CHAPTER

14

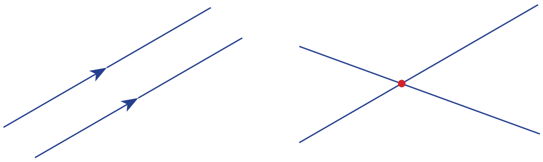
Number and Algebra

Simultaneous linear equations

The pair of equations $x - y = 4$ and $x + y = 6$ are called simultaneous equations. We wish to find the numbers x and y that satisfy both equations simultaneously. In this case $x = 5$ and $y = 1$.

We know that the x- and y-coordinates of any point on a straight line satisfy the linear equation of the line. Hence the solution $x = 5$, $y = 1$ to the linear equations $x - y = 4$ and $x + y = 6$ is the point where the two lines meet.

In this chapter we will be solving simultaneous linear equations. Since the two linear equations correspond to two distinct lines, the geometry of the situation is very simple – the lines are either parallel or they meet at a single point.



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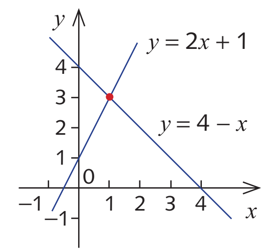
14A Solving simultaneous equations by drawing graphs

This introductory section shows how it is possible to solve simultaneous equations by drawing graphs. Care must be taken to make the graphs accurate, and it is important to check that the values found for x and y correspond to a point lying on both lines.



For example, the graph of $y = 2x + 1$ shown opposite consists of all points (x, y) that satisfy $y = 2x + 1$.

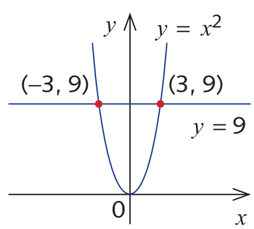
The graph of $y = 4 - x$ is drawn on the same set of axes. We see that the two lines intersect at the point $(1,\ 3)$. Since $(1,\ 3)$ is on both lines, $x = 1$ and $y = 3$ satisfy both equations. We can verify this by substitution.

We say that $x = 1$ and $y = 3$ satisfy the pair of equation $\begin{cases} y = 2x + 1 \\ y = 4 - x \end{cases}$ **simultaneously**.

Another way of saying this is that $x = 1$, $y = 3$ is the **solution** to the pair of **simultaneous equations** $\begin{cases} y = 2x + 1 \\ y = 4 - x \end{cases}$

In this chapter we will only be considering linear equations, but note what happens when we look at the simultaneous equations $y = x^2$ and $y = 9$. One solution is $x = 3$ and $y = 9$, and another solution is $x = -3$ and $y = 9$.

So in general, two simultaneous equations may have more than one solution.



Example 1

Solve the simultaneous equations

$y = x + 3$

$y = 2x + 1$

by drawing the graphs of the corresponding lines on the one set of axes.

Solution

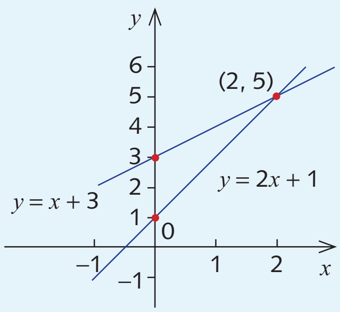
To sketch the graph of a line, we plot two points and join them (we can plot a third point as a check).

For $y = x + 3$:

x	−1	0	1
y	2	3	4

For $y = 2x + 1$:

x	−1	0	1
y	−1	1	3



We see from the diagram that the two lines intersect at $(2,5)$.

That is, the solution to the simultaneous equations is $x = 2$ and $y = 5$.

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We must check by substitution that $x = 2$ and $y = 5$ satisfy both equations.

$y = x + 3$	gives	$5 = 2 + 3$	✓
$y = 2x + 1$	gives	$5 = 2 \times 2 + 1$	✓

Note: If the check does not work, then the graphs are not drawn accurately enough.

Example 2

Solve the simultaneous equations

$x + 2y = 7$

$y = x - 1$

by drawing the graphs of the corresponding lines on the one set of axes.

Solution

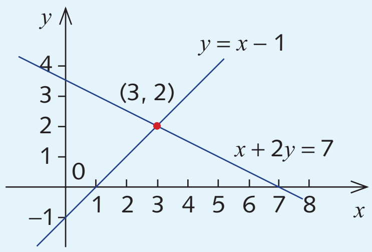
This time we draw the graphs by finding the two intercepts of each line.

If $x + 2y = 7$ and $x = 0$, then $y = 3\frac{1}{2}$

If $x + 2y = 7$ and $y = 0$, then $x = 7$

Hence $\left(0, 3\frac{1}{2}\right)$ and $(7, 0)$ are two points on the line.

Similarly, $(0, -1)$ and $(1, 0)$ determine the line $y = x - 1$.



From the graph, the lines meet at $(3, 2)$, so the solution to the simultaneous equations is $x = 3$ and $y = 2$.

Check by substitution:

$x + 2y = 7$	gives	$3 + 2 \times 2 = 7$	✓
$y = x - 1$	gives	$2 = 3 - 1$	✓

Exercise 14A

In this exercise it is important that you sketch the graphs accurately. The use of grid paper is recommended.

1 **a** Determine by substitution whether (1, 2) is a solution of the simultaneous equations $x + y = 3$ and $y = x + 1$.

b Determine by substitution whether $(-1, 6)$ is a solution of the simultaneous equations $x + y = 3$ and $y = x + 7$.

Example 1

2 **a** On the same set of axes, plot the graphs of:

i $y = x$ **ii** $y = 2x + 1$ **iii** $x + y = 4$

b Use graphs to solve the following pairs of equations simultaneously for x and y.

i $y = x$, $y = 2x + 1$

ii $y = x$, $x + y = 4$

iii $y = 2x + 1$, $x + y = 4$

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Example 2

3 By drawing graphs, solve simultaneously for x and y. Check each answer by substitution.

a $y = x + 2$
$y = 2x + 3$

b $y = 2x - 3$
$y = 3x - 5$

c $y = x + 2$
$y = 6 - x$

d $x + y = 4$
$2x + y = 6$

e $x - y = 1$
$3x + y = 3$

f $y = -2x + 3$
$y = x$

g $y = 2x + 1$
$y = 4 - x$

h $x - y = 2$
$x = 2y - 3$

i $x + 3y = 5$
$2x - y = -4$

4 On the same set of axes, draw the lines $y = 2x + 1$ and $y = 2x + 3$.

a What do you notice about these two lines?

b How many solutions do the simultaneous equations have?

5 On the same set of axes, draw the lines $x + y = 3$ and $2x + 2y = 6$.

a What do you notice about these two lines?

b How many solutions do the simultaneous equations have?

6 How many solutions do the simultaneous equations $y = 2x + 3$ and $2x - y = 6$ have?

7 **a** By drawing graphs, solve simultaneously for x and y.

i $y = x + 2$
$y = 3x + 1$

ii $y = 4x - 1$
$y = x + 4$

iii $3x + 2y = 6$
$2x - y = 4$

b Can you see any limitations in using graphs to solve a pair of simultaneous equations?

14B Substitution

In the previous section we learnt how to solve simultaneous linear equations by graphical means. This method clearly has its limitations and is very slow and tedious in practice. It is also difficult to draw sufficiently accurate graphs, particularly when the solutions are not integers. In the next two sections we shall introduce two algebraic methods called **substitution** and **elimination** for solving simultaneous equations.

Example 3

Solve $y = 2x + 1 \quad (1)$

$y = 3x - 2 \quad (2)$

for x and y.

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Solution

Using equation (1), $2x + 1$ can be substituted for y in equation (2).

$$2x + 1 = 3x - 2$$
$$1 = x - 2$$
$$x = 3$$

Substituting $x = 3$ into (1):

$$y = 2 \times 3 + 1$$
$$= 7$$

Check by substituting into (2):

$y = 3x - 2$ gives $7 = 3 \times 3 - 2$ ✓

Hence the solution is $x = 3$ and $y = 7$.

Note: This method is called the **substitution** method since one of the pronumerals is replaced (or substituted) by an expression involving the other pronumeral.

Example 4

Solve this pair of equations for x and y.

$y = 2x + 1$ (1)

$3x + 2y = 9$ (2)

Solution

Using equation (1), $2x + 1$ can be substituted for y in equation (2), giving:

$$3x + 2(2x + 1) = 9$$
$$3x + 4x + 2 = 9$$
$$7x + 2 = 9$$
$$7x = 7$$
$$x = 1$$

Substituting $x = 1$ in equation (1) gives:

$$y = 2 \times 1 + 1$$
$$y = 3$$

Check that (1, 3) satisfies both equations.

Hence the solution is $x = 1$ and $y = 3$.

Example 5

Solve the following simultaneous equations.

$x = 2y - 1$ (1)

$3y - 2x = 12$ (2)

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Solution

$3y - 2(2y - 1) = 12$ Substitute the expression for x from (1) into (2).

$$\begin{aligned} 3y - 4y + 2 &= 12 \\ -y + 2 &= 12 \\ -y &= 10 \\ y &= -10 \end{aligned}$$

Substituting $y = -10$ into (1) gives:

$$\begin{aligned} x &= 2(-10) - 1 \\ &= -21 \end{aligned}$$

Check that $(-21, -10)$ satisfies both equations.

Hence the solution is $x = -21$ and $y = -10$.

Note that Example 5 would be difficult to solve graphically.

Harder examples using substitution

If neither pronumeral is the subject of an equation, the method can still be used, but more algebra is involved. We make one of the pronumerals the subject of one of the equations.

Example 6

Solve $3x - 2y = 1$ (1)

$7y - 5x = 2$ (2)

for x and y.

Solution

We make x the subject of equation (1).

$$\begin{aligned} 3x &= 1 + 2y \\ x &= \frac{1}{3}(1 + 2y) \qquad (3) \end{aligned}$$

Substitute from (3) into (2).

$$7y - 5\left(\frac{1}{3}(1 + 2y)\right) = 2$$

$$\begin{aligned} 21y - 5 - 10y &= 6 \qquad \text{(Multiply both sides by 3)} \\ 11y &= 11 \\ y &= 1 \end{aligned}$$

Substituting $y = 1$ into (1) gives $x = 1$.

Check that $(1, 1)$ satisfies equations (1) and (2).

Hence the solution is $x = 1$ and $y = 1$.

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Solving simultaneous linear equations by substitution

- Make x or y the subject of one equation and substitute into the other equation.
- Solve the resulting equation.
- Substitute this value back into one of the original equations to find the value of the other pronumeral.
- Check that your solution satisfies the original equations.

Exercise 14B

Example 3

1 Solve the simultaneous equations using the substitution method.

a $y = x + 2$
$y = 2x + 3$

b $y = 2x - 1$
$y = 3x + 4$

c $y = 2x + 6$
$y = 4x - 2x$

d $y = x - 1$
$y = 3x - 5$

e $y = 1 - x$
$y = 3 - 2x$

f $y = 3x - 5$
$y = 6x - 11$

g $y = 4 - 3x$
$y = 5x - 2x$

h $y = 2x$
$y - 5x - 2x$

Example 4

2 Solve the simultaneous equations using the substitution method.

a $y = x + 1$
$x + y = 3$

b $y = x - 2$
$3x + y = 14$

c $y = 2x + 1$
$x + 2y = 12$

d $y = 1 - 2x$
$x + y = 2$

e $y = 3 - x$
$x - 3y = -5$

f $y = x - 4$
$x + y = 6$

g $y = x + 3$
$2x + 3y = 14$

h $y = 3x - 2$
$3x + y = 22$

i $y = 3 - 2x$
$x + 2y = 9$

j $y = 2x - 6$
$x - 2y = -3$

k $y = 3 - 2x$
$x + y = 0$

l $y = 5 - 2x$
$3x + 2y = 8$

Example 5

3 Solve the simultaneous equations using the substitution method.

a $x = 2y + 1$
$y = x - 2$

b $x = 3y - 1$
$2x + y = 12$

c $x = y - 1$
$x = 3y + 7$

d $y = 3x - 5$
$x = 2y + 6$

e $a = b + 3$
$2a + b = 18$

f $a + 3b = 6$
$a = 2b + 1$

g $a = 7 - 2b$
$a = b + 4$

h $3a - 2b = 2$
$a = b - 1$

Example 6

4 Solve the simultaneous equations using the substitution method.

a $3x + 4y = 11$
$7x - 2y = 3$

b $5x - 3y = 12$
$2x + 5y = 11$

c $2a + 7b = 13$
$3a + 4b = 9$

d $7a + 9b = -13$
$-2a - 6b = 11$

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14C Elimination

The other standard method of solving simultaneous equations is called **elimination**. This method creates an equation involving just one pronumeral, which can then be solved. Substitution then gives the value of the other pronumeral.

Consider the true statements:

$$6 + 3 = 9 \qquad (1)$$

$$5 + 7 = 12 \qquad (2)$$

If we add the left-hand sides of statements (1) and (2) and the right-hand sides of the statements we obtain:

$$(6 + 3) + (5 + 7) = 9 + 12$$

The resulting statement is true because statements (1) and (2) are true. We use this idea in the solving of simultaneous equations using the elimination method.

Example 7

Solve for x and y:

$$2x + y = 8 \qquad (1)$$

$$x - y = 1 \qquad (2)$$

Solution

We can eliminate the pronumeral y by adding the left-hand sides and the right-hand sides of the two equations.

(1) + (2) : $\quad (2x + y) + (x - y) = 8 + 1$

$$3x = 9$$

$$x = 3$$

Substituting $x = 3$ into (1): $\quad 6 + y = 8$

$$y = 2$$

Check mentally that (3, 2) satisfies equations (1) and (2).

Hence the solution is $x = 3$ and $y = 2$.

For the above example, substitution is almost as easy as elimination. In the next example, elimination is much easier.

Example 8

Solve for x and y:

$$3x - 4y = 6 \qquad (1)$$

$$2x + 4y = 4 \qquad (2)$$

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Solution

We eliminate the terms involving y by adding (1) and (2).

$$(1) + (2): \quad (3x - 4y) + (2x + 4y) = 6 + 4$$
$$5x = 10$$
$$x = 2$$

Substituting $x = 2$ into (1) gives $y = 0$.

Check that (2, 0) satisfies both equations.

Hence the solution is $x = 2$ and $y = 0$.

Example 9

Solve: $x + 3y = 7$ (1)

$x + y = 5$ (2)

Solution

Subtracting one equation from the other eliminates x.

$$(1) - (2): \quad (x + 3y) - (x + y) = 7 - 5$$
$$2y = 2$$
$$y = 1$$

Substituting $y = 1$ into (1), gives $x = 4$.

Check that (4, 1) satisfies both equations.

Hence the solution is $x = 4$ and $y = 1$.

Example 10

Solve: $2x - 3y = 4$ (1)

$5x - 3y = 19$ (2)

Solution

Clearly (1) – (2) eliminates y, but (2) – (1) is better as it avoids negative coefficients.

$$(2) - (1): \quad (5x - 3y) - (2x - 3y) = 19 - 4$$
$$3x = 15$$
$$x = 5$$

Substituting $x = 5$ into (1) gives: $10 - 3y = 4$

$$6 = 3y$$
$$y = 2$$

Check that (5, 2) satisfies both equations.

Hence the solution is $x = 5$ and $y = 2$.

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Harder examples using elimination

If we multiply both sides of an equation by a non-zero number, we obtain an **equivalent equation**. For example, $x + y = 5$ is equivalent to $2x + 2y = 10$.

Often, adding the equations, or subtracting one equation from the other, does not eliminate either pronumeral. In this situation we need to multiply one or both equations by a number to make the coefficients match.

Examples 11 and 12 show how to do this.

Example 11

Solve the following simultaneous equations.

$x + y = 5$ (1)

$2x + 3y = 14$ (2)

Solution

If we multiply equation (1) by 2, we can eliminate x.

(1) × 2: $2x + 2y = 10$ (3)

(2) − (3): $(2x + 3y) - (2x + 2y) = 14 - 10$

$y = 4$

Substituting $y = 4$ into (1) gives $x = 1$.

Check that $(1,4)$ satisfies equations (1) and (2).

Hence the solution is $x = 1$ and $y = 4$.

We could also have eliminated the pronumeral y by multiplying (1) by 3 and subtracting.

Example 12

Solve: $3x - 2y = 5$ (1)

$4x + 3y = 18$ (2)

Solution

Multiply both sides of equation (1) by 3 and both sides of equation (2) by 2 so that we can eliminate y.

(1) × 3: $9x - 6y = 15$ (3)

(2) × 2: $8x + 6y = 36$ (4)

(3) + (4): $17x = 51$

$x = 3$

Substituting into (1): $9 - 2y = 5$

$4 = 2y$

$y = 2$

Check that $(3, 2)$ satisfies the original equations.

Hence the solution is $x = 3$ and $y = 2$.

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General principles involved in solving by elimination

In solving simultaneous equations we have used two ideas repeatedly.

First, if $a = b$ and $c = d$, then:

$$a + c = b + d \quad \text{and} \quad a - c = b - d$$

Second, if $a = b$ is an equation and m is a non-zero constant, then $am = bm$.

This is called 'multiplying by the constant m'.

The skill is in choosing the order in which to perform these steps.

Keeping track of the steps used, such as (3) = (1) × 3, makes calculation and checking easier.

Elimination method for solving simultaneous equations

- Check if adding or subtracting the given equations will result in the elimination of a pronumeral. If so, then perform that operation to both sides of the equations.
- Otherwise, select a pronumeral to eliminate and take note of the respective coefficients. Identify the least common multiple of the two coefficients and create equivalent equations with the magnitude of this value represented in each. Add or subtract these equivalent equations so that elimination of the chosen pronumeral occurs.
- Solve the resulting equation.
- Find the value of the other pronumeral by substitution into one of the two equations (choose the simpler one where possible).
- Check your solution satisfies the original equations.

To decide which method is better to solve simultaneous equations, the following rule of thumb may be used.

Choosing the method (rule of thumb)

If one (or both) of the equations is of the form $y = \ldots$ or $x = \ldots$, use **substitution**; otherwise use **elimination**.

Exercise 14C

Example 7

1 Solve the simultaneous equations using the elimination method.

a $x - y = 2$
$3x + y = 14$

b $2x - 3y = 3$
$x + 3y = 6$

c $3x + y = 4$
$-3x + 2y = -10$

d $2x - y = 1$
$5x + y = 13$

e $3x + 2y = 14$
$4x - 2y = 14$

f $-x - y = 3$
$x + 5y = -11$

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Example 8, 9, 10

2 Solve the simultaneous equations using the elimination method.

a $x + y = 5$
$3x + y = 9$

b $3x + 4y = 15$
$x + 4y = 13$

c $2x - y = 7$
$4x - y = 15$

d $3x - 2y = 1$
$5x - 2y = 7$

e $2x + y = 6$
$2x - 3y = -4$

f $3x - 5y = 10$
$3x + 6y = 21$

g $2x + 3y = 1$
$-x + 3y = 4$

h $x - 2y = -8$
$3x - 2y = -12$

i $-2x - 5y = -3$
$4x - 5y = 21$

3 Solve the simultaneous equations using the elimination method.

a $2x + 3y = 11$
$2x - y = 7$

b $3x - y = -3$
$2x + y = -7$

c $4x - 3y = 17$
$-4x + y = -19$

d $3x - 5y = -27$
$2x - 5y = -23$

e $3x - y = -7$
$3x + 5y = -1$

f $4x - 3y = 8$
$2x + 3y = 4$

4 Solve the simultaneous equations using the elimination method.

a $x + y = 2$
$3x + 4y = 7$

b $x - y = 2$
$2x + 3y = 9$

c $4x + y = 1$
$3x - 2y = -13$

d $x - 5y = 9$
$3x + 2y = 10$

e $5x - 3y = -2$
$-x + 2y = -1$

f $3x - y = 11$
$5x + 2y = 14$

g $2x + 5y = 10$
$x - 3y = 5$

h $x - 3y = 0$
$2x - 4y = 2$

i $-3x - 2y = 4$
$x + 7y = 5$

Example 11, 12

5 Solve the simultaneous equations using the elimination method.

a $2x + 3y = 12$
$3x - 4y = -1$

b $5x - 2y = 17$
$4x + 3y = 9$

c $2x + 3y = 1$
$5x + 4y = -1$

d $3x - 5y = -21$
$5x + 3y = -19$

e $4x - y = -3$
$5x + 3y = -8$

f $6x - 5y = 8$
$3x + 2y = 13$

g $4a + 3b = 22$
$2a - 5b = -2$

h $3a - 4b = 3$
$7a - 3b = 26$

i $5a + 4b = 16$
$3a + 7b = 5$

j $4a - b = 1$
$3a + 7b = -38$

k $2a - 3b = -13$
$11a + 4b = 31$

l $8a - 3b = 14$
$5a + 7b = 62$

6 Solve the simultaneous equations. Use the most appropriate method.

a $2x + 3y = 8$
$3x - y = 1$

b $2x - 3y = 4$
$3x - 2y = 11$

c $y = 5 - 2x$
$2x - 3y = 17$

d $y = 2x + 8$
$y = -3x - 2$

e $x = 1 - 3y$
$3y - 2x = 4$

f $x = 2y + 1$
$x = 3y - 2$

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g $2x - 4y = 13$
$3x - 5y = 17$

h $4x + 5y = 4$
$2x + 4y = 3$

i $y = 5 - 9x$
$5x - 3y = 1$

j $y = 4x - 2$
$y = 6x - \frac{7}{2}$

k $x = 3y + 3$
$x = 6y + 1$

l $x = 1 - y$
$7x - y = 2$

14D Problems involving simultaneous linear equations

In this section we consider problems where it is natural to introduce two pronumerals, write down two simultaneous equations and solve them.

Example 13

At a hi-fi store, Benjamin bought a number of CDs costing \$10 each and a number of DVDs costing \$20 each. If he spent \$120 in total and bought twice as many CDs as DVDs, how many of each did he buy?

Solution

Method 1

Let x be the number of CDs Benjamin bought.

Let y be the number of DVDs he bought.

Then: $10x + 20y = 120$ (1) (total cost in dollars)

$x = 2y$ (2) (twice as many CDs as DVDs)

This pair of equations can be solved simultaneously.

Substituting from equation (2) into equation (1) gives:

$$10 \times 2y + 20y = 120$$
$$20y + 20y = 120$$
$$y = 3$$

Substituting $y = 3$ into equation (2) gives:

$$x = 6$$

Therefore Benjamin bought 6 CDs and 3 DVDs.

You should check that your answer satisfies the given information.

Method 2

This question can be done using just one pronumeral.

Benjamin bought twice as many CDs as DVDs.

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Let y be the number of DVDs he bought. He bought $2y$ CDs.

$$\begin{aligned} \text{Cost of DVDs} &= \$20y \\ \text{Cost of CDs} &= \$2y \times 10 \\ &= \$20y \\ \text{Hence} \quad 40y &= 120 \\ y &= 3 \end{aligned}$$

Benjamin bought 3 DVDs and 6 CDs.

Note: Many of the problems in Exercise 14D can be solved in both ways.

Example 14

A farmyard contains some pigs and some chickens. Altogether there are 60 heads and 200 legs. How many pigs are in the yard?

Solution

Let x be the number of pigs in the yard.

Let y be the number of chickens in the yard.

Then:
$$\begin{aligned} x + y &= 60 \quad (1) \quad \text{(number of heads)} \\ 4x + 2y &= 200 \quad (2) \quad \text{(number of legs)} \end{aligned}$$

Since only x is required, we eliminate y.

$(1) \times 2$: $\quad 2x + 2y = 120 \quad (3)$

$(2) - (3)$: $\quad 2x = 80$

$$x = 40$$

Therefore there are 40 pigs in the yard.

As a check, show that there are 20 chickens, 60 heads and 200 legs.

Exercise 14D

Solve each of the following problems by introducing two pronumerals and forming two simultaneous equations, even though some of them can be done more easily using one pronumeral.

1 Two numbers have a sum of 36 and a difference of 2. Find the numbers.

Example 13

2 Jane and Jacqui each think of a number. The number Jane thinks of is twice Jacqui's number and the sum of the two numbers is 39. What numbers did they think of?

3 Two numbers have a sum of 58. One number is two more than the other number. Find the two numbers.

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4 The sum of two numbers is 19. Three times one number minus twice the other number is 2. Find the two numbers.

5 At present, Martha is 24 years older than her daughter. In eight years' time she will be three times her daughter's age. What are Martha's and her daughter's present ages?

6 The equal sides of an isosceles triangle are 5 cm longer than the base. If the perimeter of the triangle is 34 cm, find the side lengths of the triangle.

7 Oscar is currently three times his son's age. In 14 years' time, he will be twice his son's age. What is Oscar's current age?

Example 14

8 Georgina has $8.00 to spend. She discovers that she can buy either 4 cans of soft drink and 4 chocolate bars or 2 cans of soft drink and 7 chocolate bars. How much does each item cost?

9 Romeo spends $15.60 and buys 3 hamburgers and 5 buckets of chips. Manni spends $12.40 and buys 4 hamburgers and 2 buckets of chips. How much does a hamburger cost?

10 An army transport vehicle has a carrying capacity of 5 tonnes. It can be used to transport either 20 men and 17 pieces of equipment or 35 men and 11 pieces of equipment. How many kilograms does each man weigh? (Assume that all men have the same weight and all pieces of equipment have the same weight.)

11 At a clothing store, trousers cost twice as much as shirts. A businessman buys 4 shirts and 2 pairs of trousers for a total cost of $240. How much does each shirt cost?

12 The length of a rectangular paddock is 5 m longer than twice its width. If the perimeter of the paddock is 154 m, find the dimensions of the paddock.

13 Four years ago Jie was twice Tan's age. In four years' time, Jie will be one and a quarter times Tan's age. How old are Jie and Tan now?

14 Toni's piggy bank contains $19.20, made up of 10-cent pieces and 20-cent pieces. If there are 114 coins in the piggy bank, how many coins of each denomination are there?

15 Sammi thinks of a two-digit number. The sum of the digits is 8. If she reverses the digits, the new number is 36 greater than her original number. What was Sammi's original number?

16 Jocelyn thinks of a fraction. If she adds 4 to both the numerator and the denominator, the new fraction is equal to $\frac{4}{5}$. If she subtracts 5 from both the numerator and the denominator, the new fraction is equal to $\frac{1}{2}$. What fraction did she think of?

17 Find a fraction that is equal to $\frac{7}{9}$ if 10 is added to both the numerator and the denominator, and is equal to $\frac{1}{2}$ if 5 is subtracted from both the numerator and the denominator.

18 A secretary buys a number of 45-cent and 60-cent stamps for a total cost of $22.50. If he interchanges the numbers of the two kinds of stamps, the total cost would have been $23.70. How many of each kind of stamp did he originally buy?

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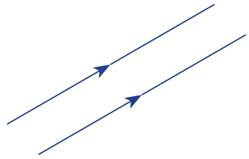
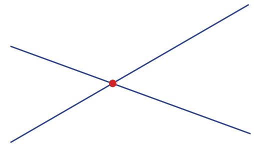
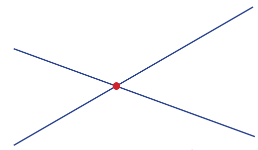
14E Geometry and simultaneous equations

Two distinct lines are either parallel or meet at a point.





There are three cases with a system of two linear equations.

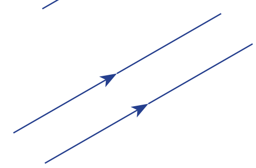


Case 1 — **There is one solution**

$x + y = 5$

$x - y = 3$

The solution to these equations is $x = 4$ and $y = 1$.
The two corresponding lines intersect at a single point.

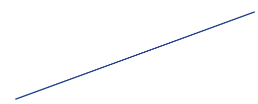


Case 2 — **There are no solutions**

$x + y = 5$

$x + y = 7$

$x + y = 5$, so $x + y$ cannot equal 7.
The two equations are parallel.



Case 3 — **There are infinitely many solutions**

$x + y = 5$

$2x + 2y = 10$

The two equations are equivalent.
The two equations represent the same line. Every point on this line represents a solution.

In case 1, the lines have different gradients and therefore intersect at a point. In case 2 and case 3, the gradients are equal and therefore the lines are either parallel or coincide.

Equivalent equations were discussed in Chapter 5.

Example 15

Determine whether the simultaneous equations have no solutions, one solution or infinitely many solutions. Illustrate these graphically.

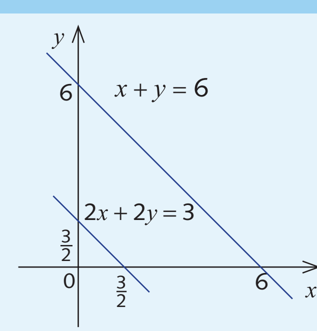
a $x + y = 6$
$2x + 2y = 3$

b $x + y = 8$
$x - y = 2$

c $x - y = 2$
$3x - 3y = 6$

Solution

a The equation $x + y = 6$ can be written as $y = -x + 6$, so the gradient of the corresponding line is -1.
The equation $2x + 2y = 3$ can be written as $y = -x + \frac{3}{2}$, so the gradient is -1.
The lines are parallel but have different y-intercepts. Therefore there are no solutions to the simultaneous equations.



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b $x + y = 8 \quad (1)$

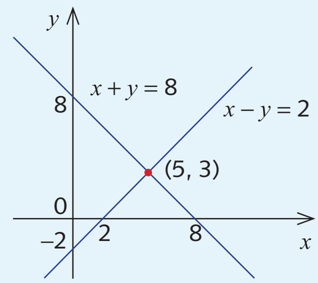
$x - y = 2 \quad (2)$

Add (1) and (2): $2x = 10$

$x = 5$

Substituting in (1) gives $y = 3$.

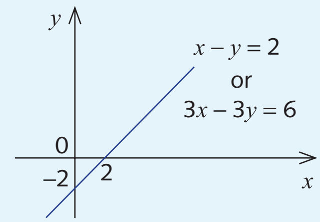
There is one solution: $x = 5$ and $y = 3$.



c $x - y = 2 \quad (1)$

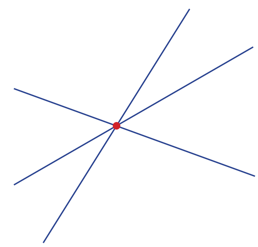
$3x - 3y = 6 \quad (2)$

These are equivalent equations. They determine the same straight line. All the points on the line satisfy both equations. There are infinitely many solutions.



Concurrence

Three or more lines are said to be **concurrent** if they intersect at a single point.



Example 16

Show that the lines $y = 3x + 2$, $y = 2x + 3$ and $y = -3x + 8$ are concurrent.

Solution

Let: $y = 3x + 2 \quad (1)$

$y = 2x + 3 \quad (2)$

$y = -3x + 8 \quad (3)$

We solve equations (1) and (2) simultaneously.

Substitute for y from (1) into (2):

$$3x + 2 = 2x + 3$$

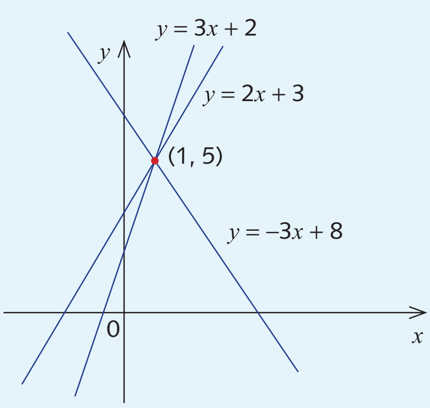
$$x = 1$$

Substituting $x = 1$ into equation (1) gives $y = 5$.

Finally check whether (1, 5) satisfies equation (3).

$$5 = -3 \times 1 + 8 \quad ✓$$

(1, 5) satisfies all three equations, hence the lines are concurrent.



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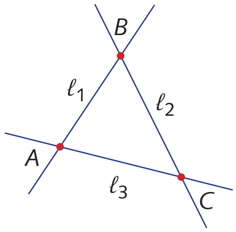
Areas of triangles



If we are given the equations of three lines ℓ_1, ℓ_2 and ℓ_3 that form a triangle, we can use the techniques developed in this chapter to find the coordinates of the vertices A, B and C of the triangle.

We can determine the area of a triangle given the coordinates of the vertices. In this section we will deal only with cases in which one line is parallel to one of the coordinate axes. The general case is dealt with in the Challenge exercise.

Example 17

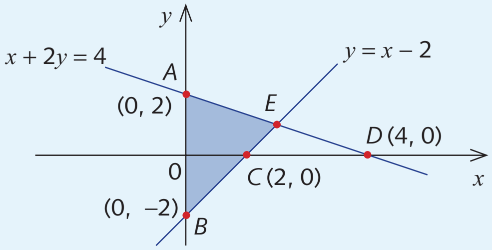
Find the area of the triangle formed by the lines $x + 2y = 4, y = x - 2$ and the y-axis.

Solution

It is essential to draw a diagram.

The line $y = x - 2$ meets the axes at $C(2,0)$ and $B(0,-2)$.

The line $x + 2y = 4$ meets the axes at $D(4,0)$ and $A(0,2)$.



We want the area of triangle ABE.

To find the coordinates of E, we solve the simultaneous equation:

$$y = x - 2 \quad (1)$$

$$x + 2y = 4 \quad (2)$$

Substituting (1) into (2):

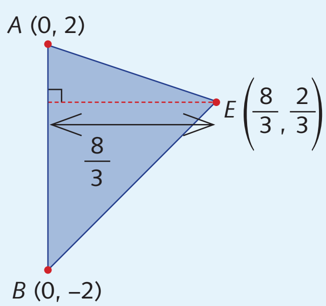
$$x + 2(x - 2) = 4$$

$$3x - 4 = 4$$

$$x = \frac{8}{3}$$

And from (1):

$$y = \frac{2}{3}$$



The length of AB is 4.

The distance from E to AB = x-coordinate of E

$$= \frac{8}{3}$$

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So area of triangle $ABE = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times 4 \times \frac{8}{3}$$

$$= \frac{16}{3}$$

Geometry and simultaneous equations

- There are three cases with a system of two linear equations:

  Case 1 **There is one solution**. The two corresponding lines meet at a single point.

  Case 2 **There are no solutions**. The two corresponding lines are parallel.

  Case 3 **There are infinitely many solutions**. The two equations are equivalent. They represent the same line.

- Three or more lines are said to be **concurrent** if they intersect at a single point.

Exercise 14E

1 For each pair of equations, find the coordinates of the point where the lines intersect.

a $2x + y = 4$, $3x - y = 1$ **b** $2x - 2y = 7$, $3x - 2y = 11$ **c** $y = 5 + x$, $2x - y = 3$

d $y = 2x - 12$, $y = -3x - 2$ **e** $x - 4y = 13$, $3x - 5y = 18$ **f** $2x + y = 4$, $3x - 4y = 6$

g $2x - 4y = 13$, $3x - 5y = 17$ **h** $y = 5 - 9x$, $5x - 3y = 1$ **i** $x - 3y + 3 = 0$, $x - 6y + 1 = 0$

Example 15

2 For each pair of equations, draw the two straight lines and determine whether there are no solutions, one solution or infinitely many solutions.

a $y = 2x + 3$ and $y = -x + 3$ **b** $x + 2y = 6$ and $y = 4x + 3$

c $y = 2x + 6$ and $y - 2x = 3$ **d** $x + y = 6$ and $y = -x + 6$

e $x + y = 6$ and $x + y = 8$ **f** $3x + 3y = 12$ and $y = -x + 4$

Example 16

3 Show that the three lines are concurrent and give the coordinates of their point of intersection.

a $y = 3x + 2,\ y = -x + 2,\ y = 2x + 2$ **b** $x + y = 2,\ y = 4x - 8,\ 2x + 3y = 4$

c $2x + y = 0,\ y = -x,\ y - 5x = 0$ **d** $y = 2x + 1,\ x + y = 4,\ 2x - y = -1$

4 Show that the lines $y = 2x + 1$, $y = 3x - 2$ and $y = 4x + 6$ are not concurrent.

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Example 17

5 Find the area of the triangle formed by the lines $x + y = 4, y = x$ and the:

a x-axis **b** y-axis

6 Find the area of the triangle formed by the lines $x + 2y = 4, y = x + 1$ and the:

a x-axis **b** y-axis

7 Find the area of the triangle formed by the lines $3x - 2y = 12, y = x - 5$ and the:

a x-axis **b** y-axis

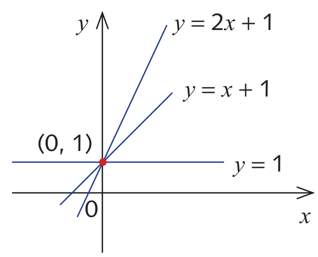
8 Find the area of the triangle formed by the lines $x - y = 4, y = -x$ and the line:

a $x = 5$ **b** $y = 3$

14F Families of straight lines

We have drawn on the right the lines with the following equations:

$$y = 1$$
$$y = x + 1$$
$$y = 2x + 1$$



These are examples of lines of the form $y = mx + 1$. All of these lines pass through $(0, 1)$.

Conversely, every non-vertical line passing through $(0, 1)$ has equation $y = mx + 1$ for some value of m.

We can describe this family geometrically as the family of all non-vertical lines through $(0, 1)$.

Example 18

The family of lines $y = mx + 1$ with varying gradient m all pass through the point $(0,1)$.

a For what value of m does the line $y = mx + 1$ not intersect the line $y = 2x + 5$?

b For what values of m does the line $y = mx + 1$ intersect the line $y = 2x + 5$?

c If the line $y = 2x + 5$ intersects the line $y = mx + 1$ at $(-1, 3)$, find m.

Solution

a The y-intercepts of the two lines are 5 and 1 respectively, so the lines are different. If the lines do not intersect they must be parallel. So $m = 2$.

b When $m \neq 2$ the lines will intersect.

c Substituting $x = -1$ and $y = 3$ into $y = mx + 1$

$$3 = -m + 1$$

Hence $m = -2$

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Exercise 14F

1 **a** Sketch the lines $y = x + 1$, $y = x + 2$ and $y = x - 3$, and geometrically describe the family of lines $y = x + b$.

b Sketch the lines $y = 2x + 1$, $y = -x + 1$ and $y = -2x + 1$, and geometrically describe the family of lines $y = mx + 1$.

c Sketch the lines $y = 2x + 2$, $y = -x + 2$ and $y = 3x + 2$, and geometrically describe the family of lines $y = mx + 2$.

Example 18

2 The equations of two lines are $y = x + 3$ and $y = mx - 2$.

a For what values of m will the two lines not intersect?

b For what values of m will the two lines intersect?

c Given that the lines intersect at $(5, 8)$, find m.

3 The equations of two lines are $y = x - 1$ and $mx - 2y = 4$.

a For what values of m will the two lines not intersect?

b For what values of m will the two lines intersect?

c Given that the lines intersect at $(2, 1)$, find m.

4 The equations of two lines are $y = mx + 5$ and $6x + 3y = 15$.

a For what values of m will the two lines not intersect?

b For what values of m will the two lines intersect?

c For the values of m found in part **b**, comment on the intersection of the lines.

5 The equations of two lines are $x + 2y = 7$ and $2x + 4y = k$.

a For what values of k will the two lines not intersect?

b For what values of k will the two lines intersect?

c For the values of k found in part **b**, comment on the intersection of the lines.

6 The lines with equations $y = x + k$ and $y = mx + 3$ intersect at $(1, 2)$. Find m and k.

7 The lines with equations $y = nx + 2$ and $y = mx + 3$ intersect at $(3, -1)$. Find m and n.

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Review exercise

1 Solve the simultaneous equations using the substitution method.

a $y = x + 4$, $y = 2x + 3$ **b** $y = 3x - 2$, $y = 2x - 1$ **c** $y = 2x - 3$, $y = 3x - 5$ **d** $y = x + 1$, $y = 4 - 2x$

2 Solve the simultaneous equations using the substitution method.

a $y = 2x + 1$, $x + y = 4$ **b** $y = 3x - 2$, $3x + 2y = 5$ **c** $y = 2x - 3$, $x + 2y = 9$ **d** $y = 7 - 2x$, $x + y = 5$

3 Solve the simultaneous equations using the substitution method.

a $x = 3y + 2$, $y = x - 6$ **b** $x = 2y - 1$, $2x + y = 11$ **c** $x = y - 3$, $x = 3y - 13$ **d** $y = 3x - 2$, $x = 2y + 3$

4 Solve the simultaneous equations using the elimination method.

a $x + y = 2$, $3x - y = 10$ **b** $2x - 3y = 1$, $3x + 3y = 9$ **c** $3x + y = 4$, $-3x + 2y = -10$ **d** $2x - y = 1$, $x + y = 5$

5 Solve the simultaneous equations using the elimination method.

a $x + y = 2$, $2x + y = 5$ **b** $3x + 2y = 13$, $x + 2y = 9$ **c** $2x - y = 9$, $3x - y = 11$ **d** $2x - 4y = 1$, $5x - 4y = 8$

6 Solve the simultaneous equations using the elimination method.

a $3x + 2y = 10$, $3x - y = 7$ **b** $2x - y = -3$, $5x + y = -11$ **c** $x - 3y = 5$, $-x + y = -19$ **d** $3x - 7y = -17$, $2x - 7y = -13$

7 Solve the simultaneous equations using the elimination method.

a $x + y = 2$, $2x + 3y = 7$ **b** $x - y = 2$, $3x + 4y = 6$ **c** $2x + y = 3$, $5x - 2y = -15$ **d** $x - 5y = 9$, $2x + 3y = 10$

8 Solve the simultaneous equations using the elimination method.

a $2x + 5y = 12$, $3x - 4y = -5$ **b** $3x - 2y = 17$, $4x + 3y = 0$ **c** $2x + 3y = -4$, $5x + 4y = -3$ **d** $3x - 5y = -21$, $4x - 3y = 8$

9 For each pair of equations, solve for x and y using the most appropriate method.

a $x + 2y = 6$, $3x - y = 2$ **b** $2x - 5y = 10$, $5x - 2y = 12$ **c** $y = 3 - 2x$, $2x - 5y = 12$

d $y = 2x + 5$, $y = -3x - 7$ **e** $x = 1 - 2y$, $5y - 2x = 10$ **f** $x = 5y + 3$, $x = 3y - 2$

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10 For each pair of lines, find the coordinates of the point where the graphs intersect.

a $5x + y = 7$
$3x - y = 1$

b $x - 3y = 5$
$4x - 3y = 13$

c $y = 3 + x$
$3x - 2y = 6$

d $y = 2x - 10$
$y = -3x + 4$

e $x - 3y = 13$
$4x - 5y = 18$

f $5x + y = 8$
$2x - 3y = 10$

11 The equations of two lines are $y = 2x + 5$ and $mx - 2y = 4$.

a For what values of m will the two lines not intersect?

b For what values of m will the two lines intersect?

c Given that the lines intersect at $(-1, 3)$, find m.

12 The equations of two lines are $mx + y = 2$ and $2x + 5y = 10$.

a For what values of m will the two lines not intersect?

b For what values of m will the two lines intersect?

c For the values of m found in part **b**, comment on the intersection of the lines.

13 The lines with equations $y = 2x + k$ and $y = mx - 5$ intersect at $(15, 40)$. Find m and k.

14 Find the area of the triangle formed by the lines $x + 3y = 9, y = 7 - x$ and the:

a x-axis

b y-axis

15 Tickets to the circus cost \$30 for adults and \$12 for children. For a matinee performance, 960 people attended and \$19080 was collected in ticket sales. Find the number of adults and children who attended the performance.

16 The difference of two numbers is 5. The sum of three times one number and two times the other is 25. Find the two numbers.

17 Find the value of m for which the lines $y = 3x + 2, y = 2x + 3$ and $y = mx$ are concurrent.

18 **a** Find the point of intersection of the lines $y = 3x + 6$ and $y = 2 - x$.

b Find the equation of the line that passes through this point of intersection and has y-intercept 6.

19 Find the value of c for which the lines $x + y = 8, x - y = 12$ and $3x + 2y = c$ are concurrent.

20 The line $ax + by = 6$ is concurrent with the lines $4x - 3y = 17$ and $4x + y = -19$ and is parallel to the line $y = 6x$. Find the values of a and b.

21 Solve the simultaneous equations.

a $3x + y = 5x + 4$ and $3x + y = 2y$

b $3x + 2y = x - 3y$ and $4x + 6y = 3x - 7$

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22 For a certain fraction, if 1 is added to the numerator and 1 is subtracted from the denominator the result is 1 For the same fraction, if 1 is subtracted from the numerator and 1 is added to the denominator the result is $\frac{3}{5}$. Find the fraction.

23 Solve the simultaneous equations.

a $\frac{x}{3} + \frac{y}{4} = 1$

$\frac{3x}{2} - y = 4$

b $\frac{2x}{7} + \frac{y}{8} = 0$

$\frac{3x}{4} - \frac{y}{3} = 0$

24 Two lines have gradients 3, 4 respectively and y-intercepts 1, –2 respectively. Find the coordinates of the point of intersection of the two lines.

25 The triangle ABC has a perimeter of 4 m. Find the length of each side if $AB + BC = 2AC$ and $AB + AC = 3BC$.

26 The line $y = 3x + c$ intersects the line $3x - 2y = 6$ at the point $(-4, -9)$. Find the value of c.

27 The lines $y = ax + 11$ and $y = \frac{1}{2}x - 3$ are perpendicular.

a Find the value of a.

b Find the coordinates of the point of intersection of the lines.

28 Points A, B and C have coordinates $(0,\ 1)$, $(5,\ 11)$ and $(1,\ 8)$ respectively. The line from C, which is perpendicular to AB, meets AB at the point N.

a Find the equations of AB and CN.

b Find the coordinates of N.

29 The lines $y = mx + 6$ and $y = nx - 3$ meet at a point where $x = 9$. Find the values of m and n if $m + n = 7$.

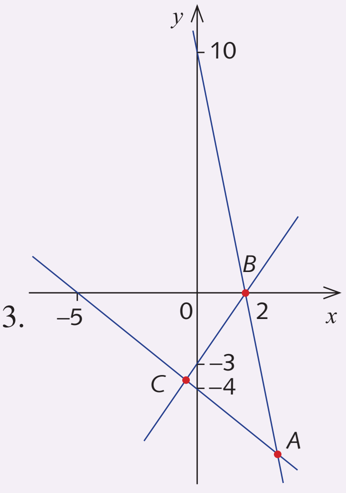
30 The equations of the lines that define ΔABC are:

AB: $5x + y = 10$

BC: $3x - 2y = 6$

CA: $4x + 5y = -20$

Find the coordinates of A, B and C.



31 In the rectangle $ABCD$,
$AB = p + 2q, BC = 3p - 10, CD = 5(p - q)$ and $DA = 2q + 3$.
Find the perimeter of the rectangle.

32 The length of the sides of a triangle are $2x - 4, y - 7$ and $2y - 4x$. If the triangle is equilateral, find the length of a side.

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Challenge exercise

1 Assume that m is positive throughout this question.

a Find the point of intersection of the lines $x + y = 4$ and $y = mx + 1$ in terms of m.

b Find the area of the region enclosed by the lines $x + y = 4$, $y = mx + 1$ and the x-axis in terms of m.

c Given the area of the triangle formed by the lines $x + y = 4$, $y = mx + 1$ and the x-axis is 8 square units, find the value of m.

2 **a** Find the point of intersection of the lines $2x + y = 2$ and $y = x + c$ in terms of c.

b Find the area of the region enclosed by the lines $2x + y = 2$, $y = x + c$ and the y-axis in terms of c.

c Given that the area of the region enclosed by the lines $2x + y = 2$, $y = x + c$ and the y-axis is 6 square units, find the value of c.

3 Solve the simultaneous equations for x and y. Assume that a and b are non-zero numbers.

a $ax - by = b$

$bx + ay = a$

b $2bx - a^2y = ab$

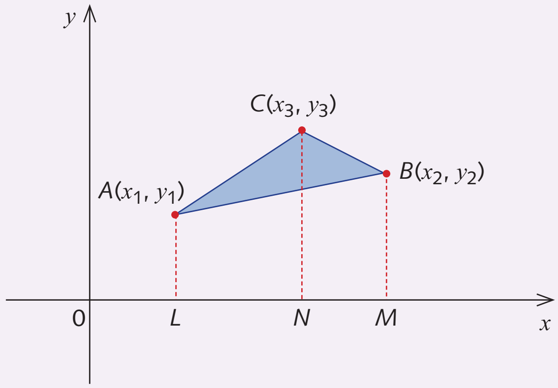
$ax + aby = a^2 + b^2$

c $\frac{x}{2a} + \frac{y}{2b} = a + b$

$x - y = -b(a + b)$

4 In the diagram shown below:

Area of triangle ABC = area of trapezium $ALNC$ + area of trapezium $CNMB$ − area of trapezium $LABM$



a Use this observation to prove that:

Area of triangle $ABC = \frac{1}{2}(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))$

b Use the formula to calculate the area of the triangle ABC where A, B and C have coordinates $(2, 3)$, $(4, 1)$ and $(3, 6)$, respectively.

c Find the area of the triangle DEF where D, E and F have coordinates $(2, 5)$, $(4, 6)$ and $(3, 2)$ respectively. Explain the answer.

d If the area determined by this formula is zero, what can be said about the points?

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5 Solve the simultaneous equations.

a
$$x + 2y - 3z = 10$$
$$y + z = 7$$
$$y - z = 3$$

b
$$x + y + z = 6$$
$$2x + 3y + 4z = 20$$
$$3x - 2y + 2z = 5$$

6 Solve the simultaneous equations.

a
$$\frac{1}{x} - \frac{1}{y} = 4$$
$$\frac{1}{y} - \frac{1}{z} = 5$$
$$\frac{1}{x} + \frac{1}{z} = 1$$

b
$$\frac{2}{x} - \frac{3}{y} + 4z = 16$$
$$\frac{1}{x} + \frac{6}{y} - z = 7$$
$$\frac{3}{x} - \frac{1}{y} + \frac{z}{3} = 15$$

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