CHAPTER

13

Measurement and Geometry

Trigonometry

In this chapter we will begin studying a very important branch of mathematics called trigonometry, which starts with the ratios of side lengths in right-angled triangles. This topic is of great practical importance to surveyors, architects, engineers and builders, and also has many other applications.

The Greeks had a form of trigonometry based on using lengths of chords of circles. This was further developed by Arab mathematicians. German scholar Regiomontanus (1436–76) wrote the first book on what we might call 'modern' trigonometry to help him in his study of astronomy.

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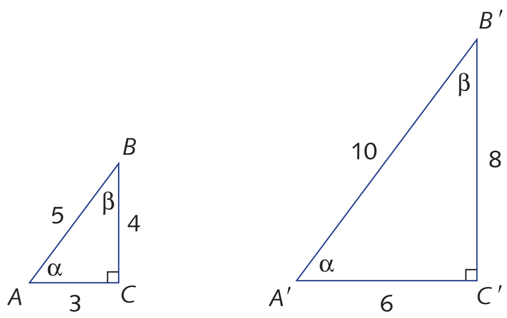
13A Introduction

Suppose we have two similar right-angled triangles as shown on the right. The angles of one match up with the angles of the other, and their matching sides are in the same ratio.



Since the matching sides are in the same ratio, then

$$\frac{BC}{B'C'} = \frac{AB}{A'B'} = \frac{1}{2}$$

Notice that this can be written as

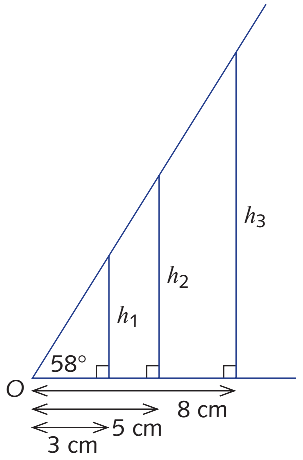
$$\frac{BC}{AB} = \frac{B'C'}{A'B'} = \frac{4}{5}$$

This tells us that in a right-angled triangle, once the angles are fixed, the ratios of sides are constant.

Trigonometry is concerned with this idea.

Activity: Similar triangles

Use your protractor to draw an angle of 58°. Place markers at distances of 3 cm, 5 cm, and 8 cm from O and draw perpendiculars as shown.



Measure the heights h_1, h_2 and h_3.

You should find that $h_1 \approx 4.8$ cm, $h_2 \approx 8.0$ cm and $h_3 \approx 12.8$ cm.

Now look at the ratio of the heights to the bases $\frac{h_1}{3}$, $\frac{h_2}{5}$ and $\frac{h_3}{8}$. These ratios are all (approximately) 1.6.

The ratios are the same since the triangles are similar.

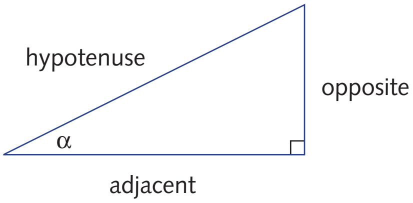
Naming the sides

To be more specific about the ratios of sides, we need to introduce some names.

You already know that the longest side in a right-angled triangle, which is the side opposite the right angle, is called the **hypotenuse**.

We now choose one of the two acute angles and give it a name. One of the Greek letters α, β, γ or θ is generally used. This angle is sometimes called the **reference angle**.

The side opposite the reference angle is called the **opposite side**, often abbreviated to the **opposite**; the remaining side, which is between the reference angle and the right angle, is called the **adjacent side** or simply **adjacent**. The word ‘adjacent’ means ‘near’ or ‘next to’. The side named the **adjacent** is next to the reference angle.



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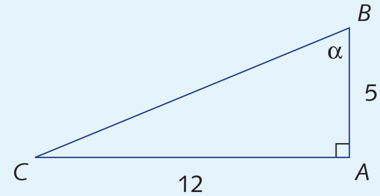
Example 1

a Label the three sides of the triangle relative to the marked reference angle.

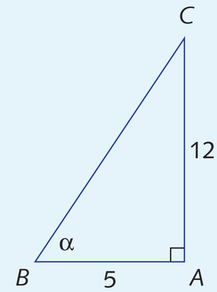
b Let $AB = 5$ and $AC = 12$.



i Find the length of BC.

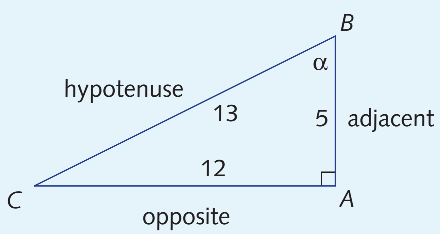
ii Write down the ratio $\dfrac{\text{opposite}}{\text{hypotenuse}}$.

iii Write down the ratio $\dfrac{\text{opposite}}{\text{adjacent}}$.



Solution

a



b i By Pythagoras' theorem:

$$BC = \sqrt{5^2 + 12^2}$$
$$= 13$$

ii $\dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{12}{13}$

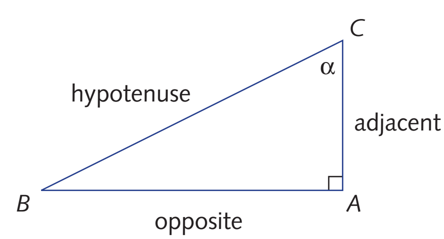
iii $\dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{12}{5}$

In this chapter we will generally leave ratios as improper fractions.

For example, we will write $\dfrac{12}{5}$ rather than $2\frac{2}{5}$.

Introduction to trigonometry

- For a right-angled triangle, once the angles are fixed, the ratios of sides are constant.
- For α, the reference angle, the sides of a right-angled triangle are named as shown on the right



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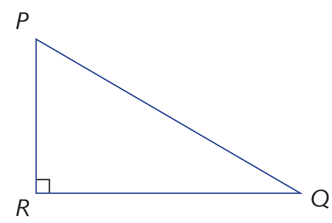
Exercise 13A

Example 1a

1 For the triangle opposite, name the opposite side, the adjacent side and the hypotenuse, using:

a $\angle PQR$ as reference angle

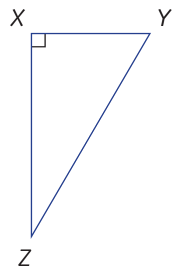
b $\angle RPQ$ as reference angle



2 For the triangle opposite, name the opposite side, the adjacent side and the hypotenuse, using:

a $\angle XYZ$ as reference angle

b $\angle XZY$ as reference angle

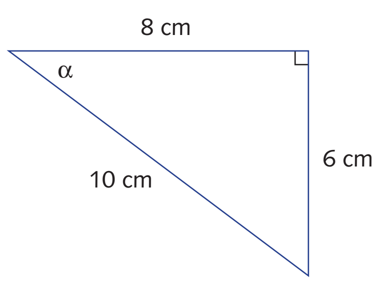


3 For the triangle opposite, relative to the reference angle α, what is the length of:

a the hypotenuse?

b the opposite side?

c the adjacent side?

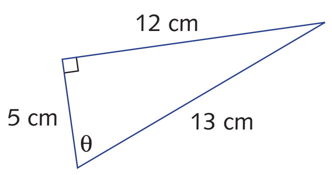


4 For the triangle opposite, relative to the reference angle θ, what is the length of:

a the hypotenuse?

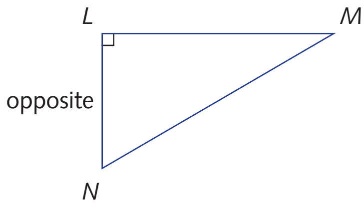
b the opposite side?

c the adjacent side?

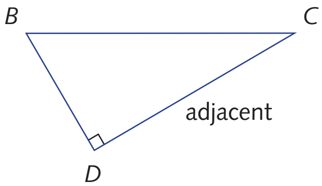


5 Name the angle that has been used as the reference angle for the naming of the sides.

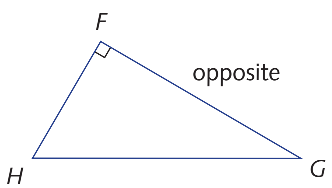
a



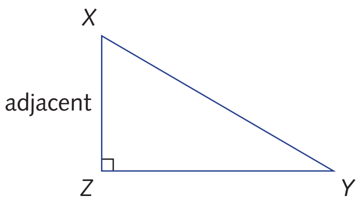
b



c



d



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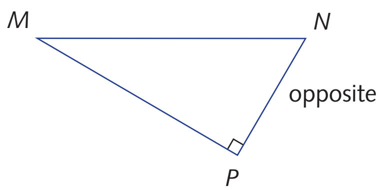
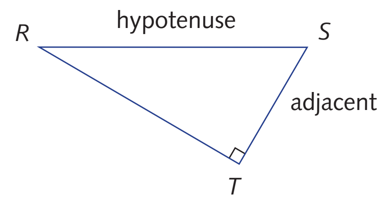
e 

f 

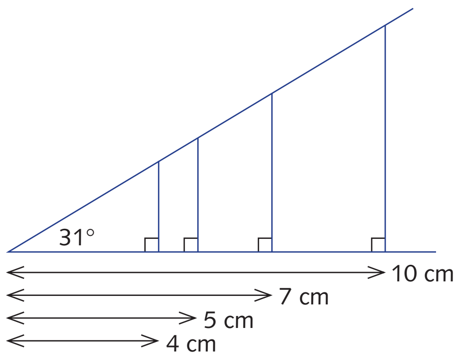
6 Draw diagrams of right-angled triangles with:

a reference angle 30° and the opposite side of length 8 cm

b reference angle 42° and the adjacent side of length of 5 cm

c reference angle 40° and the hypotenuse of length of 10 cm

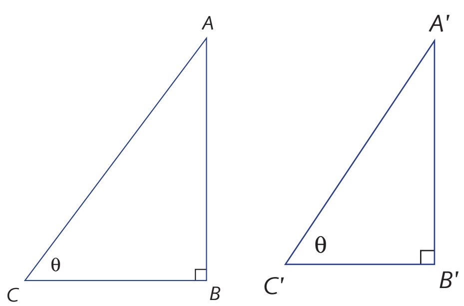
7 Use your protractor to draw an angle of 31°. Mark off distances of 4 cm, 5 cm, 7 cm and 10 cm. Draw a perpendicular at each point. Measure the heights and find, approximately, the ratio $\dfrac{\text{opposite}}{\text{adjacent}}$ in each triangle.



13B The three basic trigonometric ratios

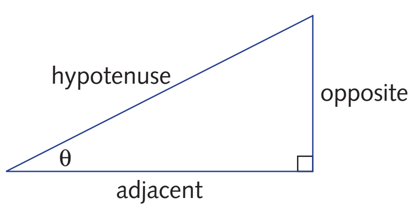
We saw in the previous section that once we fix an acute angle in a right-angled triangle, the ratios of the various sides remain the same, irrespective of the size of the triangle. This happens because the triangles are similar.

So in the diagrams on the right:



$$\frac{AB}{AC} = \frac{A'B'}{A'C'} \qquad \frac{BC}{AC} = \frac{B'C'}{A'C'} \qquad \frac{AB}{BC} = \frac{A'B'}{B'C'}$$

The above ratios have special names based on their relationship to the reference θ angle. These names are sine, cosine and tangent respectively, and in the context of their application to θ, we write:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

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Note: Collectively, the sine, cosine and tangent rations are known as the **basic trigonometric** ratios. Their names stem from Latin words that reflect a relationship they have with circles.

You will need to learn the three ratios for sine, cosine and tangent carefully. A simple mnemonic is

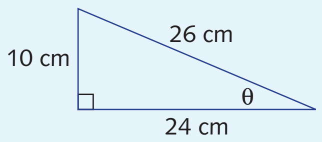
SOH CAH TOA

Sine – **O**pposite / **H**ypotenuse
Cosine – **A**djacent / **H**ypotenuse
Tangent – **O**pposite / **A**djacent

Example 2

For the triangle shown, write down the value of:

a sin θ **b** cos θ **c** tan θ



Solution

a $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
$= \dfrac{10}{26}$
$= \dfrac{5}{13}$

b $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
$= \dfrac{24}{26}$
$= \dfrac{12}{13}$

c $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
$= \dfrac{10}{24}$
$= \dfrac{5}{12}$

Example 3

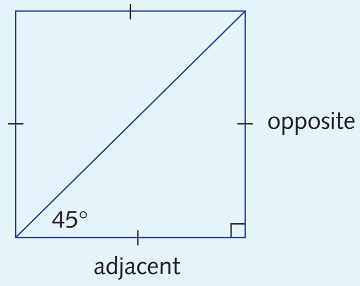
Draw the diagonal of a square and find the value of tan 45°.

Solution

Two right-angled triangles are formed, each with an angle of 45°. Each triangle is isosceles (angle sum of a triangle).

Hence the length of the opposite equals the length of the adjacent, so

$\tan 45° = \dfrac{\text{opposite}}{\text{adjacent}}$
$= 1$



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To find the tangent ratio for other angles, we could draw a triangle and measure lengths. This is what we did in the activity 'Similar triangles' on page 369 when we found the ratio of the height to the base in the triangle containing 58°. Hence, $\tan 58° \approx 1.6$.

Finding the values of trigonometric ratios in this way can only be approximate. Fortunately, your calculator, using clever mathematics, gives you these values with good accuracy. For example, your calculator gives $\tan 58° \approx 1.600\,334\,529$, which is approximately 1.6.

Make sure your calculator is in *degree* mode. Try entering tan 45 into your calculator and check that you get 1.

Example 4

Use your calculator to find, correct to 3 decimal places:

a $\tan 15°$ **b** $\tan 63°$

Solution

a $\tan 15° \approx 0.268$ **b** $\tan 63° \approx 1.963$

Values of the trigonometric ratios

Notice that since the hypotenuse is the longest side in a right-angled triangle, the ratios $\dfrac{\text{opposite}}{\text{hypotenuse}}$ and $\dfrac{\text{adjacent}}{\text{hypotenuse}}$ are always less than 1.

That is, if $0° < \theta < 90°$, then

$$0 < \sin\theta < 1 \quad \text{and} \quad 0 < \cos\theta < 1$$

Example 5

Use a calculator to find, correct to 3 decimal places:

a $\sin 15°$ **b** $\cos 63°$ **c** $\sin 23\frac{1}{2}°$

Solution

A calculator gives:

a $\sin 15° \approx 0.259$ **b** $\cos 63° \approx 0.454$ **c** $\sin 23\frac{1}{2}° \approx 0.399$

Finding the lengths of sides

We can use sine, cosine and tangent to find the side lengths of right-angled triangles.

Depending on which sides are given, we choose the appropriate ratio: either sine, cosine or tangent. You should always write down the ratio since the unknown side may be in the denominator.

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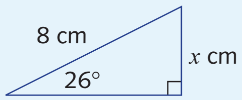
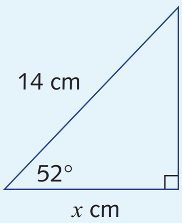
Example 6

Find the value of x, correct to 4 decimal places.

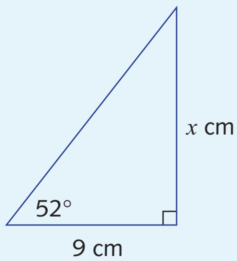
a



b



c



Solution

a This problem involves opposite and hypotenuse, so use sine.

$$\frac{x}{8} = \sin 26^\circ$$
$$x = 8 \times \sin 26^\circ$$
$$\approx 3.5070$$

b This problem involves adjacent and hypotenuse, so use cosine.

$$\frac{x}{14} = \cos 52^\circ$$
$$x = 14 \times \cos 52^\circ$$
$$\approx 8.6193$$

c This problem involves opposite and adjacent, so use tangent.

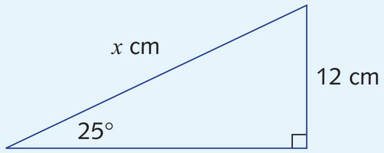
$$\frac{x}{9} = \tan 52^\circ$$
$$x = 9 \times \tan 52^\circ$$
$$\approx 11.5195$$

If the unknown is the length of the hypotenuse, an extra step is needed in the calculation.

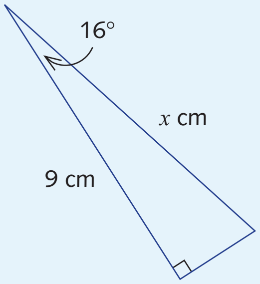
Example 7

Find the value of x, correct to 4 decimal places.

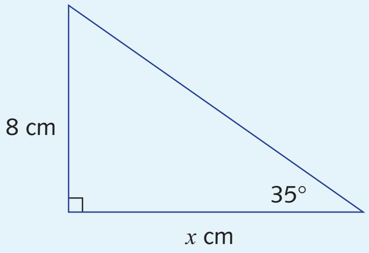
a



b



c



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Solution

a This problem involves opposite and hypotenuse, so use sine.

$$\sin 25° = \frac{12}{x}$$
$$x \sin 25° = 12$$
$$x = \frac{12}{\sin 25°}$$
$$\approx 28.3944$$

b This problem involves adjacent and hypotenuse, so use cosine.

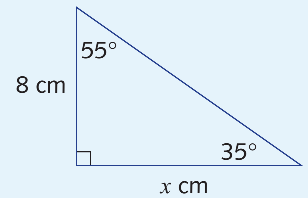
$$\cos 16° = \frac{9}{x}$$
$$x \cos 16° = 9$$
$$x = \frac{9}{\cos 16°}$$
$$\approx 9.3627$$

c This problem involves adjacent and opposite, so use tangent.

$$\tan 35° = \frac{8}{x}$$
$$x \tan 35° = 8$$
$$x = \frac{8}{\tan 35°}$$
$$\approx 11.4252$$

or

$$\frac{x}{8} = \tan 55° \text{ (55° is the complement of 35°)}$$
$$x = 8 \tan 55°$$



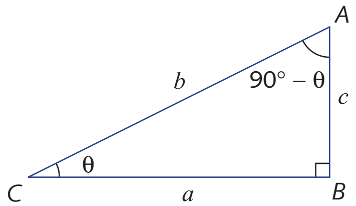
Complementary angles

In the triangle ABC, $\sin\theta = \frac{c}{b}$ and $\cos(90° - \theta) = \frac{c}{b}$. Hence

$$\sin\theta = \cos(90° - \theta)$$

Also, $\cos\theta = \frac{a}{b}$ and $\sin(90° - \theta) = \frac{a}{b}$. Hence

$$\cos\theta = \sin(90° - \theta)$$



For example:

$\sin 70° = \cos 20°$ $\quad$ $\cos 40° = \sin 50°$

$\cos 15° = \sin 75°$ $\quad$ $\sin 2° = \cos 88°$

The three basic trigonometric ratios

Let $0° < \theta < 90°$. Then:

- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\quad \cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\quad \tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
- $\sin\theta = \cos(90° - \theta)$, $\cos\theta = \sin(90° - \theta)$
- $0 < \sin\theta < 1$ $\quad 0 < \cos\theta < 1$ $\quad \tan\theta > 0$

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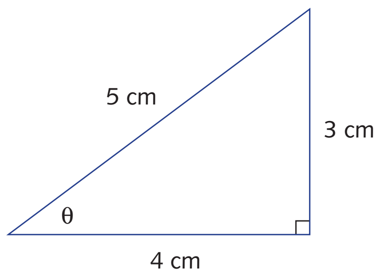
Exercise 13B

Example 2

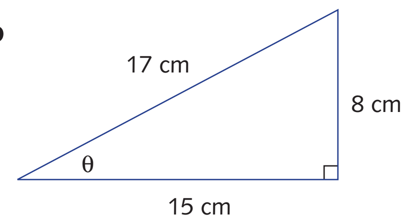
1 Write down the value, as a fraction in simplest form, of:

i sin θ **ii** cos θ **iii** tan θ

a



b



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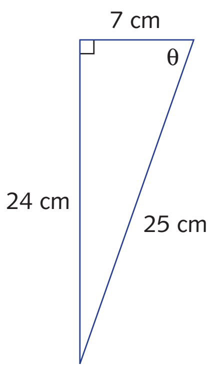

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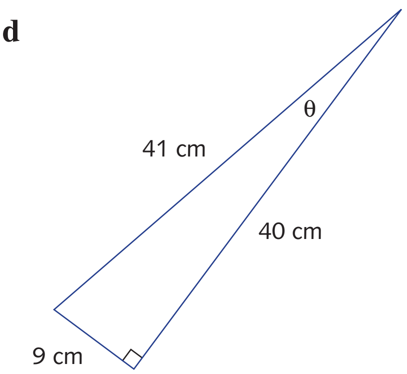

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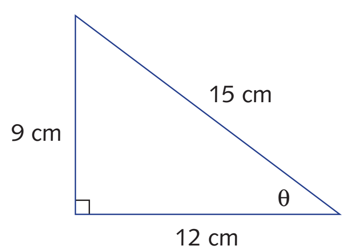

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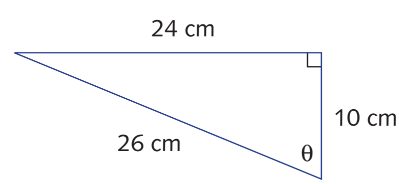


Example 4, 5

2 Using your calculator, give the value, to 4 decimal places, of:

a sin 10° **b** sin 20° **c** sin 30° **d** cos 46°

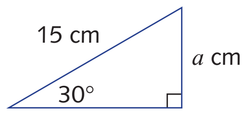
e cos 75° **f** tan 49° **g** tan 81° **h** sin 1°

i cos 37° **j** cos 88° **k** tan 3° **l** tan 45°

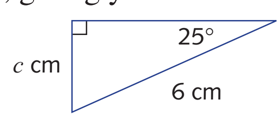
Example 6

3 Find the value of each pronumeral, giving your answer correct to 2 decimal places.

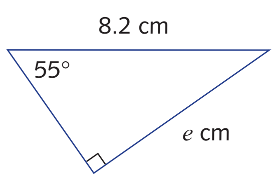
a



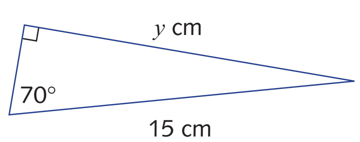
b



c



d



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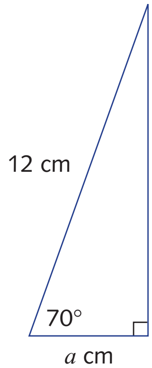
4 Find the value of each pronumeral, giving your answer correct to 3 decimal places.

a



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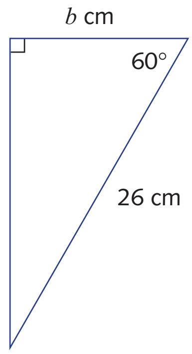

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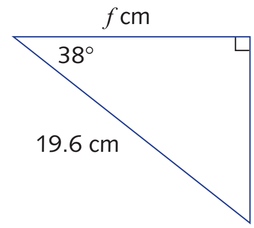
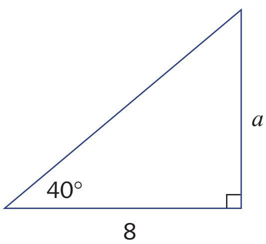


5 Find the value of each pronumeral, giving your answer correct to 4 decimal places.

a



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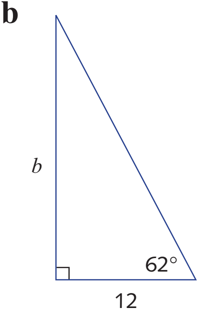

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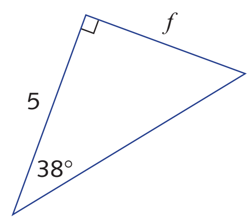
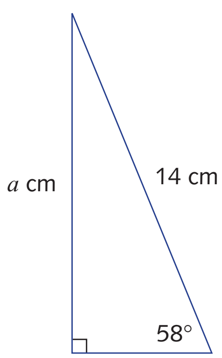


6 Find the value of each pronumeral, correct to 4 decimal places.

a



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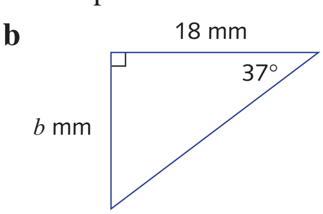

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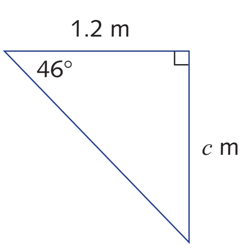

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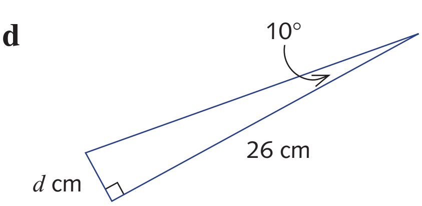

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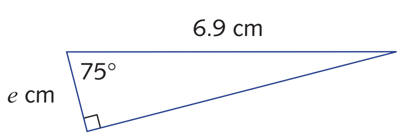

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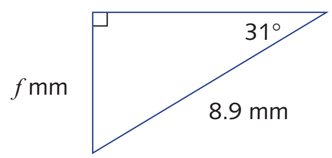

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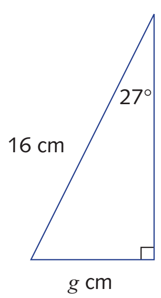

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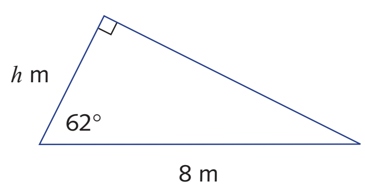

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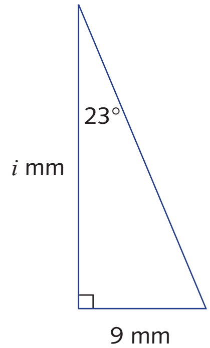
i



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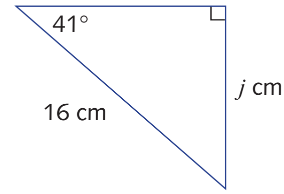

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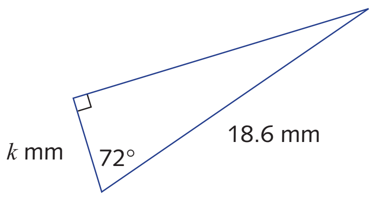

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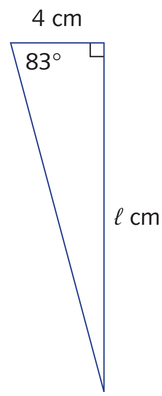
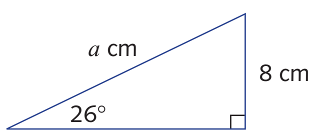


Example 7

7 Find the value of each pronumeral, correct to 4 decimal places.

a



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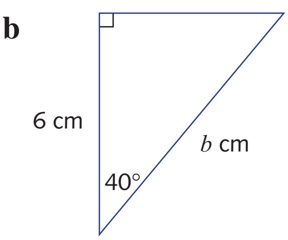

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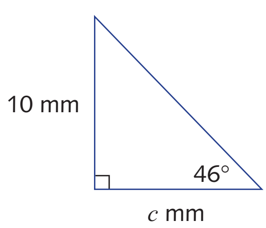

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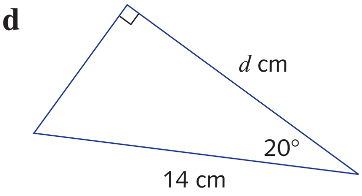

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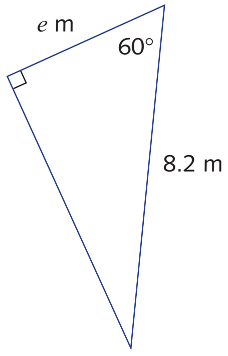

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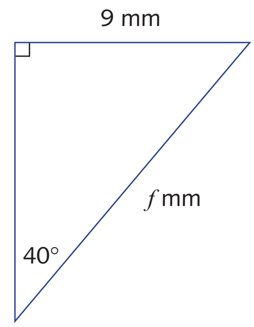

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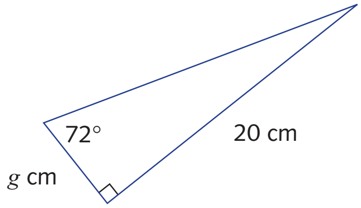

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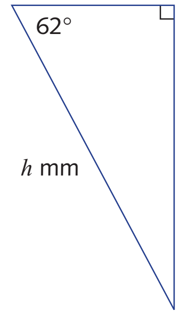

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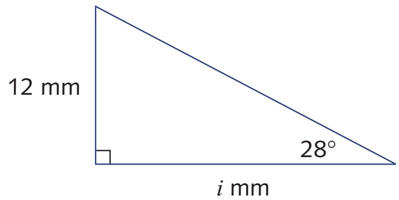

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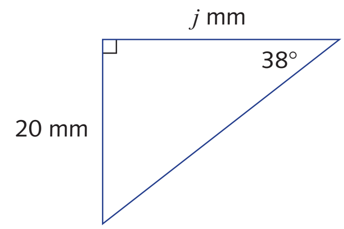

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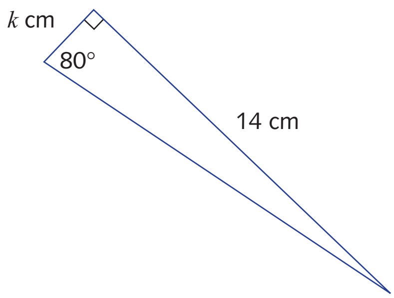

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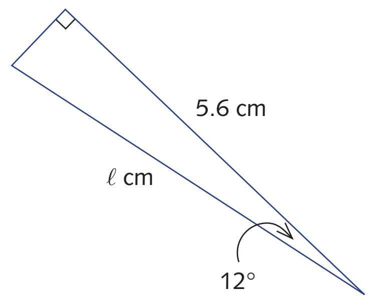
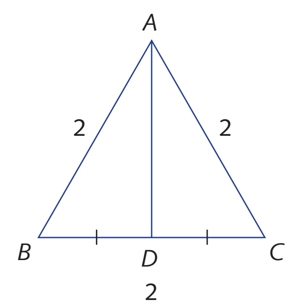


8 Draw a right-angled triangle with an angle of 45°. If one of the shorter sides has length 1, write down the lengths of the other sides without using trigonometry. Hence find the exact values of sin 45°, cos 45° and tan 45°.

9 *ABC* is an equilateral triangle, each of whose sides is 2 units. *D* is the midpoint of *BC*.



a What are the angles $\angle ABD, \angle BAD$ and $\angle ADB$?

b What is the length of *BD*?

c Find the length of *AD* in surd form.

d Use triangle *ABD* to complete the following table with exact values (in surd form where necessary).

θ	30°	60°
sin θ		
cos θ		
tan θ		

10 Use complementary angles to complete the following.

a sin 5° = cos …

b cos 72° = sin …

c sin 65° = cos …

d cos 46° = sin …

11 Draw a diagram and explain why $\tan(90° - \theta) = \dfrac{1}{\tan\theta}$.

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In the following questions, $0° < \theta < 90°$.

12 Find a value of θ for which $\sin\theta = \cos\theta$.

13 **a** Which is the largest side of a right-angled triangle?

b Is it possible for values of $\sin\theta$ to be greater than 1? Why?

c Is it possible for values of $\cos\theta$ to be greater than 1? Why?

d Is it possible for values of $\tan\theta$ to be greater than 1? Why?

14 **a** What happens to the value of $\sin\theta$ as θ gets closer and closer to $0°$?

b What happens to the value of $\sin\theta$ as θ gets closer and closer to $90°$?

c Can you explain why the value of $\sin\theta$ behaves like this?

15 Repeat Question **14** for:

i $\cos\theta$ **ii** $\tan\theta$.

13C Finding angles

We have seen that in any right-angled triangle with reference angle θ, there are three basic ratios associated with that angle and that the value of each ratio can be found using the calculator. In order to use the trigonometric ratios to find angles in a right-angled triangle, we need to reverse the process. Thus, given the sine, cosine or tangent of some angle between $0°$ and $90°$, we want to find the angle with the given ratio.

We have seen that $\tan 45° = 1$. We say that $45°$ is the **inverse tangent** of 1.

This is written as $\tan^{-1} 1 = 45°$

We use the calculator to find $\tan^{-1} x$. For example, $\tan 37° = 0.735\ldots$ and $\tan^{-1} 0.735$ is approximately equal to $37°$, correct to 2 decimal places.

Similarly, $\sin 30° = \frac{1}{2}$. We say $30°$ is the **inverse sine** of $\frac{1}{2}$ and write $\sin^{-1} 0.5 = 30°$

This notation is standard, but can be rather misleading. The index -1 does NOT mean 'one over' as it normally does in algebra. To help you avoid confusion, you should always read $\sin^{-1} x$ as *inverse sine of x* and $\tan^{-1} x$ as *inverse tan of x* and so on.

Similarly, a calculator gives $\cos 35° \approx 0.8192$ and $\cos^{-1} 0.8192 \approx 35°$ (read this as **inverse cosine** of 0.8192).

Finding angles in right-angled triangles

- If $x > 0$, then $\tan^{-1} x$ is the angle whose tangent is x.
- If $0 < x < 1$, then $\sin^{-1} x$ is the angle whose sine is x.
- If $0 < x < 1$, then $\cos^{-1} x$ is the angle whose cosine is x.
- Tangent, sine and cosine are applied to a reference angle to determine a specific ratio of side lengths in a right-angled triangle.
- Inverse tangent, inverse sine and inverse cosine are applied to a ratio to determine the size of a specific reference angle.

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Example 8

Find, correct to the nearest degree:

a $\sin^{-1} 0.6$ **b** $\cos^{-1} 0.412$ **c** $\tan^{-1} 2$

d the angle θ for which $\sin\theta = 0.8$

e the angle θ for which $\cos\theta = 0.2$ **f** $\cos^{-1} 2$

Solution

a By calculator

$\sin^{-1} 0.6 = 36.869\,89...°$

$\approx 37°$

b $\cos^{-1} 0.412 = 65.669\,46\ldots°$

$\approx 66°$

c $\tan^{-1} 2 = 63.434\,94\ldots°$

$\approx 63°$

d if $\sin\theta = 0.8$

then $\theta = \sin^{-1} 0.8$

$= 53.13...°$

$\approx 53°$

e If $\cos\theta = 0.2$

then $\theta = \cos^{-1} 0.2$

$= 78.463...°$

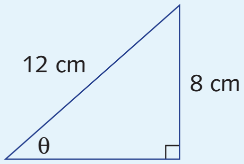
$\approx 78°$

f $2 > 1$, so $\cos^{-1} 2$ is not defined.

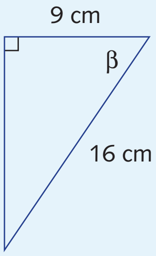
Example 9

Find each angle, correct to the nearest degree.

a



b



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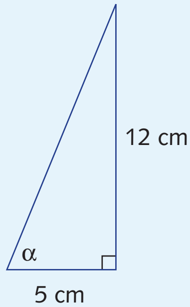

Solution

a $\sin\theta = \frac{8}{12} = \frac{2}{3}$

so $\theta = \sin^{-1}\frac{2}{3}$

$\approx 42°$

b $\cos\beta = \frac{9}{16}$

so $\beta = \cos^{-1}\frac{9}{16}$

$\approx 56°$

c $\tan\alpha = \frac{12}{5}$

so $\alpha = \tan^{-1}\frac{12}{5}$

$\approx 67°$

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Exercise 13C

Example 8a, b, c

1 Find, correct to the nearest degree:

a $\sin^{-1} 0.7$ **b** $\sin^{-1} 0.732$ **c** $\cos^{-1} 0.713$

d $\cos^{-1} 0.1234$ **e** $\tan^{-1} 0.1$ **f** $\tan^{-1} 12$

Example 8d, e

2 Find θ, correct to the nearest degree.

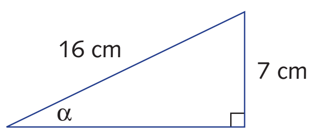
a $\sin \theta = 0.4067$ **b** $\sin \theta = 0.8480$ **c** $\cos \theta = 0.9816$

d $\cos \theta = 0.5299$ **e** $\tan \theta = 1.5399$ **f** $\tan \theta = 4.0108$

Example 9

3 Find the size of each marked angle, correct to the nearest degree.

a



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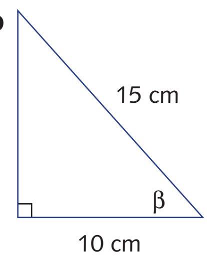

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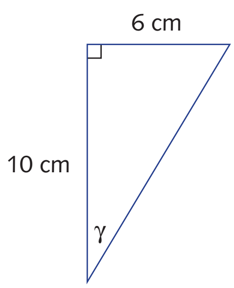

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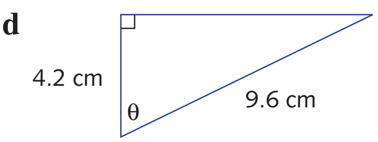

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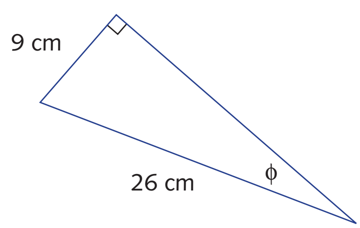

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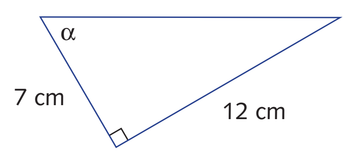

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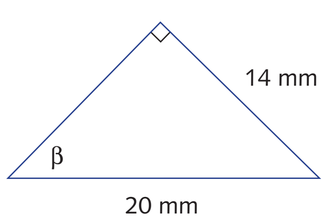

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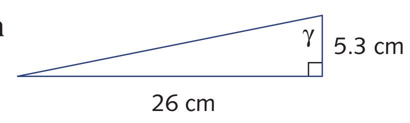

4 What are the sizes of the angles, correct to the nearest degree, in a:

a 3, 4, 5 right-angled triangle?

b 5, 12, 13 right-angled triangle?

c 7, 24, 25 right-angled triangle?

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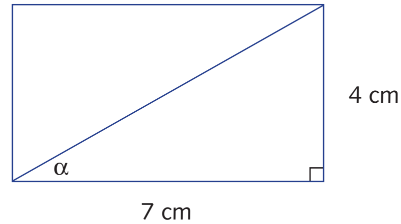
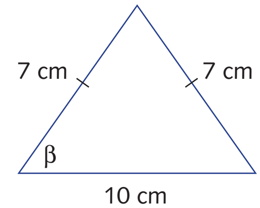
5 Find the angle between the diagonal and the base of the rectangle, correct to 2 decimal places.



6 Find the base angle in the isosceles triangle, correct to 2 decimal places.

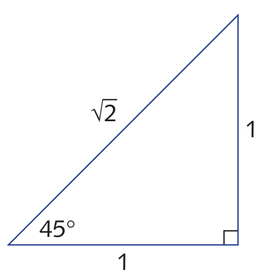


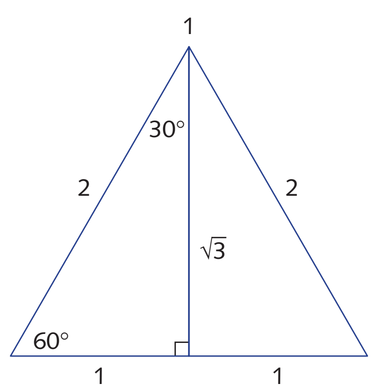
7 The diagonals of a rhombus have lengths 7 cm and 10 cm. Find the angles of the rhombus correct to 2 decimal places.

13D Miscellaneous exercises

The following exercise reinforces the techniques introduced in the previous sections.

You will have found the exact values of sine, cosine and tangent for 30º, 45º and 60º in previous exercises. These values are exemplified in the following triangles. One is a right-angled isosceles triangle with shorter sides of 1 unit, and the other is an equilateral triangle with side lengths 2 and with an angle bisector drawn.





These trigonometric ratios are given in the table below.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

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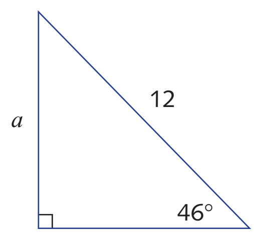
Check the details in the triangles and the entries in the table. We can either learn the table or remember the diagrams to construct the table. Note that, by complementary angles, $\sin 30° = \cos 60°$, $\sin 60° = \cos 30°$ and $\sin 45° = \cos 45°$.

Knowing that $\cos 60° = \frac{1}{2}$ and $\tan 45° = 1$, we can quickly reconstruct the triangles to find all the values.

Exercise 13D

1 Find the value of the pronumerals, giving angles to the nearest degree and lengths correct to 2 decimal places.

a



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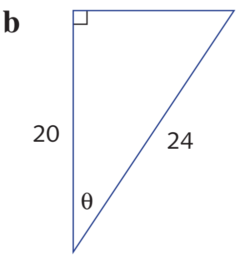

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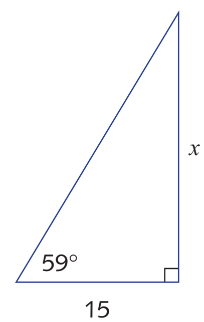

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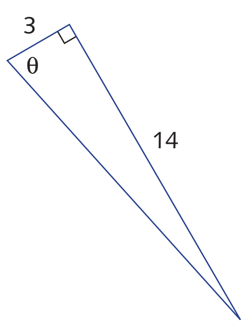

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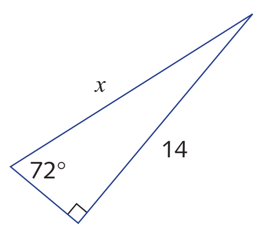

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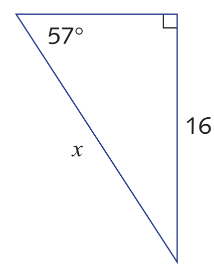

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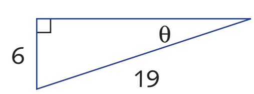

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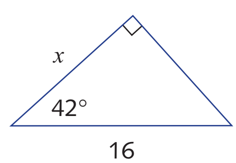

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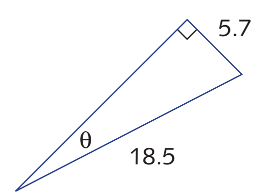

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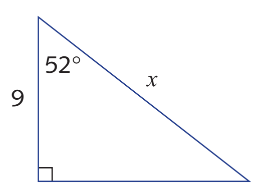

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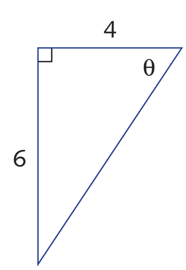

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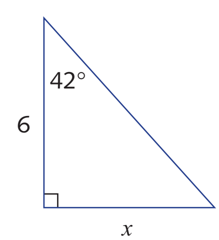

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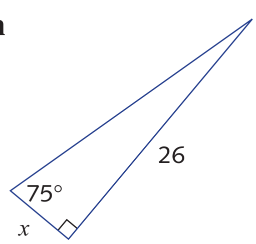

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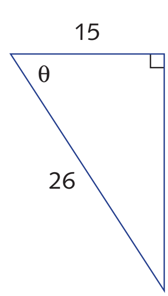

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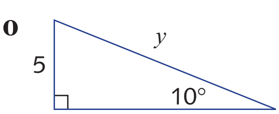

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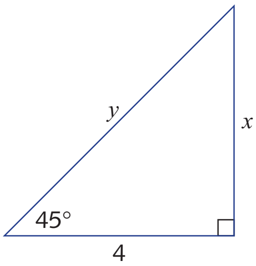
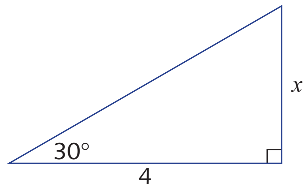
2 Find the exact value of the pronumerals.

a



b



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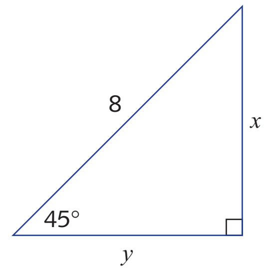

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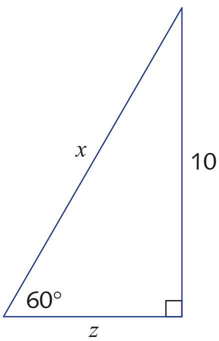

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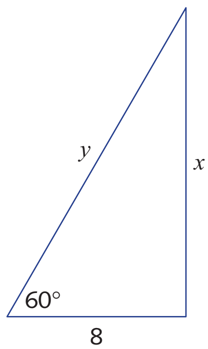

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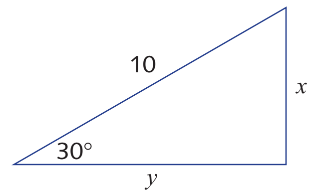

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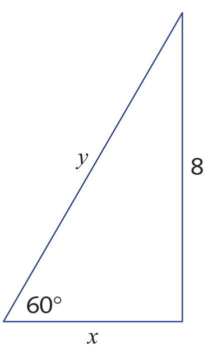

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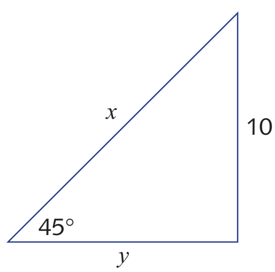

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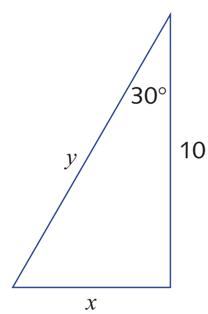

13E Solving problems using trigonometry

The results of the previous sections can be applied to practical problems.

In tackling each problem you should draw a diagram and clearly identify the side or angle you are asked to find.

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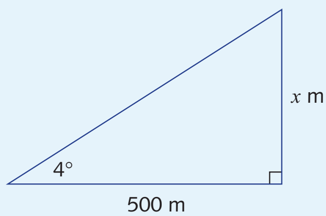
Example 10

A plane climbs at an angle of 4° after take-off. Find its altitude when it has flown a horizontal distance of 500 m.

Solution

Let the altitude be x m.

$$\text{Then } \tan 4° = \frac{x}{500}$$
$$x = 500 \tan 4°$$
$$\approx 35$$



Thus, the plane's altitude is 35 m (to the nearest metre).

Example 11

A rope, 4 m long, is attached to a vertical pole. The rope, held taut, is pegged into the ground 2 m from the base of the pole.

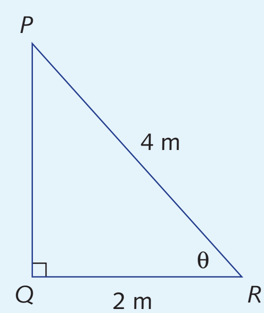
a Find the angle the rope makes with the ground.

b Find the angle the rope makes with the pole.

Solution

a Let PQ be the pole and θ be the angle between the rope and the ground.

$$\text{Then } \cos\theta = \frac{2}{4}$$
$$= \frac{1}{2}$$
$$\text{Hence } \theta = \cos^{-1}\frac{1}{2}$$
$$= 60°$$



Thus, the rope makes an angle of 60° with the ground.

b $\angle QPR = 90° - 60°$

$= 30°$

Thus, the rope makes an angle of 30° with the pole.

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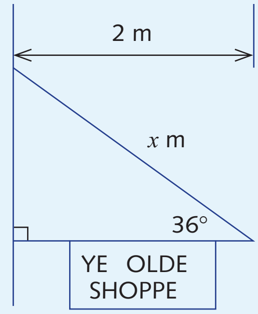
Example 12

A shop sign is supported by two rods. One rod is horizontal and is of length 2 m. The other rod is of length x m and makes an angle of 36° with the horizontal. Find its length correct to 2 decimal places.



Solution

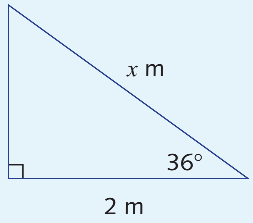
$$\text{From the diagram} \quad \cos 36° = \frac{2}{x}$$

$$x \cos 36° = 2$$

Hence

$$x = \frac{2}{\cos 36°}$$

$$\approx 2.47$$



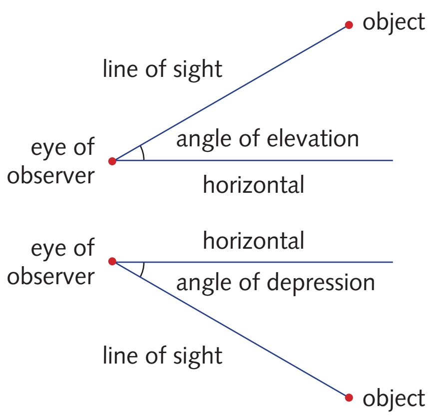
Therefore the rod is of length 2.47 m (correct to 2 decimal places).

Angles of elevation and depression

When a person looks at an object that is higher than the person, the angle between the line of sight and the horizontal is called the **angle of elevation**.

On the other hand, when the object is lower than the person, the angle between the horizontal and the line of sight is called the **angle of depression**.

Angles of elevation and angles of depression are always measured from the horizontal.



Example 13

From a point P, 30 m from a building, the angle of elevation of the top of the building is 41°. Find the height of the building, correct to the nearest metre.

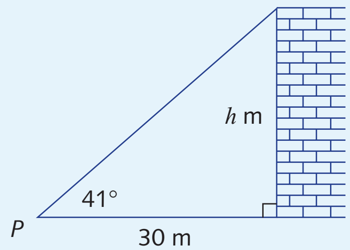
Solution

Let h m be the height of the building.

$$\text{Now} \quad \tan 41° = \frac{h}{30}$$

$$\text{hence} \quad h = 30 \tan 41°$$

$$\approx 26$$



So the building is 26 m high (correct to the nearest metre).

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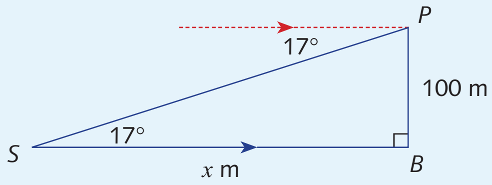
Example 14

From the top of a cliff, 100 m above sea level, the angle of depression to a ship sailing below is 17°. How far is the ship from the base of the cliff, to the nearest metre?

Solution

The diagram to the right shows the top of the cliff P, the ship S and the base of the cliff B. Let $SB = x$ m be the distance of the ship from the cliff. By alternate angles, $\angle PSB = 17°$.



Hence $\tan 17° = \dfrac{100}{x}$

so $x = \dfrac{100}{\tan 17°}$

$= 327.085$

The distance is 327 m (correct to the nearest metre).

Compound problems

Example 15

From a point A, 10 m from the base of a tree of height h m, the angle of elevation of the top of a tree is 36°. From a point B, x m further away from the base of the tree, the angle of elevation is 20°.

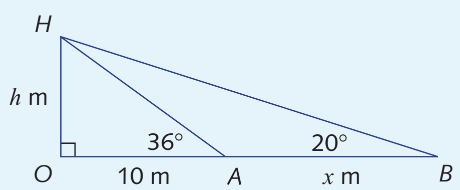
a Draw a diagram to illustrate the information given.

b Find the height of the tree, correct to the nearest tenth of a metre.

c Find the distance x m, correct to the nearest tenth of a metre.

Solution

a



b In ΔOHA $\tan 36° = \dfrac{h}{10}$

so $h = 10 \tan 36°$

$= 7.2654\ldots$ (leave this answer in your calculator)

≈ 7.3 (correct to the nearest tenth of a metre)

(*Continued over page*)

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c In ΔHOB $\quad \dfrac{h}{OB} = \tan 20°$

Hence $\quad OB = \dfrac{h}{\tan 20°}$

$= \dfrac{10 \tan 36°}{\tan 20°}$

$= 19.96...$

so $\quad x = AB$

$= OB - OA$

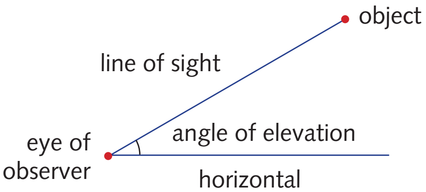
$= 9.96$

≈ 10.0 (correct to the nearest tenth of a metre)

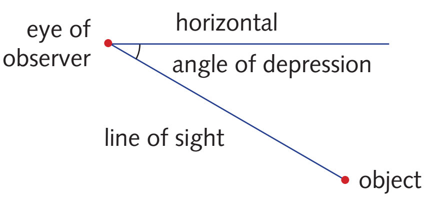
Note: Do not re-enter approximations. Keep values in your calculator.

Angles of elevation and depression

- When the object is higher than the person, the angle between the line of sight and the horizontal is called the angle of elevation.



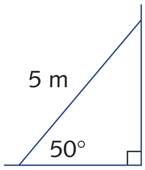
- When the object is lower than the person, the angle between the line of sight and the horizontal is called the angle of depression.



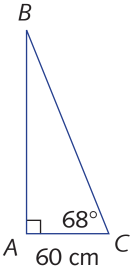
Exercise 13E

Example 10

1 A 5 m ladder leans against a wall, making an angle of 50° with the floor. How far up the wall (correct to the nearest centimetre) is the top of the ladder?



2 A girl's shadow is 60 cm long. If the line joining the top of the girl's head and the end of the shadow makes an angle of 68° with the ground, how tall is the girl (correct to the nearest centimetre)?



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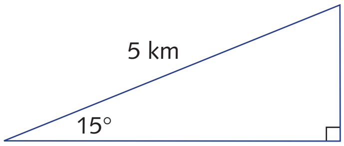
3 Gregor drives up a hill that is inclined at an angle of 15° to the horizontal. If he drives for 5 km, through what vertical distance has he risen? Give your answer correct to the nearest metre.



4 A tower is held in place by a rope of length 25 m. The rope is attached to the top of the tower and makes an angle of 76° with the ground. Find:

a the height of the tower, correct to the nearest centimetre

b how far, to the nearest centimetre, the base of the tower is from the anchor point

5 A ladder of length 4 m is leaning against a wall so that the top of the ladder is at the top of the wall. If the ladder makes an angle of 7° with the wall, find, correct to the nearest centimetre:

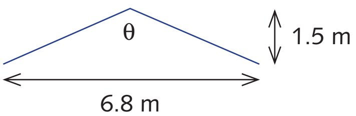
a the height of the wall

b how far the base of the ladder is from the wall

Example 11 **6** A skateboard ramp is 5 m long and rises a vertical distance of 2 m. What angle does the ramp make with the horizontal ground? Give your answer correct to the nearest degree.

7 An isosceles triangle has base length 10 cm and the other two sides are each of length 6 cm. Find the angles in the triangle, giving your answers correct to the nearest degree.

8 A roof truss is 6.8 m wide and 1.5 m high. Calculate the angle θ between the two beams, correct to the nearest degree.



9 A horizontal awning that has a depth of 80 cm is attached to the top of a window of height 2 m. What is the least angle that a ray of sunlight can make with the window if no light is to shine directly into it? Give your answer correct to the nearest tenth of a degree.

Example 13 **10** The angle of elevation from a point on a horizontal surface to the top of a vertical pole 20 m away is 35°. Find the height of the pole, correct to the nearest metre.

Example 14 **11** From a point on top of a tower of height 5.8 m, the angle of depression to a swimmer is 16°. How far from the base of the tower is the swimmer? Give your answer correct to the nearest tenth of a metre.

12 Two points in space 25 m apart are such that the vertical distance between them is 18 m. Find the angle of elevation or depression between the two points. Give your answer correct to the nearest degree.

13 Two bushwalkers, Gore and Tess, are testing new electronic measuring devices. Gore is on a mountain side at the 2300 m contour line. Tess is at the 2150 m contour line on a different mountain. Their measuring devices say that the direct distance between them is 1.35 km. Find the angle of elevation, correct to 2 decimal places, from Tess to Gore.

14 Carlos is standing 16 m from the base of a tree that is 14 m high. Carlos's eye is 1.6 m from the ground.

a What is the angle of elevation from Carlos's eye to the top of the tree, correct to the nearest degree?

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b If Carlos looks up at an angle of 26°, how far up the tree will he see? Give your answer correct to the nearest centimetre.

c What is the angle of elevation from Carlos's eye to a point halfway up the tree, correct to the nearest degree?

15 From a point 100 m from the base of a building, the angle of elevation to the top of the building is 31°.

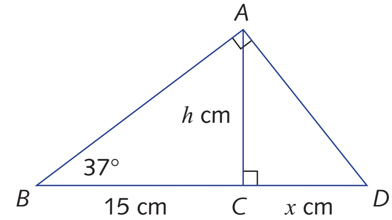
a How high is the building, correct to the nearest tenth of a metre?

b If there is a vertical flag pole of height 15 m on top of the building, what is the angle of elevation of the top of the flag pole? Give your answer correct to the nearest degree.

16 For the diagram on the right:

a find the value of h, correct to 2 decimal places

b use this value to find x, correct to 2 decimal places

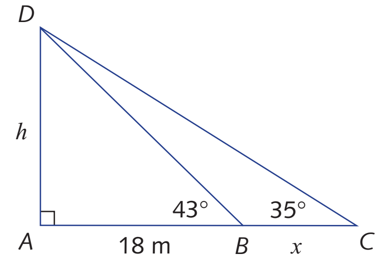


Example 15 **17** For the diagram on the right:

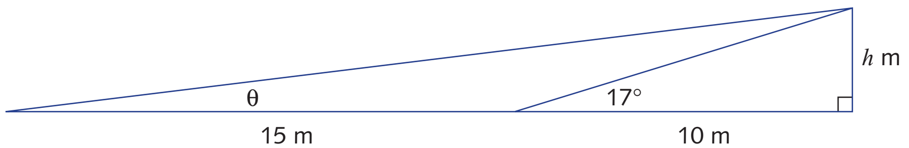
a find the value of h, correct to the nearest centimetre

b use this value to find the length AC, correct to the nearest centimetre

c hence find x, correct to the nearest centimetre



18 For the diagram below:



a find the value of h, correct to the nearest centimetre

b hence find the value of θ, correct to the nearest degree

19 The top of a 25 m tower is observed from two points, A and B, in line with the base of the tower, but on opposite sides of it. The angle of elevation of the top of the tower is 26° from A and 34° from B.

a Draw a diagram to represent the information above.

b How far apart are the points A and B, correct to the nearest centimetre?

20 At a point A on the ground, 12 m from the base of a building, the angle of elevation of the top of the building is 51°. From a point B on the ground x m further out from A, in line with the base of the building, the angle of elevation is 40°.

a Draw a diagram to represent the information above.

b Find the height of the building to the nearest centimetre.

c Find the value of x correct to the nearest centimetre.

21 The line $y = 3x + 6$ crosses the x-axis at an angle of θ with the positive direction of the x-axis. Find the value of θ, correct to 1 decimal place.

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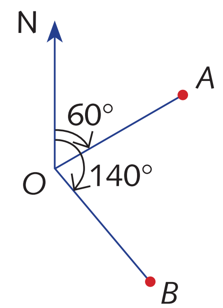
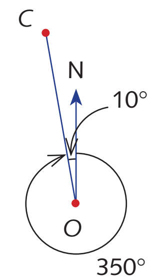
13F True bearings

Bearings are used to indicate the direction of an object from a fixed reference point. **True bearings** give the angle $\theta°$ from the north, measured clockwise. We write a true bearing of $\theta°$ as $\theta°$T, where $\theta°$ is an angle between 0° and 360°. It is customary to write the angle using three digits, so 0°T is written 000°T, 15°T is written 015°T and so on.

For example, the true bearing of A from O is 060°T, and the true bearing of B from O is 140°T.

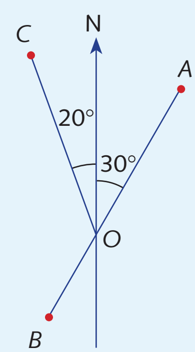


The true bearing of C from O is 350°T.



Example 16

Write down the true bearings of A, B and C from O.



Solution

The true bearing of A from O is 030°T.

For B, 30° + 180° = 210°, so the true bearing of B from O is 210°T.

For C, 360° − 20° = 340°, so the true bearing of C from O is 340°T.

Example 17

Anthony walks for 490 m on a true bearing of 140°T from point A to point B.

Find, correct to the nearest metre:

a how far east the point B is from point A

b how far south the point B is from point A

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Solution

a Distance east $= 490 \sin 40°$

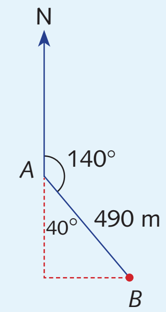
$= 314.96...$

≈ 315 m

b Distance south $= 490 \cos 40°$

$= 375.36...$

≈ 375 m



Example 18

The bearing of B from A is 075°. Find the bearing of A from B.

Solution

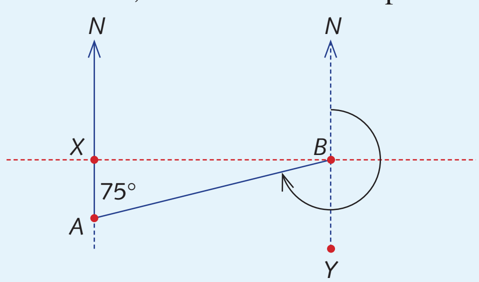
Mark the direction of B from A with respect to the direction of north, then re-draw a compass cross with origin at B.

$\angle ABY = 75°$ (alternate)

$\angle ABX = 15°$ (sum of angles in triangle ABX)

$180° + 75° = 255°$ (or $270° - 15° = 255°$)

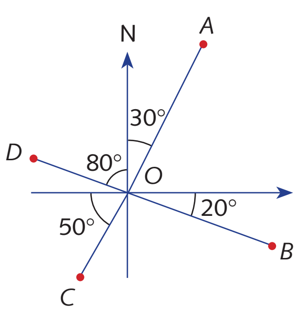
So the true bearing of A from B is 255°.



Exercise 13F

Example 16

1 For the diagram opposite, write down the true bearing of points A, B, C and D from O.



2 Draw a diagram to illustrate the fact that:

a the bearing from O to Z is 160°T

b the bearing from P to Q is 340°T

c to get from C to D, you travel on a bearing of 210°T

d to get from A to Y, you travel on a bearing of 170°T

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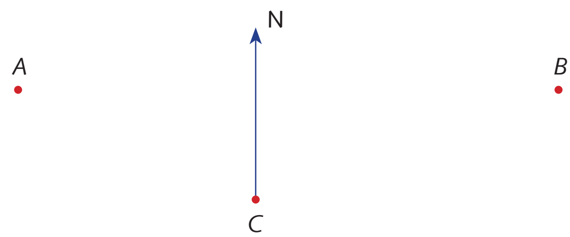
3 Using a protractor, measure the bearing of:

a A from C

b B from C



Example 17

4 Jo hikes for 369 m on a bearing of 053°T from point A to point B. Find, correct to the nearest metre:

a how far east point B is from point A

b how far north point B is from point A

5 A ship leaves port and sails on a bearing of 160°T for 200 km. Find, correct to the nearest tenth of a kilometre:

a how far south the ship is from the port

b how far east the ship is from the port

6 During an orienteering competition, Fran runs 1600 m on a bearing of 316°T from checkpoint 1 to checkpoint 2.

a How far west of checkpoint 1 is checkpoint 2, correct to the nearest metre?

b How far north of checkpoint 1 is checkpoint 2, correct to the nearest metre?

Example 18

7 Alan is standing 100 m due east of a landmark. Benny is standing 250 m due south of the landmark. What is the bearing, correct to the nearest tenth of a degree, of:

a Benny from Alan?

b Alan from Benny?

8 Christos is 200 m from a tree and Danielle is 400 m from the same tree. What is the bearing of Danielle from Christos, to the nearest degree, if:

a Christos is due north of the tree and Danielle is due east of the tree?

b Christos is due west of the tree and Danielle is due south of the tree?

c Christos is due east of the tree and Danielle is due south of the tree?

9 A hiker starts at point O and travels for 6 km on a bearing of 070°T to a point A. The hiker wishes to travel from A to a point B, which is 15 km due east of O. Give answers correct to the nearest hundredth of a kilometre and to the nearest degree.

a How far east of O is A?

b How far north of O is A?

c How far east of A is B?

d How far south of A is B?

e How far is B from A?

f On what bearing does the hiker travel if she walks directly from A to B?

10 A car travels on a bearing of 280°T on a very long straight road at 85 km/h, starting from an initial point at 12 p.m. Find how far north the car is from its original point at 12:30 p.m., correct to one-tenth of a kilometre.

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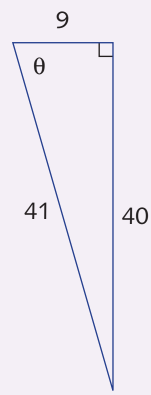
Review exercise

1 Write down the sine, cosine and tangent ratio of the indicated angle. Leave answers in exact form.

a

b



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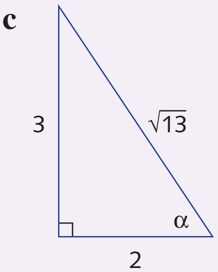
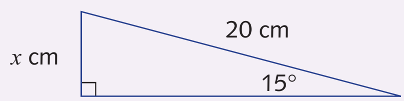


2 Find the value of each pronumeral, correct to 2 decimal places.

a



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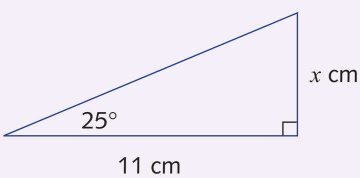

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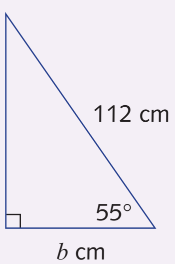

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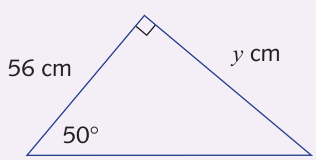

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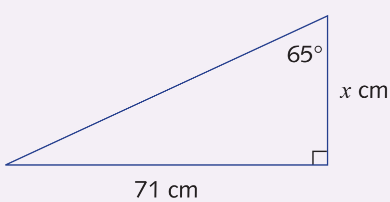

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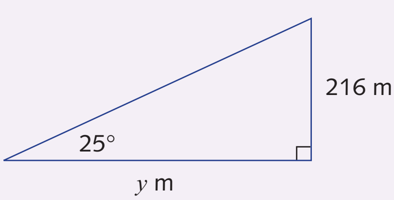
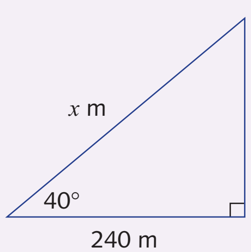


g



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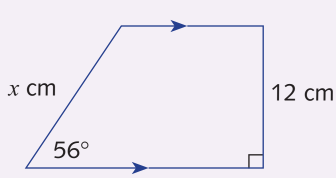

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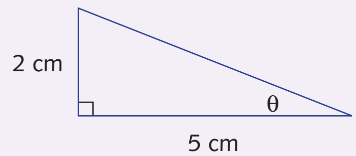
3 Find the value of each pronumeral, correct to the nearest degree.

a



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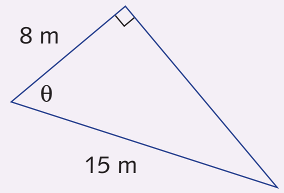

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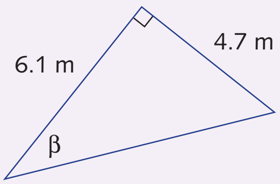

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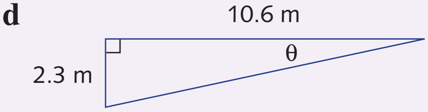


4 A ladder 9 m long is resting against a wall, making an angle of 55° with the ground.

a How high up the wall does the ladder reach, correct to the nearest centimetre?

b How far away from the wall is the foot of the ladder, correct to the nearest centimetre?

5 A ship is travelling due north at 15 km/h. The navigator sees a light due east, and eight minutes later the light has a bearing of 125°T from the ship. What is the distance from the ship to the light each time the navigator sees it, correct to the nearest metre?

6 The sides of a rectangle are 4 cm and 5 cm in length.

a Find the angles that a diagonal makes with the sides, correct to the nearest degree.

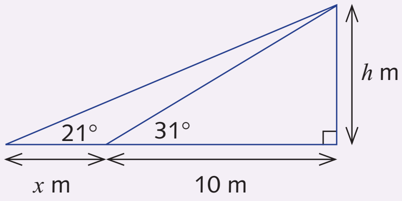
b What are the angles of intersection of the diagonals, correct to the nearest degree?

7 The angle of elevation of the top of a vertical cliff is observed to be 41° from a boat 450 m from the base of the cliff. What is the height of the cliff, correct to the nearest metre?

8 Find the angle of depression, to the nearest degree, of a boat 500 m out to sea as observed from the top of a cliff of height 90 m.

9 A ship sails on a bearing of 156°T until it is 45 km south of its starting point. How far has it travelled, correct to 2 decimal places?

10



a Find the height h, correct to the nearest centimetre.

b Find the value of x, correct to the nearest centimetre.

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Challenge exercise

1 The 'good ship' is at A, 1 km due north of the 'enemy ship' at 12 p.m. After this time the good ship moves on a bearing of 045°T while the enemy ship remains stationary.

a The good ship has travelled 1.5 km from A. Find, to the nearest metre:

i how far east it is from its position at 12 p.m.

ii how far north it is from its position at 12 p.m.

iii how far it is from the enemy ship

b Suppose that the good ship travels from A on the same bearing for x km. Find the distance between the ships d km, in terms of x.

c Finally, suppose that the good ship travels from A at a speed of 10 km/h. How far apart are the ships at 2:45 p.m. that day, correct to 2 decimal places?

2 A police helicopter, hovering at an altitude of 1000 m, observes a car travelling along a straight highway. At that instant, the angle of depression of the car from the helicopter is 27°. Five seconds later, the angle of depression of the car from the helicopter is 26°.

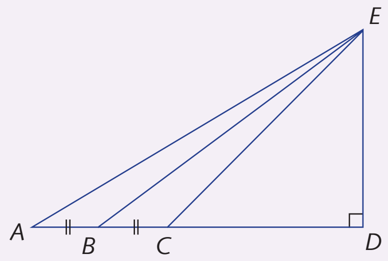
a What is the horizontal distance the car is from the helicopter when it is first sighted, giving your answer correct to the nearest tenth of a metre?

b Calculate the horizontal distance the car is from the helicopter 5 seconds after it is first sighted, giving your answer correct to the nearest tenth of a metre.

c Find the distance the car has travelled in the 5 seconds, giving your answer correct to the nearest tenth of metre.

d What is the speed at which the car is travelling in metres per second, correct to 1 decimal place?

3 Consider the diagram shown below.



a Prove that $BD = \frac{1}{2}(AD + CD)$.

b Hence, deduce that $\tan DEB = \frac{1}{2}\tan\angle AED + \frac{1}{2}\tan\angle CED$.

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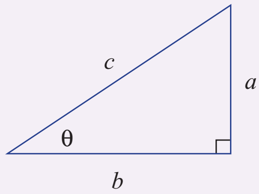
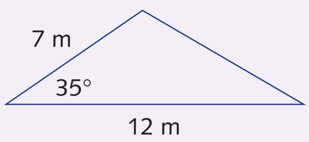
4 Consider the diagram shown below.

a Write down the values of $\sin\theta$ and $\cos\theta$.

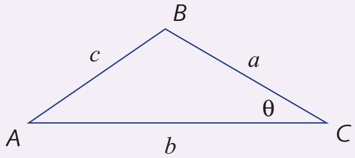
b Show that $(\sin\theta)^2 + (\cos\theta)^2 = 1$.



5 **a** Find the area in square metres of the triangle shown below, correct to 2 decimal places.

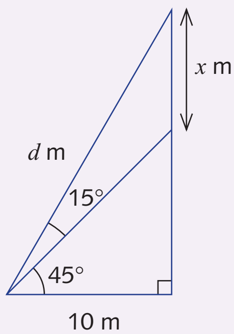


b Prove that the area of triangle ABC is $\frac{1}{2}ab\sin\theta$.



6 **a** For the diagram shown below, find the value of d.

b Show that $x = 10(\sqrt{3} - 1)$.



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