CHAPTER

12

Statistics and Probability

Probability

Probability deals with how likely it is that something will happen. It is an area of mathematics with many diverse applications. Probability is used in weather forecasting and in insurance to calculate risk factors and premiums. It is also used in science to predict the risks of new medical treatments and to forecast the effects of climate change.

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12A An introduction to probability

Probability measures the likelihood of an event occurring on a scale from 0 to 1, inclusive.

In this section we look at methods for determining probabilities. We recall some ideas from *ICE-EM Mathematics Year 8*.

Sample space

A box contains 12 identical marbles numbered from 1 to 12. The box is shaken and a marble is randomly taken from it and its number noted. This is an example of doing a random **experiment**. The numbers 1, 2,......, 12 are called the **outcomes** of this experiment. The outcomes for this experiment are **equally likely**. The probability of each outcome is $\frac{1}{12}$. The complete set of possible outcomes (or sample points) for any experiment is called the **sample space** of that experiment. For example, we can write down the sample space ξ for this experiment as:

$$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

In this chapter, all of the experiments have finite sample spaces with equally likely outcomes. For a sample space with n equally likely outcomes, the probability of each outcome is $\frac{1}{n}$.

Events

An **event** is a collection of outcomes. It is a subset of the sample space.

Suppose that for the experiment above we are interested in getting a prime number. In this case 'the number is prime' is the event that interests us. Some of the outcomes will give rise to this event. For instance, if the outcome of the experiment is 2, the event 'the number is prime' takes place. We say that the outcome 2 is **favourable to the event** 'the number is prime'. If the outcome is 4, the event 'the number is prime' does not occur. The outcome 4 is **not favourable to the event**.

Of the 12 outcomes, these are the ones that are favourable to the event 'the number is prime':

$$\{2, 3, 5, 7, 11\}$$

In many situations, 'success' means 'favourable to the event' and 'failure' means 'not favourable to the event'.

Events are named by capital letters. For instance, we talk about:

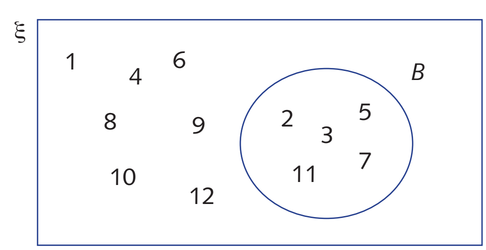
B is the event 'a prime is obtained' from the experiment described previously.
C is the event 'obtaining an even number' when a die is tossed.

An outcome is favourable to an event if it is a member of that event. For example:

5 is a member of B and 6 is a member of C.

These sample spaces and events can be illustrated with **Venn diagrams**. Venn diagrams were used in *ICE-EM Mathematics Year 8* and were introduced in *ICE-EM Mathematics Year 7*.

Here is the sample space ξ and the event B. In this context ξ is the universal set for the experiment of withdrawing a marble and observing the number on it, as described previously.



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Probability of an event

The probability of the event A is written $P(A)$.

Since the probability of an event is a number between 0 and 1 inclusive, we can assert:

$$0 \leq P(A) \leq 1 \text{ for all events } A$$

For an experiment in which all of the outcomes are equally likely:

$$\text{Probability of an event} = \frac{\text{number of outcomes favourable to that event}}{\text{total number of outcomes}}$$

For the event B described above, $P(B) = \frac{5}{12}$.

In general, the probability of an event is the sum of the probabilities of the outcomes that are favourable to that event.

The total probability is 1

The sum of the probabilities of the outcomes of an experiment is 1.

For the previous experiment of taking a marble, each outcome has probability $\frac{1}{12}$. The sum of these probabilities is 1.

The words 'random' and 'randomly'

In probability, we frequently hear these words, as in the following situation.

A classroom contains 23 students. A teacher comes into the room and chooses a student at random to answer a question about history.

What does this mean? It means that the teacher chooses the student as if the teacher knows nothing at all about the students. Another way of interpreting this is to imagine that the teacher had her eyes closed and had no idea who was in the class when she choose a student. Each student is equally likely to be chosen. The probability of a particular student being chosen is $\frac{1}{23}$.

Example 1

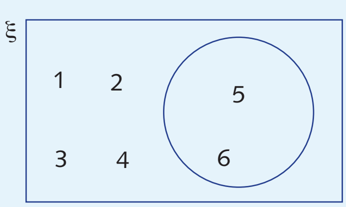
A die is rolled once. Draw a Venn diagram for the experiment and circle the outcomes favourable to the event 'the number is greater than 4'. What is the probability of a number greater than 4 being obtained?

Solution

$\xi = \{1, 2, 3, 4, 5, 6\}$

The outcomes favourable to the event are a 5 or a 6 appearing.

$$P(\text{number greater than 4}) = \frac{2}{6} = \frac{1}{3}$$



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Example 2

A standard pack of playing cards consists of four suits: Hearts, Diamonds, Clubs and Spades. Each suit has 13 cards consisting of an Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. The pack is shuffled and a card is drawn. What is the probability of drawing:

a a King **b** a heart?

Solution

The sample space ξ is the 52 playing cards.

a Let K be the event 'a King is drawn'.

$$P(K) = \frac{4}{52} = \frac{1}{13}$$

b Let H be the event 'a heart is drawn'.

$$P(H) = \frac{13}{52} = \frac{1}{4}$$

Exercise 12A

Example 1

1 One letter is chosen at random from the word 'SALE'. What is the probability that it is L?

2 What is the probability of choosing a prime number from the numbers 5, 6, 7, 8, 9, 10, 11, 12, 13?

3 What is the probability of randomly picking the most expensive car from a range of eight new and differently priced cars in a showroom?

4 What is the probability of choosing an integer that is exactly divisible by 5 from the set $\{5, 6, 7, 8, 9, 10, 11, 12\}$?

5 In a raffle, 500 tickets are sold. If you have bought one ticket, what is the probability that you will win first prize?

6 One card is chosen at random from a pack of 52 ordinary playing cards. What is the probability that it is the King of Hearts?

7 A number is chosen from the first 25 positive whole numbers. What is the probability that it is exactly divisible by both 3 and 4?

Example 2

8 One card is drawn at random from a pack of playing cards. What is the probability that it is:

a a King? **b** a red card?

c a Spade? **d** a picture card? (A picture card is a King, Queen or Jack.)

9 A book of 420 pages has pictures on 60 of its pages. If one page is chosen at random, what is the probability that it has a picture on it?

10 One counter is picked at random from a bag containing 20 red counters, 5 white counters and 15 yellow counters. What is the probability that the counter removed is:

a red? **b** yellow?

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11 On a spinning wheel, the numbers go from 0 to 36. The wheel is spun. What is the probability that when it stops the resulting number will be:

a an even number?

b an odd number?

c a number less than 15, excluding 0?

12 A number is chosen at random from the first 25 positive whole numbers. What is the probability that it is not a prime number?

12B The complement, union and intersection

The complement of an event

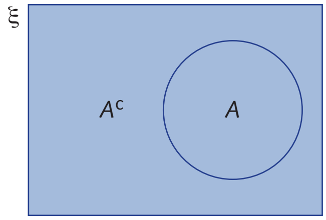
In some problems the outcomes in the event A can be difficult to count, whereas the event 'not A' may be easier to deal with.

The event 'not A' consists of every possible outcome in the sample space ξ that is not in A. The set 'not A' is called the **complement** of A and is denoted by A^c.

Every outcome of a sample space is contained in exactly one of A or A^c.

Therefore $P(A) + P(A^c) = 1$ *and* $P(A^c) = 1 - P(A)$.

This can be illustrated with a Venn diagram, as shown.



Example 3

A card is drawn from a standard pack. What is the probability that it is not the King of Hearts?

Solution

Let A be the event 'the King of Hearts is drawn'.

Then A^c is the event 'the King of Hearts is not drawn'.

$$P(A) = \frac{1}{52}$$

$$P(A^c) = 1 - P(A) = \frac{51}{52}$$

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Union and intersection

Sometimes, rather than just considering a single event, we want to look at two or more events.

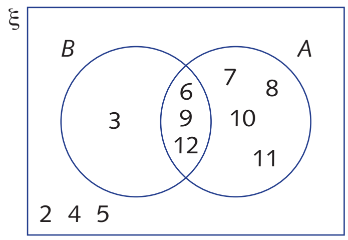
A bowl contains 11 marbles numbered from 2 to 12. One marble is withdrawn.

The sample space $\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Let A be the event 'a number greater than 5 is chosen'.

Let B be the event 'a number divisible by 3 is chosen'.

The Venn diagram illustrating these events is as shown.

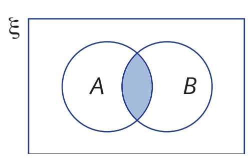


The outcomes favourable to the event 'the number is divisible by 3 and greater than 5' is the **intersection** of the sets A and B. That is, $A \cap B$. The event $A \cap B$ is often called 'A and B'.

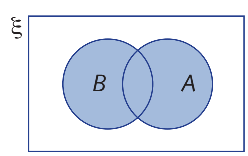
In this example, $A \cap B = \{6, 9, 12\}$.

The outcomes favourable to the event 'the number is divisible by 3 or greater than 5' is the **union** of the sets A and B. That is, $A \cup B$. The event $A \cup B$ is often called 'A or B'.

In this example, $A \cup B = \{3, 6, 7, 8, 9, 10, 11, 12\}$



$A \cap B$ is shaded



$A \cup B$ is shaded

For an outcome to be in the event $A \cup B$, it must be in *either* the set of outcomes for A or the set of outcomes for B. Of course, it could be in both sets.

For an outcome to be in the event '$A \cap B$', it must be in *both* the sets of outcomes for A and the set of outcomes for B.

We recall the **addition rule** for probability. For any two events, A and B:

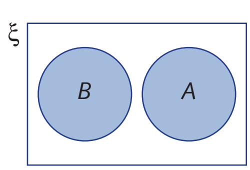
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We subtract $P(A \cap B)$ from $P(A) + P(B)$ because $A \cap B$ is a subset of both A and B and would be counted twice otherwise.

Two events are **mutually exclusive** if they have no outcomes in common. That is:

$$A \cap B = \emptyset$$

where $\emptyset$ is the empty set.



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The addition rule becomes $P(A \cup B) = P(A) + P(B)$ if A and B are mutually exclusive.

Here are some examples using these ideas.

Example 4

Twenty plastic discs numbered from 1 to 20 are placed in a bowl. A disc is randomly removed, its number noted and then replaced.

a Find the probability of a disc with an even number *or* a number greater than or equal to 15 being obtained.

b Find the probability of a disc with a number that is even *and* divisible by 5 being obtained.

Solution

The sample space is:

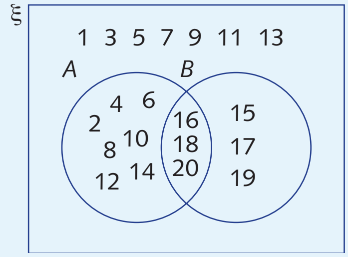
$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

Let A be the event 'an even number'.

Let B be the event 'a number greater than or equal to 15'.

Let C be the event 'a number is divisible by 5'.

a $A \cup B = \{2, 4, 6, 8, 10, 12, 14, 15, 16, 17, 18, 19, 20\}$

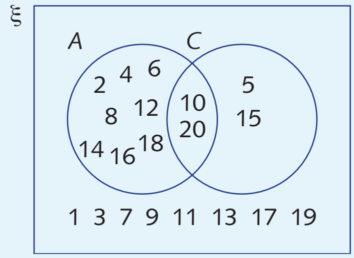


The number of outcomes in $A \cup B$ is 13.

$$P(A \cup B) = \frac{13}{20}$$

b $A \cap C = \{10, 20\}$

The number of outcomes in $A \cap C$ is 2.



$$P(A \cap C) = \frac{2}{20} = \frac{1}{10}$$

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Example 5

There are 250 students in Year 9 in a school: 60 study music, 90 study French, and 23 study both French and music.

If a student is chosen at random, what is the probability that they study:

a both music and French?

b music or French?

c music but not French?

d French but not music?

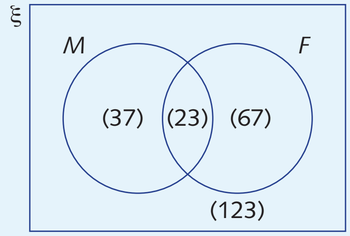
e music or French but not both?

Solution

Let M be the event 'study music'.

Let F be the event 'study French'.

The brackets in the Venn diagram are used to indicate that it is the number of elements in the region.



a From the Venn diagram, 23 students study French and music.

$$P(M \cap F) = \frac{23}{250}$$

b From the Venn diagram, 127 students study French or music.

$$P(M \cup F) = \frac{127}{250}$$

c From the Venn diagram, 37 students study music but not French.

'Not French' is the event F^c.

$$P(M \cap F^c) = \frac{37}{250}$$

d From the Venn diagram, 67 students study French but not music.

'Not music' is the event M^c.

$$P(M^c \cap F) = \frac{67}{250}$$

e From the Venn diagram, 104 students study French or music but not both.

The events $M \cap F^c$ and $M^c \cap F$ are mutually exclusive.

Probability of a student studying French or music but not both:

$$\frac{37}{250} + \frac{67}{250} = \frac{52}{125}$$

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Example 6

The eye colour and gender of 150 people were recorded. The results are shown in the table below.

Gender \ Eye colour	Blue	Brown	Green	Grey
Male	20	25	5	10
Female	40	35	5	10

What is the probability that a person chosen at random from the sample:

a has blue eyes **b** is male **c** is male and has green eyes

d is female and does not have blue eyes **e** has blue eyes or is female?

Solution

The sample space ξ has 150 outcomes, or the size of $\xi = 150$.

Let: A be the event 'has blue eyes'

B be the event 'has brown eyes'

G be the event 'has green eyes'

M be the event 'is male'

F be the event 'is female'.

a The probability that a person has blue eyes is

$$P(A) = \frac{20 + 40}{150} = \frac{60}{150} = \frac{2}{5}$$

b The probability that a person is male is

$$P(M) = \frac{20 + 25 + 5 + 10}{150} = \frac{60}{150} = \frac{2}{5}$$

c The probability that a person is male and has green eyes is

$$P(M \cap G) = \frac{5}{150} = \frac{1}{30}$$

d The probability that a person is female and does not have blue eyes is

$$P(F \cap A^c) = \frac{35 + 5 + 10}{150} = \frac{50}{150} = \frac{1}{3}$$

e The probability that a person has blue eyes or is female is

$$P(A \cup F) = \frac{20 + 40 + 35 + 5 + 10}{150} = \frac{110}{150} = \frac{11}{15}$$

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Complement, or and and

- The event 'not A' includes every possible outcome of the sample space ξ that is not in A
  The event 'not A' is called the **complement** of A and is denoted by A^c.
  $P(A^c) = 1 - P(A)$
- An outcome that is in the event '$A \cup B$' is in either A *or* B, or both.
- An outcome that is in the event '$A \cap B$' is in both A *and* B.
- For any two events, A and B
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Two events A and B are **mutually exclusive** if $A \cap B = \varnothing$ and
  $P(A \cup B) = P(A) + P(B)$

Exercise 12B

1 A number is chosen at random from the first 15 positive whole numbers. What is the probability that it is not a prime number?

Example 3

2 A card is drawn at random from an ordinary pack of 52 playing cards. What is the probability that it is not a King?

3 A number is chosen at random from the first 30 positive whole numbers. What is the probability that it is not divisible by 7?

4 In a raffle, 1000 tickets are sold. If you buy 50 tickets, what is the probability that you will not win the first prize?

5 A letter is chosen at random from the 10 letters of the word 'COMMISSION'. What is the probability that the letter is:

a N? **b** S? **c** a vowel? **d** not S?

Example 4

6 A card is drawn at random from a well-shuffled pack of playing cards. Find the probability that the card chosen:

a is a Club

b is a court card (that is, Ace, King, Queen or Jack)

c has a face value between 2 and 9 inclusive

d is a Club and a court card

e is a Club or a court card

f has a face value between 2 and 5 inclusive and is a court card

g has a face value between 2 and 5 inclusive or is a court card

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7 A six-sided die is thrown and the uppermost number is noted. Find the probability that the number is:

a even and a 6

b even or a 6

c less than or equal to 4 and a 6

d less than or equal to 3 or a 6

e even and less than or equal to 4

f odd or less than or equal to 3

Example 6

8 A survey of 200 people was carried out to determine hair and eye colour. The results are shown in the table below.

Eyecolour \ Hair colour	Fair	Brown	Red	Black
Blue	25	9	6	18
Brown	16	16	18	22
Green	15	17	22	16

What is the probability that a person chosen at random from this group has:

a blue eyes?

b red hair?

c fair or brown hair?

d blue or brown eyes?

e red hair and green eyes?

f eyes that are not green?

g hair that is not red?

h eyes that are not blue or hair that is not fair?

Use an appropriate Venn diagram in the following questions.

Example 5

9 In a group of 100 students, 60 study mathematics, 70 study physics and 30 study both mathematics and physics.

a Represent this information on a Venn diagram.

b One student is selected at random from the group. What is the probability that the student studies:

i mathematics and not physics?

ii physics but not mathematics?

iii neither physics nor mathematics?

10 In a group of 40 students, 26 play tennis and 19 play soccer. Assuming that each of the 40 students plays at least one of soccer and tennis, find the probability that a student chosen at random from this group plays:

a both tennis and soccer

b only tennis

c only one sport

d only soccer

11 In a group of 65 students, 30 students study geography, 42 study history and 20 study both history and geography. If a student is chosen at random from the group of 65 students, find the probability that the student studies:

a history or geography

b neither history nor geography

c history but not geography

d only one of history or geography

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12C Relative frequency

When we toss a coin, we can assume that the two outcomes are equally likely, so each has a probability of $\frac{1}{2}$ of occurring. This is reasonable, provided that the coin is well made and evenly balanced. We then make a 'probability model' consisting of $\{H, T\}$, where H and T both have a probability of $\frac{1}{2}$. This is what we have done so far in this chapter.

Estimates of probability

If we find a coin on the street that is badly worn and bent by the weather and passing traffic, it would not be reasonable to assume that a Head and a Tail are equally likely to occur when we toss the coin. So we would have no idea of what the probability of a Head would be.

One way of attempting to get around the difficulty is to toss the coin a large number of times and record how many Heads occur. If we tossed the coin 100 times, we might get these results:

Heads: 40   Tails: 60

Then we could divide to get $\frac{40}{100} = \frac{2}{5}$ as an estimate of the probability of getting a Head.

This is called **relative frequency**.

The relative frequency of getting a Tail is therefore $\frac{3}{5}$.

We are going to use relative frequency as an estimate of the probability for future experiments. So in this case, our estimate for the probability of getting a Head is $\frac{2}{5}$.

Limitations

If we think about relative frequency, we will see there are some tricky issues. For example:

- Would we get the same relative frequency if we performed another 100 coin tosses?
- If we toss a perfectly new coin from the mint 100 times, would the relative frequency of a Head be $\frac{1}{2}$?

The answer to both questions is: 'No, unless you are very lucky'. There will always be **random variation** between the outcomes of repeated trials like these.

The idea of relative frequency is useful and important. Statisticians use it very effectively by combining it with techniques of **sampling**. This is a sophisticated area of probability and statistics. Random sampling was considered in *ICE-EM Mathematics Year 8*. We are going to use relative frequency in simple ways in this section.

Relative frequency

The relative frequency of the event A is defined to be

$$\frac{\text{number of times } A \text{ occurs}}{\text{total number of times the experiment is performed}}$$

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Example 7

Clara has a die. She does not know if the values 1, 2, 3, 4, 5 and 6 are equally likely. She rolls the die 2000 times and obtains the following results.

Outcome	1	2	3	4	5	6
Frequency	290	370	330	280	340	390

Use the table to calculate the relative frequency to estimate the probability that the next roll will be:

a a 2 **b** an even number **c** a number greater than 4

Solution

a The relative frequency of obtaining a 2 is $\frac{370}{2000} = \frac{37}{200}$

This can be taken as an estimate of the probability of obtaining a 2.

b Of the 2000 rolls, $370 + 280 + 390 = 1040$ yielded an even number.

The relative frequency of obtaining an even number $= \frac{1040}{2000}$

$= \frac{13}{25}$

This can be taken as an estimate of the probability of obtaining an even number.

c Of the 2000 rolls, $340 + 390 = 730$ yielded a number greater than 4.

The relative frequency of obtaining a number greater than 4 $= \frac{730}{2000}$

$= \frac{73}{200}$

This can be taken as an estimate of the probability of obtaining a number greater than 4.

'or', 'and' and relative frequency

We can use a two-way table to help determine relative frequencies.

Example 8

A survey of 200 people was carried out to determine hair and eye colour. The results are shown in the table below.

Hair colour / Eyecolour	Fair	Brown	Red	Black
Blue	25	9	6	18
Brown	16	16	18	22
Green	15	17	22	16

What is the relative frequency of:

a fair or brown hair? **b** green eyes and black hair?

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Solution

a Relative frequency of fair or brown hair $= \dfrac{25 + 16 + 15 + 9 + 16 + 17}{200}$

$= \dfrac{49}{100}$

b Relative frequency of green eyes and black hair $= \dfrac{16}{200} = \dfrac{2}{25}$

Exercise 12C

1 A market researcher interviews 200 people and discovers that 120 of them use a particular brand of toothpaste. Using this data, what is the relative frequency of a person:

a using the particular brand of toothpaste

b not using the particular brand of toothpaste?

2 A quality-control inspector discovers that, of 1500 electrical components tested, 25 are faulty. Using this data, what is an estimate for probability that the next component tested is:

a faulty? **b** not faulty?

3 A meteorologist's records indicate that it has rained on 30% of the occasions when a particular weather pattern occurred. What is the estimated probability that it will rain the next time the particular weather pattern occurs?

4 A bag contains a number of counters, each of which is coloured blue, yellow, red or green. A student chooses a counter from the bag at random, notes its colour and returns it to the bag. She repeats this process 100 times. The results of the experiment are in the following table.

Colour	Blue	Yellow	Red	Green
Frequency	18	21	53	8

a What is the estimated probability that the next counter she withdraws is:

i blue? **ii** red? **iii** not green?

b If the bag contains 10 counters, how many counters of each colour do you think are in the bag?

5 Abdul chose a card at random from a not-necessarily standard pack of cards, noted its suit and returned it to the pack. He repeated this process 500 times. The results of his experiment are in the following table.

Suit	Spades	Hearts	Diamond	Clubs
Frequency	112	119	135	

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a Fill in the missing entry in the table.

b What is the estimated probability that next card Abdul withdraw is:

i a Spade? **ii** a Heart? **iii** not a Club? **iv** a red card?

6 One thousand people were surveyed about their favourite sport. That results of the survey are given in the table below.

Sport	AFL	Rugby Union	Rugby League	Soccer	Baseket ball
Frequency	188	160	145		210

a Fill in the missing entry in the table.

b What is the relative frequency of:

i AFL?

ii Rugby Union?

iii a sport other than soccer?

7 A survey of 600 people was carried out to determine hair and eye colour. The results are shown in the table below.

Eyecolour \ Hair colour	Fair	Brown	Red	Black
Blue	125	30	15	45
Brown	40	40	45	55
Green	68	42	55	40

What is the relative frequency of:

a fair or red hair?

b blue or green eyes?

c red hair and green eyes?

d green eyes and brown hair?

8 The 700 subjects volunteering for a medical study are classified by blood pressure (high, normal and low) and gender.

Gender \ Blood pressure	High	Normal	Low
Male	352	100	52
Female	60	88	48

Find the relative frequency of:

a male and high blood pressure

b male or low blood pressure

c female and low blood pressure

d male or normal blood pressure

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12D Multi-stage experiments

Rolling two dice, tossing three coins and drawing six lottery numbers are examples of multi-stage experiments.

Tossing a coin twice

The possible outcomes of this experiment are:

HH HT TH TT

where H stands for 'heads' and T stands for 'tails', and the order of the letters indicates the outcomes of the two separate tosses. The four outcomes of the experiment are equally likely, so each has a probability of $\frac{1}{4}$.

Example 9

A coin is tossed twice. What is the probability of getting:

a one head and one tail in any order?

b at least one tail?

Solution

a Here is the sample space, with circles around the favourable outcomes.

HH (HT) (TH) TT

$$\begin{aligned} P(\text{one head and one tail}) &= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

b Here is the sample space, with the favourable outcomes circled.

HH (HT) (TH) (TT)

$$P(\text{at least one tail}) = \frac{3}{4}$$

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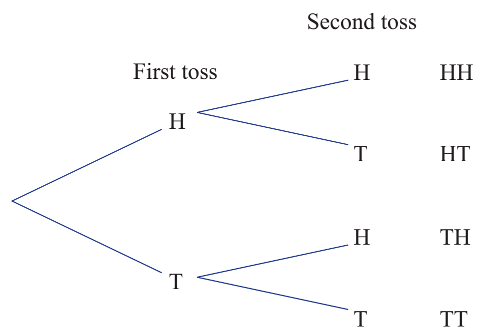
Tree diagrams

Another way of illustrating the outcomes of the experiment of tossing a coin twice is a **tree diagram**.

Each path, starting from the far left and ending with H or T on the far right, represents a possible outcome.

The sample space is $\xi = \{HH, HT, TH, TT\}$.

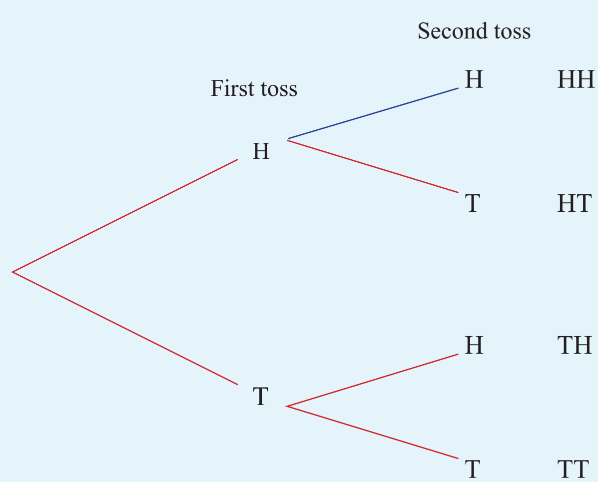


Example 10

Using a tree diagram, calculate the probability of getting at least one tail when a coin is tossed twice.

Solution

The tree diagram, with the favourable paths indicated with red lines, is shown below.



There are 3 paths that give at least one tail and 4 $(= 2 \times 2)$ possible paths in total, so the event 'at least one tail' is $\{HT, TH, TT\}$ and $P(\text{at least one tail}) = \frac{3}{4}$.

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Tossing a coin three times

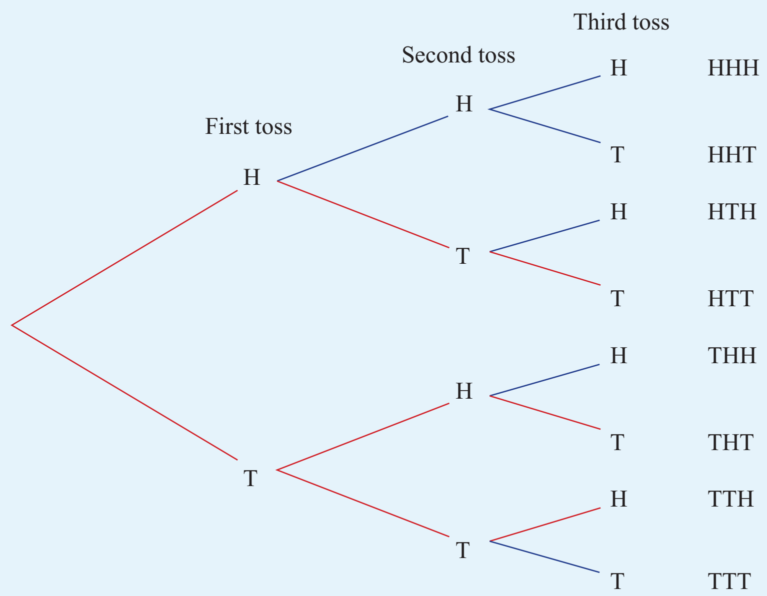
Tree diagrams are especially useful when the experiment consists of more than two stages.

Example 11

A coin is tossed three times. What is the probability of getting exactly one head?

Solution

The paths that contain exactly one head are indicated with red lines in the tree diagram below.



There are 3 of these and 8 (= $2 \times 2 \times 2$) possible paths in total, so the event 'exactly one head' $= \{$HTT, THT, TTH$\}$ and P (exactly one head) $= \frac{3}{8}$.

Tossing a die twice

If a die is tossed twice, there are $6 \times 6 = 36$ outcomes. It would be possible to draw the corresponding tree diagram, but that would be cumbersome. A better approach is to use an **array diagram**.

Each die has six faces, marked with dots from 1 to 6. The outcomes of the experiment can be set out in an array, as shown below.

Die 1 \ Die 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

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Each entry in the array represents a pair of possible results, the first from die 1 and the second from die 2.

For example, the entry $(3,\ 4)$ in the array corresponds to the outcome:

die $1 = 3$ die $2 = 4$

That is, the first die shows a 3, the second a 4.

Each outcome is equally likely, and there are $6 \times 6 = 36$ of them in total, so:

$$\text{probability of each outcome} = \frac{1}{36}$$

Example 12

Two dice are tossed. What is the probability of getting a 6 on at least one of the dice?

Solution

The favourable outcomes are shaded in the array diagram below.

Die 1 \ Die 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

$$P(\text{at least one } 6) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$= \frac{11}{36}$$

How could a solution to the above problem be provided by the addition rule?

Example 13

Two dice are tossed. What is the probability that the sum of their faces lies between 4 and 7 inclusive?

Solution

Here we can take a shortcut by making a diagram showing the sums of the faces. The sums for the favourable outcomes are shaded.

(continued over page)

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Die 1 \ Die 2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are 18 favourable outcomes out of a total of 36, so:

$$P(\text{sum lies between 4 and 7 inclusive}) = \frac{18}{36} = \frac{1}{2}$$

Multi-stage experiments

The outcomes of a multi-stage experiment can be represented by:

- systematic listing
- a tree diagram
- an array (only suitable for two-stage experiments).

Exercise 12D

1 A 10-cent coin is tossed and an ordinary six-sided die is rolled. Copy and complete the following array, which shows the possible outcomes.

10-cent coin \ Die	1	2	3	4	5	6
H		(H, 2)				
T				(T, 4)		

2 One bag contains 1 red counter, 2 yellow counters and 1 blue counter. Another bag contains 1 yellow counter, 2 red counters and 1 blue counter. One counter is taken at random from each bag. Copy and complete the following array to show all the possible outcomes for this experiment.

1st bag \ 2nd bag	R	R	Y	B
R	(R, R)			
Y				(Y, B)
Y				
B	(B, R)			

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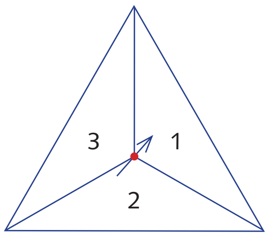
3 A spinner like the one in the diagram is spun twice. Copy and complete the array to show all the possible outcomes for this experiment.

1st spin \ 2nd spin	1	2	3
1			
2			
3			



Example 12

4 A coin and a six-sided die are tossed. Construct an array that shows all the equally likely possible outcomes of this random experiment, and use it to find the probability of obtaining:

a a head and a six

b a head and an even number

c a tail

d a four

5 One bag of coins contains one 10-cent coin and one 50-cent coin. Another bag contains three 10-cent coins and two 50-cent coins. One coin is removed at random from each bag. Construct an array to show all the equally likely outcomes of this experiment, and use it to find the probability that:

a a 50-cent coin is taken from each bag

b a 10-cent coin is taken from each bag

c the coins are of different value

6 One bookshelf contains 3 storybooks and 1 textbook, while the next shelf holds 2 storybooks and 3 textbooks. Draw an array showing the various ways in which you could pick a pair of books, one from each shelf. Use this array to find the probability that:

a both books are storybooks

b both books are textbooks

7 The four Aces and the four Kings are removed from a pack of playing cards. One card is taken from the set of four Aces and one card is taken from the set of four Kings. Construct an array for the outcomes of this experiment, and use it to find the probability that the two cards chosen:

a are both black

b are both Spades

c include at least one black card

d are both the same suit

Example 13

8 Two fair six-sided dice are tossed and the uppermost numbers are noted. Construct an array for the possible outcomes (or use the table given in Example 13) to find the probability of obtaining:

a two sixes

b a total of 7

c the same number on both dice

d a total less than or equal to 10

9 A special six-sided die has two blank faces and the other faces numbered 1, 3, 4 and 6. This die is rolled along with an ordinary six-sided die with faces numbered 1, 2, 3, 4, 5 and 6. Make an array showing the values of the pairs of uppermost faces of the dice, and use it to find the probability of getting a total score of:

a 6

b 10

c 1

d at least 6

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Example 10 **10** Two coins are tossed.

a Draw a tree diagram showing all the outcomes.

b What is the probability that 'a head and a tail' are showing?

Example 11 **11** A coin is tossed three times.

a Draw a tree diagram showing all the outcomes.

b What is the probability that three heads are obtained?

c What is the probability that exactly one head is obtained?

d What is the probability that at least one head is obtained?

12 A painter is going to paint a child's bedroom. He has two choices of colour for the ceiling, blue or white; he has three choices of colour for the walls, blue, grey or white; and he has two choices of colour for the skirting boards, grey or white. He has to choose the colour for the ceiling, the walls and the skirting boards.

a Draw a tree diagram showing all the outcomes.

b If he chooses at random, what is the probability that the ceiling, the walls and the skirting boards are painted different colours?

13 A family has three children. By drawing a tree diagram that shows all the possible combinations of children, find the probability that:

a all three children are girls

b only one child is a boy

c at least two of the children are girls

12E Two-step experiments involving replacement

With replacement

A bag contains three red balls, R_1, R_2 and R_3, and two black balls, B_1 and B_2. A ball is chosen at random and its colour recorded. It is then put back in the bag, the balls are mixed thoroughly and a second ball is chosen. Its colour is also noted. The sample space is shown in the array below.

First ball \ Second ball	R_1	R_2	R_3	B_1	B_2
R_1	(R_1, R_1)	(R_1, R_2)	(R_1, R_3)	(R_1, B_1)	(R_1, B_2)
R_2	(R_2, R_1)	(R_2, R_2)	(R_2, R_3)	(R_2, B_1)	(R_2, B_2)
R_3	(R_3, R_1)	(R_3, R_2)	(R_3, R_3)	(R_3, B_1)	(R_3, B_2)
B_1	(B_1, R_1)	(B_1, R_2)	(B_1, R_3)	(B_1, B_1)	(B_1, B_2)
B_2	(B_2, R_1)	(B_2, R_2)	(B_2, R_3)	(B_2, B_1)	(B_2, B_2)

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The sample space contains 25 pairs. They are equally likely. Each outcome has the probability $\frac{1}{25}$ of occurring.

Let A be the event 'both balls are red':

$$A = \{(R_1, R_1), (R_1, R_2), (R_1, R_3), (R_2, R_1), (R_2, R_2), (R_2, R_3), (R_3, R_1), (R_3, R_2), (R_3, R_3)\}$$

Therefore $P(A) = \frac{9}{25}$

Without replacement

Again, we start with a bag containing three red and two black balls. A ball is chosen at random and its colour recorded. The ball is *not* put back in the bag. A second ball is chosen at random from the remaining balls and its colour recorded.

The sample space is listed in the array below. There are 5×4 outcomes in the sample space.

Second ball / First ball	R_1	R_2	R_3	B_1	B_2
R_1	–	(R_1, R_2)	(R_1, R_3)	(R_1, B_1)	(R_1, B_2)
R_2	(R_2, R_1)	–	(R_2, R_3)	(R_2, B_1)	(R_2, B_2)
R_3	(R_3, R_1)	(R_3, R_2)	–	(R_3, B_1)	(R_3, B_2)
B_1	(B_1, R_1)	(B_1, R_2)	(B_1, R_3)	–	(B_1, B_2)
B_2	(B_2, R_1)	(B_2, R_2)	(B_2, R_3)	(B_2, B_1)	–

The '–' indicates that the pair cannot occur.

Let A be the event 'both balls are red':

$$A = \{(R_1, R_2), (R_1, R_3), (R_2, R_1), (R_2, R_3), (R_3, R_1), (R_3, R_2)\}$$

Therefore $P(A) = \frac{6}{20} = \frac{3}{10}$

Exercise 12E

1 The letters O, R, I, G, I, N are printed on plastic squares and placed in a hat.

a A square is randomly taken from the hat, observed and replaced. Then a second square is removed.

i What is the probability of obtaining two 'I's?

ii What is the probability of obtaining an 'I' on the first and an 'R' on the second?

iii What is the probability of an 'R' on the first and an 'I' on the second?

b A square is randomly taken from the hat, observed and not replaced. Then a second square is removed.

i What is the probability of obtaining two 'I's?

ii What is the probability of obtaining an 'I' on the first and an 'R' on the second?

iii What is the probability of an 'R' on the first and an 'I' on the second?

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2 A fishpond has 2 gold fish and 3 black fish. Two fish are taken without replacement, one after another. What is the probability of:

a two gold fish?

b two black fish?

c a black and a gold in any order?

3 A jar contains 5 blue and 3 red jelly beans. Joe randomly takes one out and eats it. Leanne then takes one out. What is the probability that:

a Joe obtains a blue jelly bean and Leanne a red?

b Joe obtains a red jelly bean and Leanne a red?

c Joe obtains a blue jelly bean and Leanne a blue?

d Leanne obtains a red jelly bean?

4 A two-digit number is randomly created from the set of digits: 3, 4, 5, 6 and 7.

a If each digit is allowed to be used more than once, what is the probability that it will be:

i an even number greater than 50?

ii an odd number less than 60?

b If each digit is allowed to be used only once, what is the probability that it will be:

i an even number greater than 50?

ii divisible by 11?

Review exercise

1 There are 10 marbles, numbered 1 to 10, in a bowl. A marble is randomly taken out. What is the probability of getting:

a a 6?

b an even number?

c a number divisible by 5?

d a number greater than 4?

2 Twenty cards are put in a hat. Fifteen of the cards are blue and numbered 1 to 15. The other five cards are red and numbered 16 to 20. A card is taken randomly from the hat. Find the probability of taking out:

a a blue card

b a red card

c a red card with an odd number

d a number greater than 16

e an odd number

f a number divisible by 3

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3 A ball is drawn from a box containing 6 red balls, 4 white balls and 10 blue balls. Find the probability of:

a getting a blue ball or a red ball

b not getting a red ball

4 Clare and David made a record of the colour of handbags carried by customers in a restaurant. This is what they recorded.

Colour	White	Red	Yellow	Green	Blue	Black
Frequency	20	8	6	5	12	16

a What was the total number of handbags recorded?

b If a person with a handbag is selected at random, what is the probability that the handbag is:

i red? ii blue? iii white? iv not green?

5 The names Arthur, Con, Christine, Anna, Donald and Enid are written separately on cards and put in a hat. One card is drawn out. What is the probability that:

a a girl is chosen?

b a boy is chosen?

c Christine in chosen?

d Con is not chosen?

6 All students at a school were asked what their method of transport to school was. The results were: 40% walk to school, 20% come by car, 12% use public transport and the rest ride a bicycle.

A student is chosen at random. Find the probability that this student:

a walks to school

b rides a bicycle to school

c does not use public transport

7 If a letter of the alphabet is chosen at random, what is the probability that it is:

a a vowel?

b a consonant?

c one of the letters of AMSI?

d one of the letters of ICE-EM?

e one of the consonants of ICE-EM?

f one of the vowels of ICE-EM?

8 A coin is tossed three times. What is the probability of getting:

a three heads b two heads c one head d no heads?

9 A die is rolled and a coin is tossed. The pair of results is written down.

a Draw a tree diagram for the experiment.

b Draw an array diagram for the experiment.

c What is the probability of obtaining a tail and a 5?

d List the favourable outcomes for the event ‘obtain a head following an even number’.

e What is the probability of obtaining a head following an even number?

f What is the probability of obtaining a tail and a number greater than 3?

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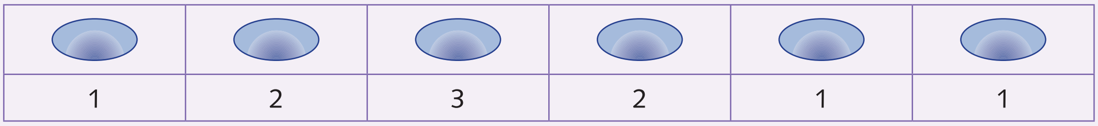
10 A ball is dropped at random into one of six holes, numbered as shown in the diagram below. The number under each hole gives the score obtained when the ball drops into that hole.



a State the probability of scoring 1.

b If the ball is dropped twice and the scores are added, find the probability of obtaining:

i a total of 5 **ii** a total of 4

11 A green die and a blue die are rolled and the results showing on the top faces are noted. What is the probability of getting:

a a 1 on the blue and a 2 on the green

b a 1 and a 2

c a 5 on the green and an odd number on the blue

d at least one 5

e an odd number on the blue and an even number on the green

12 Josephine has written 3 different letters and she correctly addresses 3 different envelopes. She is not at all careful in putting the letters in the envelopes, to the extent that it can be considered totally random. There are 6 ways of putting the letters in the envelopes. The 6 possibilities may be shown in a table, as below. In the second column, one possibility has been completed for you. It represents that letter 2 has been put in envelope 1, letter 3 has been put in envelope 2, and letter 1 has been put in envelope 3.

E1	L2					
E2	L3					
E3	L1					

a Complete the table.

b What is the probability that all letters are in the correct envelopes?

c What is the probability that exactly one letter is in the correct envelope?

d What is the probability that no letter is in the correct envelope?

13 Two dice are rolled and the results are noted. What is the probability that:

a the sum of the numbers is odd?

b the product of the numbers is even?

c the difference of the two numbers is 2?

d the sum of the numbers is greater than 9?

e one of the numbers is 3 and the sum of the numbers is less than 6?

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14 A game consists of choosing a card from a set of 5 cards and tossing a coin. The 5 cards are labelled from 'A' to 'E'.

a Draw a tree diagram to show the different outcomes of the experiment.

b Find the probability of getting:

i a head and an 'A'

ii a tail and an 'E'

iii a head and a vowel

iv a tail and a consonant

Challenge exercise

1 **a** A coin is tossed twice. By drawing a tree diagram, calculate the probability that at least one head is obtained.

b A coin is tossed three times. By drawing a tree diagram, calculate the probability that at least one head is obtained.

c A coin is tossed four times.

i How many outcomes are there? (*Hint*: Try to picture what the tree diagram would look like without actually drawing it.)

ii What is the probability that at least one head is obtained?

d A coin is tossed n times.

i How many outcomes are there?

ii What is the probability that at least one head is obtained?

2 **a** If a die is rolled, what is the probability that a six is not obtained?

b If two dice are rolled, what is the probability that no sixes are obtained?

3 For each event state the following:

i How many outcomes are there?

ii What is the probability that no sixes are obtained?

iii What is the probability that at least one six is obtained?

a Three dice are rolled.

b Four dice are rolled.

c n dice are rolled.

4 In a certain town there are 100 000 cars, each having a licence plate number from 1 to 100 000. What is the probability that the first car you see on a given day does not have a 6 or 7 in its number plate?

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