{BY}$

Thus $\frac{PX}{AY} = \frac{QX}{BY}$

So the triangles are similar by the SAS test.

Hence the corresponding angles PQX and ABY are equal and the lines are parallel.

Example 20

Show that the line passing through the points $A(6,\ 4)$ and $B(7,\ 11)$ is parallel to the line passing through $P(0,\ 0)$ and Q(1, 7)

Solution

$$\begin{aligned} \text{Gradient of } AB &= \frac{11-4}{7-6} \\ &= 7 \\ \text{Gradient of } PQ &= \frac{7-0}{1-0} \\ &= 7 \end{aligned}$$

The two lines have the same gradient and so they are parallel.

Example 21

Find the equation of the line that is parallel to the line with equation $y = -2x + 6$ and that passes through the point $A(1,\ 10)$.

Solution

The gradient of the line $y = -2x + 6$ is -2.

Therefore the line through the point $A(1,\ 10)$ parallel to $y = -2x + 6$ also has gradient -2 and hence the equation is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 10 &= -2(x - 1) \\ y - 10 &= -2x + 2 \\ y &= -2x + 12 \end{aligned}$$

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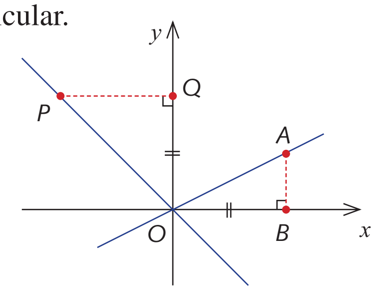
Perpendicular lines

We are now going to show that if two lines are **perpendicular**, then the product of their gradients is −1 (or one is vertical and the other horizontal). The converse is also true. That is, if the product of the gradients of two lines is −1, then they are perpendicular.

We first consider the case where both lines pass through the origin.

Draw two lines passing through the origin, with one of the lines having positive gradient and the other negative gradient.



Form right-angled triangles OPQ and OAB with $OQ = OB$.

$$\text{Gradient of the line } OA = \frac{AB}{BO}$$

$$\text{Gradient of the line } OP = -\frac{OQ}{PQ}$$

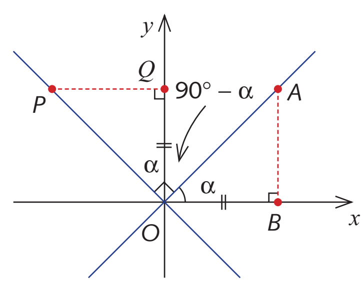
$$\begin{aligned}\text{Product of the gradient} &= -\frac{OQ}{PQ} \times \frac{AB}{BO}\\ &= -\frac{OQ}{PQ} \times \frac{AB}{OQ} \quad (\text{since } OQ = OB)\\ &= -\frac{AB}{PQ}\end{aligned}$$

Proof that if two lines are perpendicular, then the product of their gradients is –1

If the lines are perpendicular, $\angle POQ = \angle AOB$ because when each of these angles is added to $\angle AOQ$, the result is 90°.

Therefore triangles OPQ and OAB are congruent (AAS).

So $PQ = AB$ and the product of the gradients, $-\frac{AB}{PQ}$, is −1.



Proof that if the product of the gradients is –1, then the lines are perpendicular

If the product is −1, $AB = PQ$ since the product of the gradients $= -\frac{AB}{PQ}$.

So the triangles OBA and OQP are congruent (SAS).

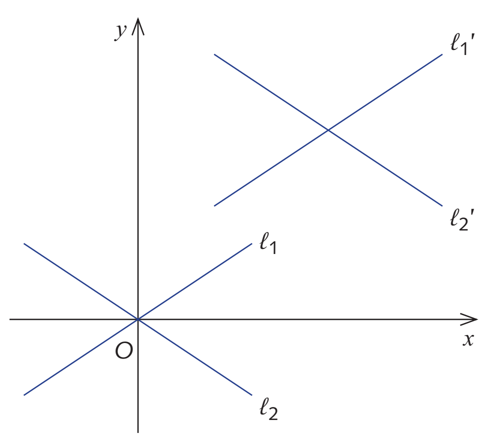
Therefore $\angle POQ = \angle AOB$ and $\angle AOP = 90° - \alpha + \alpha = 90°$

Hence the lines are perpendicular.

Lines that do not meet at the origin

If we are given two lines anywhere in the plane, we can draw lines through the origin parallel to the original two lines.

In the given diagram, $\ell_1 \parallel \ell_1'$ and $\ell_2 \parallel \ell_2'$. Therefore, lines ℓ_1 and ℓ_1' have the same gradient, and lines ℓ_2 and ℓ_2' have the same gradient.



Hence, any relationship between the gradients of ℓ_1 and ℓ_2 is shared by ℓ_1' and ℓ_2' and vice-versa. This means the above result established for lines through the origin holds for any pair of lines (not parallel with the axes).

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Example 22

Show that the line through the points $A(6,\ 0)$ and $B(0,\ 12)$ is perpendicular to the line through $P(8,\ 10)$ and $Q(4,\ 8)$.

Solution

$$\begin{aligned}\text{Gradient of } AB &= \frac{12-0}{0-6}\\ &= -2\\ \text{Gradient of } PQ &= \frac{10-8}{8-4}\\ &= \frac{2}{4}\\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}(\text{Gradient of } AB) \times (\text{gradient of } PQ) &= -2 \times \frac{1}{2}\\ &= -1\end{aligned}$$

Hence the line AB is perpendicular to the line PQ, which we can write as $AB \perp PQ$.

Parallel and perpendicular lines

- Two non-vertical lines are **parallel** if they have the same gradient. Conversely, if two non-vertical lines are parallel, then they have the same gradient.
- Two lines are **perpendicular** if the product of their gradients is –1 (or if one is vertical and the other is horizontal). Conversely, if two lines are perpendicular, then the product of their gradients is –1 (or one is vertical and the other is horizontal).

Example 23

Find the equation of the line that passes through the point (1, 3) and is perpendicular to the line whose equation is $y = 2x + 1$.

Solution

The gradient of the line $y = 2x + 1$ is 2.

Hence the gradient of a line perpendicular to this line is $-\frac{1}{2}$.

The required equation is:

$$\begin{aligned}y - 3 &= -\frac{1}{2}(x-1)\\ 2(y-3) &= -(x-1)\\ 2y + x &= 1 + 6\\ 2y + x &= 7\end{aligned}$$

Thus the equation of the required line is $2y + x = 7$ or $y = \frac{1}{2}x + \frac{7}{2}$.

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Exercise 11G

1 The equations of eight lines are given below. State which lines are parallel.

a $y = 2x - 3$ **b** $3x + y = 7$ **c** $y = 4 - 2x$

d $x = \frac{1}{2}y + 1$ **e** $y = \frac{1}{3}x - 3$ **f** $2y - x = 7$

g $3y + x = 8$ **h** $y = -2x + 5$

Example 21

2 Find the equation of the line that:

a is parallel to the line $y = 2x - 3$ and passes through the point $(1,\ 5)$

b is parallel to the line $y = 4 - x$ and passes through the point $(-2, -1)$

c is parallel to the line $y - 3x = 4$ and passes through the point $(0, -3)$

d is parallel to the line $2y + x = 3$ and passes through the point $(4, -2)$

3 If $y = (2a - 3)x + 1$ is parallel to $y = 3x - 4$, find the value of a.

4 If $y = (3a + 2)x - 1$ is parallel to $y = ax - 4$, find the value of a.

5 In each part, lines ℓ_1 and ℓ_2 are perpendicular. In the table, the gradient of ℓ_1 is given. Find the gradient ℓ_2.

	Gradient of ℓ_1	Gradient of ℓ_2
a	$\frac{1}{2}$	
b	2	
c	$\frac{4}{3}$	
d	3	
e	$-\frac{3}{4}$	
f	-5	
g	$-\frac{1}{2}$	
h	$\frac{2}{3}$	

6 The equations of eight lines are given below. State which lines are perpendicular.

a $y = 2x - 4$ **b** $4y + 3x = 7$ **c** $y = x + 2$

d $3y - x = 5$ **e** $y = 4 - x$ **f** $2y + x = 7$

g $3y - 4x = 8$ **h** $y = -3x + 5$

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Example 23

7 Find the equation of the line that:

a is perpendicular to the line $y = 2x - 3$ and passes through the point $(1, 4)$

b is perpendicular to the line $y = 4 - 2x$ and passes through the point $(-2, -1)$

c is perpendicular to the line $x + 3y = 7$ and passes through the point $(2, -3)$

d is perpendicular to the line $y - 3x = 4$ and passes through the point $(1, -3)$

8 *ABCD* is a quadrilateral with vertices $A(1, 2)$, $B(3, 5)$, $C(7, -1)$ and $D(5, -5)$. Draw a diagram showing all points and lines.

a Find the coordinates of M, the midpoint of AB.

b Find the coordinates of N, the midpoint of BC.

c Calculate the gradient of MN.

d If P is the midpoint of CD and Q is the midpoint of DA, find the gradient of PQ.

e What can be concluded about the intervals MN and PQ?

f Find the gradients of QM and PN.

g What type of quadrilateral is $MNPQ$?

9 If $y = (a + 1)x + 7$ is perpendicular to $y = 2x - 4$, find the value of a.

10 If $y = (2a + 3)x + 1$ is perpendicular to $y = 2x - 4$, find the value of a.

11 Let AB be the interval with endpoints $A(2, 3)$ and $B(6, 11)$

a Find the gradient of AB.

b Find the coordinates of C, the midpoint of AB.

c Find the equation of the line that is perpendicular to AB and passes through C. This line is the **perpendicular bisector** of AB.

d What is the y-intercept of this perpendicular bisector?

12 Triangle ABC has vertices $A(-1, -3)$, $B(4, 2)$ and $C(12, -6)$. By calculating gradients, show that ΔABC is right-angled.

13 Quadrilateral $ABCD$ has vertices $A(2, -2)$, $B(5, 2)$, $C(9, -1)$ and $D(6, -5)$. Show that the diagonals AC and BD are perpendicular.

14 The coordinates of the vertices of ΔPQR are $P(1, -2)$, $Q(3, 6)$ and $R(7, 0)$. Find:

a the coordinates of S, the midpoint of PQ

b the gradient of SR

c the equation of the line SR (This line is called a **median**. It passes through a vertex and the midpoint of the side opposite that vertex.)

d the equations of the other two medians of the triangle

15 In triangle ABC, the **altitude** through A is the line through A perpendicular to BC. The coordinates of the vertices of the triangle are $A(0, 1)$, $B(4, 7)$ and $C(6, -1)$. Find the equation of the altitude through:

a C **b** B **c** A

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16 $A(5, 6)$ and $B(6, 7)$ are adjacent vertices of a square $ABCD$.

a Find the length of each side of the square.

b Find the gradient of AB.

c Find the gradient of BC.

d The coordinates of C are $(7, c)$. Find the value of c.

e Find the coordinates of D.

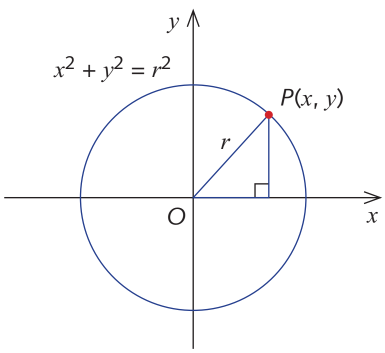
11H The equation of a circle

Circles with centre the origin

Consider a circle in the coordinate plane with centre the origin and radius r.

If $P(x, y)$ is a point on the circle, its distance from the origin is r and, by Pythagoras' theorem, $x^2 + y^2 = r^2$.

Conversely, if a point $P(x, y)$ in the plane satisfies the equation $x^2 + y^2 = r^2$, its distance from $O(0, 0)$ is r, so it lies on a circle with centre the origin and radius r.



The circle with centre $O(0, 0)$ and radius r has equation

$$x^2 + y^2 = r^2$$

Example 24

Sketch the graph of the circle $x^2 + y^2 = 25$ and test whether the points $(3, -4)$, $(-3, 4)$, $(1, 4)$ and $(3, 2)$ lie on the circle.

Solution

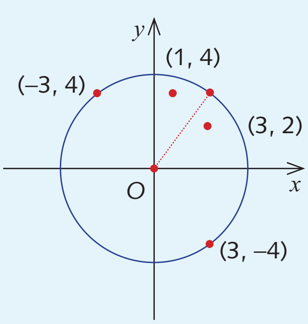
The circle has centre the origin and radius 5.

The point $(3, -4)$ lies on the circle as $3^2 + (-4)^2 = 5^2$

The point $(-3, 4)$ lies on the circle as $(-3)^2 + 4^2 = 5^2$

The point $(1, 4)$ does not lie on the circle as $1^2 + 4^2 = 17 \neq 5^2$

The point $(3, 2)$ does not lie on the circle as $3^2 + 2^2 = 13 \neq 5^2$



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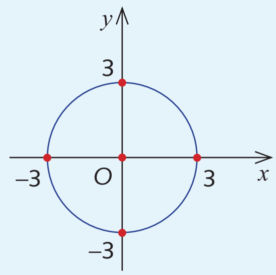
Example 25

Sketch the graphs of the circles with the following equations.

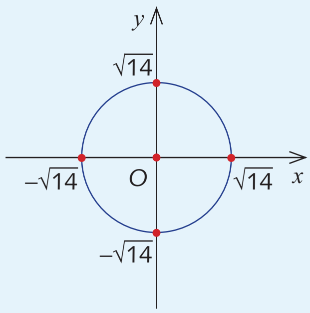
a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 14$

Solution

a $x^2 + y^2 = 3^2$ is the equation of a circle with centre the origin and radius 3.



b $x^2 + y^2 = 14$ is the equation of a circle with centre the origin and radius $\sqrt{14}$.



Exercise 11H

Example 24

1 Test whether or not the point with the given coordinates is on the circle $x^2 + y^2 = 16$.

a $(4,\ 0)$ **b** $(4,\ 4)$ **c** $(0, -4)$
d $(2\sqrt{2}, 2\sqrt{2})$ **e** $(2,\ 2)$ **f** $(2\sqrt{2}, -2\sqrt{2})$
g $(-2,\ 2)$ **h** $(2, 2\sqrt{3})$ **i** $(\sqrt{5}, \sqrt{11})$

2 Test whether or not the point with the given coordinates is on the circle $x^2 + y^2 = 75$.

a $(5\sqrt{3}, 0)$ **b** $(5, 5\sqrt{2})$ **c** $(-5\sqrt{2}, 5)$
d $(0, -5\sqrt{3})$ **e** $(6, \sqrt{39})$ **f** $(5,\ 5)$
g $(25,\ 3)$ **h** $(8, \sqrt{11})$ **i** $(-\sqrt{11}, 8)$

Example 25

3 Sketch the graph of the circle, labelling the x- and y-intercepts.

a $x^2 + y^2 = 16$ **b** $x^2 + y^2 = 3$ **c** $x^2 + y^2 = 25$
d $x^2 + y^2 = 20$ **e** $x^2 + y^2 = 10$ **f** $x^2 + y^2 = 36$
g $y^2 = 8 - x^2$ **h** $x^2 = 15 - y^2$ **i** $y^2 = 11 - x^2$

4 Write down the equation of the circle with centre the origin and radius:

a 11 **b** $\sqrt{7}$ **c** $2\sqrt{3}$ **d** $5\sqrt{3}$

5 Find the equation of the circle with centre the origin passing through the point:

a $(1,\ 1)$ **b** $(2,\ 1)$ **c** $(1,\ 7)$ **d** $(\sqrt{2}, \sqrt{2})$
e $(-1,\ 6)$ **f** $(-2, -7)$ **g** $(-12,\ 5)$ **h** $(2\sqrt{3}, 2\sqrt{3})$

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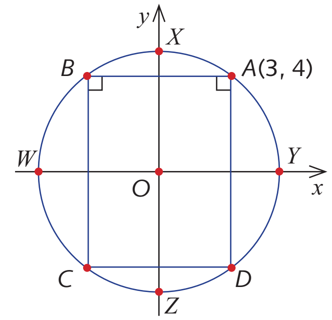
6 Find the equation of the circle with centre the origin and diameter:

a 6 **b** 12 **c** 7 **d** 11 **e** $\frac{3}{4}$

7 The point $A(3, 4)$ lies on the circle $x^2 + y^2 = 25$. The rectangle $ABCD$ has axes of symmetry the x- and y-axes.



a Find the coordinates of B, C and D.

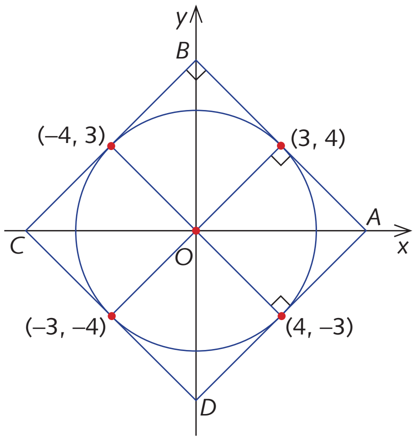
W and Y are the x-intercepts and X, Z are the y-intercepts of the circle.

b Find the distance XY.

c Find the area of the square $WXYZ$.

d Determine which has a greater area: $WXYZ$ or $ABCD$? By how much?

8 The circle to the right has equation $x^2 + y^2 = 25$.



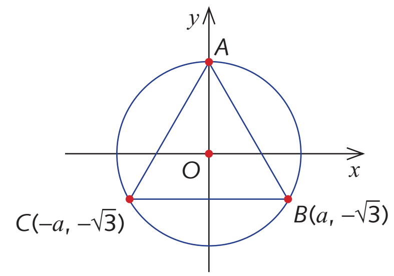
a Find the gradient AB.

b Find the equation of the line AB.

c Find the distance AB.

d Find the area of the square $ABCD$.

9 In the diagram to the right, ΔABC is inscribed in the circle $x^2 + y^2 = 12$.



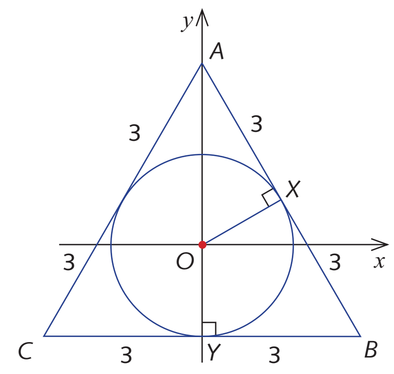
a Find the value of a.

b Find AB.

c Find BC.

d What type of triangle is ΔABC?

10 A circle is inscribed in the equilateral triangle ABC.



a Find AY.

b Give reasons why ΔAXO is similar to ΔAYB.

c Find OX and hence find the equation of the circle.

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Review exercise

1 Find the distance between the two points.

a $(-3, 4)$ and $(-3, 13)$ **b** $(-5, 2)$ and $(1, 2)$

c $(-2, -6)$ and $(3, 6)$ **d** $(-2, -7)$ and $(13, 1)$

2 Find the coordinates of the midpoint of the interval AB with endpoints:

a $(-3, 4)$ and $(-3, 13)$ **b** $(-5, 2)$ and $(1, 2)$

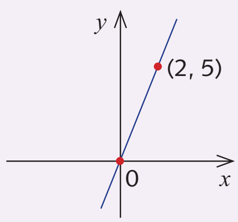
c $(-2, -6)$ and $(3, 6)$ **d** $(-2, -7)$ and $(13, 1)$

3 Find the gradient of each interval AB.

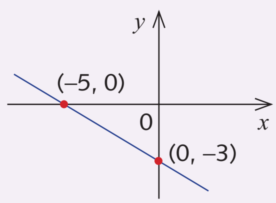
a $A(5, 4), B(1, 0)$ **b** $A(-5, 3), B(0, 12)$

4 Find the gradient of each of the following lines.

a



b



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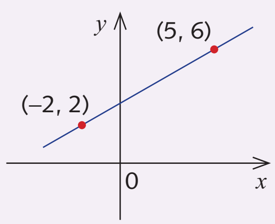
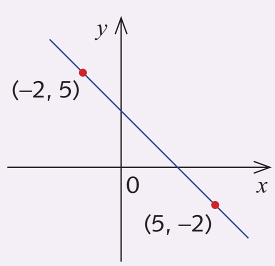


d



5 Find the gradient of the line passing through the two points.

a $(0, 0)$ and $(4, -3)$ **b** $(-3, -2)$ and $(-5, -6)$

6 A line passes through the point $(2, 3)$ and has gradient 4.

a Find the value of x for the point on the line with $y = 11$.

b Find the value of y for the point on the line with $x = 5$.

7 A line passes through the point $(4, 12)$ and has gradient 2. Find where the line crosses the x- and y-axes.

8 A line passes through the point $(1, 5)$ and crosses the y-axis at the point $(0, 3)$. At what point does it cross the x-axis?

9 Write down the gradient and y-intercept of each line.

a $y = 3x + 2$ **b** $y = -3x + 4$ **c** $y = \frac{1}{4}x - 7$

d $y = -\frac{2}{5}x + 6$ **e** $y = -8x$ **f** $y = 2 - 9x$

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10 Write down the equation of the line that has:

a gradient 3 and y-intercept 5

b gradient −1 and y-intercept 4

c gradient $\frac{3}{4}$ and y-intercept −2

d gradient $-\frac{1}{7}$ and y-intercept 0

11 Rewrite in the form $y = mx + c$ and then write down the gradient and y-intercept.

a $3x + y = 12$ **b** $9x + 4y = 6$ **c** $2x - 3y = 8$ **d** $4y - 3x = 9$

e $x = 7y - 2$ **f** $x = -9y$ **g** $y + 2x = 0$ **h** $x - 11y = 0$

12 Find the equation of the line that:

a passes through the point (2, 4) and has gradient 1

b passes through the point (−2, 0) and has gradient 4

c passes through the point (3, −1) and has gradient $\frac{1}{2}$

d passes through the point (−2, −5) and has gradient $-\frac{2}{5}$

13 Sketch the graph of each of the following lines by using the y-intercept and finding another point.

a $y = 4x - 3$ **b** $y = 5 - 6x$ **c** $y = \frac{1}{3}x + 2$

14 Sketch the graph of each of the following lines by finding the coordinates of the x- and y-intercepts.

a $3x + y = 4$ **b** $x + 2y = 5$ **c** $3x + 4y = 6$

d $x - y = 5$ **e** $5x - y = 9$ **f** $2y - x = 8$

g $3y - 4x = 5$ **h** $\frac{x}{3} + \frac{y}{4} = 1$ **i** $\frac{2x}{5} - \frac{3y}{7} = 2$

15 Sketch the graph of:

a $y = 2$ **b** $x = -5$ **c** $x + 7 = 0$

d $y - 4 = 0$ **e** $9 - y = 0$ **f** $6 + x = 0$

16 The equations of six lines are given below. Which pairs of lines are parallel?

a $4y + x = 8$ **b** $-3x + y = 6$ **c** $y = 5 - 3x$

d $x = \frac{1}{3}y + 2$ **e** $y = 5 - \frac{1}{4}x$ **f** $y = -3x + 10$

17 Find the equation of the line parallel to the line $y = 3x - 4$ and passing through the point (2, 6).

18 The equations of six lines are given below. State which pairs of lines are perpendicular.

a $y = 5x - 1$ **b** $5y + 2x = 7$ **c** $5y - 2x = 8$

d $2y - x = 5$ **e** $y = -2x + 5$ **f** $5y + x = -2$

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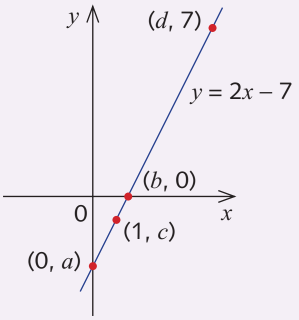
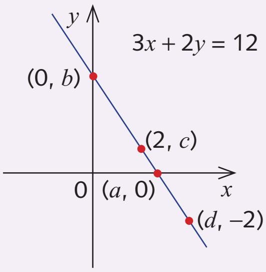
19 Find the equation of the line that is perpendicular to the line with equation $y = 2x - 4$ and passes through the point $(3,\ 8)$.

20 The graph of $y = 2x - 7$ is shown opposite. Find the values of a, b, c and d.



21 The graph of $3x + 2y = 12$ is shown opposite. Find the values of a, b, c and d.



22 The interval AB has endpoints $A(1,\ 7)$ and $B(-1,\ -11)$.

a Find the gradient AB.

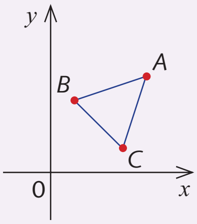
b Find the distance between points A and B.

c Find the equation of the line that passes through A and B.

d Find the coordinates of the midpoint of interval AB.

e Find the equation of the perpendicular bisector of AB.

23 ΔABC is isosceles with $AC = BC$. The coordinates of C, B and A are $(3, 1), (1, 3)$ and $(4, a)$ respectively. Find the value of a.



24 The line through the points $A(0,\ 7)$ and $B(11,\ -6)$ is parallel to the line through the points $C(6,\ 12)$ and $D(-11, d)$. Find the value of d.

25 Find the equation of the line with x-intercept 6 and y-intercept 11.

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Challenge exercise

1 For the points $A(-1, 3)$ and $B(4, 2)$, find the coordinates of the point P on the interval AB such that $AP : PB$ equals:

a $1 : 1$

b $2 : 1$ (P is closer to B.)

c $2 : 3$ (P is closer to A.)

2 Show that the points $(1, -1)$, $(-1, 1)$ and $(-\sqrt{3}, -\sqrt{3})$ are the vertices of an equilateral triangle.

3 Show that the points $A(1, -1)$, $B(7, 3)$, $C(3, 5)$ and $D(-3, 1)$ are the vertices of a parallelogram and find the length of its diagonals.

4 If $(3, -1)$, $(-4, 3)$ and $(1, 5)$ are three vertices of a parallelogram, find the coordinates of the fourth vertex if it lies in the first quadrant.

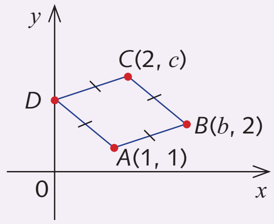
5 Find the equation of the line whose intercepts are twice those of the line with equation $2x - 3y - 6 = 0$.

6 In the rhombus $ABCD$, A has coordinates $(1, 1)$ and the coordinates of B are $(b, 2)$.

The gradient of AB is $\frac{1}{2}$.

a Find the value of b.

b Find the length of AB.



The point C has coordinates $(2, c)$, where c is a positive integer.

c Find the value of c.

d Find the gradient of BC.

e State the gradient of:

i CD **ii** AD

f Find the coordinates of D.

7 Let ABC be any triangle. Take the x-axis to be along BC and the y-axis through the midpoint O of BC perpendicular to BC. Prove that $AB^2 + AC^2 = 2AO^2 + 2OC^2$. (This result is known as Apollonius' theorem.)

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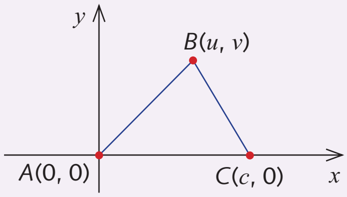
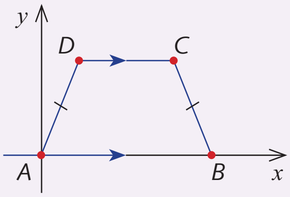
8 Use coordinate geometry to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length. Take the base of the triangle to be on the x-axis with one vertex at the origin.

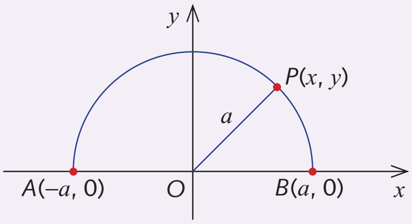


9 Use coordinate geometry to prove that the diagonals of an isosceles trapezium $ABCD$ are equal in length. Choose coordinates appropriately.



10 Use coordinate geometry to prove that the medians of any triangle are concurrent. *Hint*: On each median find the coordinates of the point that divides the median in the ratio $2:1$.

11 The semicircle shown below is drawn with centre O at the origin and radius a. $P(x, y)$ is any point on the semicircle.



a Show that $x^2 + y^2 = a^2$.

b Show that PA is perpendicular to the line PB.

c Show that $PA^2 + PB^2 = 4a^2$.

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