CHAPTER

15

Number and Algebra

Further factorisation

Consider the identity $(x - 3)(x - 4) = x^2 - 7x + 12$. Going from left to right is called expansion. Going from right to left is called factorisation. Once a quadratic is factorised, it is easy to write down the solutions to the corresponding equation.

In this chapter we review methods of factorising, including taking out common factors, factorising monic quadratics and recognising the difference of two squares or perfect squares.

We then introduce two more methods of factorising: factorising by grouping and extending the method of factorising simple quadratics to the general case $ax^2 + bx + c$. We then apply these methods in the sections on algebraic fractions.

In the last section we introduce the new technique of 'completing the square', which applies to all quadratics.

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15A Grouping

When factorising an expression with four terms, it sometimes helps to group the terms in pairs. It may happen that after factorising each pair, the resulting expression has a common factor. We may have to rearrange the four terms first.

Example 1

Factorise $3ax - a + 12x - 4$.

Solution

$$\begin{aligned} 3ax - a + 12x - 4 &= (3ax - a) + (12x - 4) \\ &= a(3x - 1) + 4(3x - 1) \\ &= (3x - 1)(a + 4) \end{aligned}$$

The rearrangement of the expression can take place in more than one way to give the result.

Example 2

Factorise $ab - 6 + 2a - 3b$.

Solution

$$\begin{aligned} ab - 6 + 2a - 3b &= ab + 2a - 3b - 6 \\ &= a(b + 2) - 3(b + 2) \\ &= (b + 2)(a - 3) \end{aligned}$$

Alternatively:

$$\begin{aligned} ab - 6 + 2a - 3b &= ab - 3b - 6 + 2a \\ &= b(a - 3) + 2(a - 3) \\ &= (a - 3)(b + 2) \end{aligned}$$

Example 3

Factorise $6x^2 - x - 12$ by writing it in the form $6x^2 - 9x + 8x - 12$.

Solution

We group the terms into pairs and factorise each pair.

$$\begin{aligned} 6x^2 - x - 12 &= (6x^2 - 9x) + (8x - 12) \\ &= 3x(2x - 3) + 4(2x - 3) \\ &= (2x - 3)(3x + 4) \end{aligned}$$

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Exercise 15A

1 Factorise by grouping in pairs.

a $x(x + 2) + 5(x + 2)$
b $2x(3 - x) + 5(3 - x)$
c $4x(2x + 3) + 6(2x + 3)$
d $2x(4 - x) - 5(4 - x)$
e $4x(2x - 3) - 6(2x - 3)$
f $4x(x - 2) - 7(x - 2)$
g $2x(5 - 2x) + 5(5 - 2x)$
h $7x(11x + 5) + (11x + 5)$

Example 1

2 Factorise by grouping in pairs.

a $mx + 2m + 3x + 6$
b $ax + 2a + 4x + 8$
c $mx + 3m + 2x + 6$
d $5xy + 10x + 5y + 10$
e $8ab - 16a + b - 2$
f $x - 2 + 4xy - 8y$
g $ab - 7a - 9b + 63$
h $3ab + 6a + 5b + 10$
i $2ab - 12a - 5b + 30$
j $12x - 8xy - 15 + 10y$

Example 2, 3

3 Factorise by grouping in pairs.

a $x^2 - 3x - 4x + 12$
b $2x^2 - 2x + 5x - 5$
c $6x^2 - 4xy + 3xy - 2y^2$
d $10x^2 + 15xy - 4xy - 6y^2$
e $x^2 - 3x + 4x - 12$
f $4x^2 + 20x + x + 5$
g $3x^2 + 12x + x + 4$
h $6x^2 - 9x - 8x + 12$
i $6x^2 + 15x - 4x - 10$
j $2x^2 - 2x + x - 1$

4 Copy and complete by grouping and factorising.

a $x^2 + 5x + 6 = x^2 + 2x + 3x + 6$
$= \ldots$

b $x^2 + 5x - 14 = x^2 + 7x - 2x - 14$
$= \ldots$

c $x^2 - 7x - 18 = x^2 - 9x + 2x - 18$
$= \ldots$

d $x^2 - 5x - 36 = x^2 - 9x + 4x - 36$
$= \ldots$

e $x^2 + 11x + 30 = x^2 + 6x + 5x + 30$
$= \ldots$

f $x^2 + 29x + 100 = x^2 + 25x + 4x + 100$
$= \ldots$

15B Factorising the general quadratic $ax^2 + bx + c$

A quadratic expression is an expression of the form $ax^2 + bx + c$, where a, b and c are given numbers (and $a \neq 0$). In Chapter 4 we introduced a technique for factorising **monic quadratic** expressions where $a = 1$. This was extended to the class of quadratic expressions where $a = 1$, but where a was a common factor of b and c. For example:

$$-2x^2 + 28x - 96 = -2(x^2 - 14x + 48) \quad \text{(factor out } -2\text{)}$$
$$= 2(x - 6)(x - 8) \quad \text{(find factors of 48 that sum to } -14\text{: } -6 \text{ and } -8\text{)}$$

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Consider the quadratic expression $4x^2 + 5x - 6$. The coefficient of x^2 is not 1 and the coefficients have no common factor. Here is a method for factorising such quadratics.

Step 1: Multiply the coefficient of x^2, 4, by the constant term, −6.

$$4 \times (-6) = -24$$

Step 2: Find two numbers whose product is −24 and whose sum is 5, the coefficient of x. The numbers are 8 and −3.

$$8 \times (-3) = -24 \text{ and } 8 + (-3) = 5$$

Step 3: Split the x term using these two numbers.

$$5x = 8x - 3x$$

Step 4: Next use grouping to factorise the quadratic.

$$\begin{aligned} 4x^2 + 5x - 6 &= 4x^2 + 8x - 3x - 6 \\ &= 4x(x + 2) - 3(x + 2) \\ &= (x + 2)(4x - 3) \end{aligned}$$

Note: Grouping in a different way also works.

$$\begin{aligned} 4x^2 - 3x + 8x - 6 &= x(4x - 3) + 2(4x - 3) \\ &= (4x - 3)(x + 2) \end{aligned}$$

That is, $5x = 8x - 3x$ and $5x = -3x + 8x$ both work.

Factorising the general quadratic

To factorise a quadratic of the form $ax^2 + bx + c$, find two numbers α and β whose sum is b and whose product is ac. Split the middle term as $\alpha x + \beta x$ and factorise by grouping.

Note: This method also works for monic quadratics.

Example 4

Factorise $2x^2 + 9x + 4$.

Solution

Multiply 2 by 4 to obtain 8. Look for two numbers whose product is 8 and whose sum is 9. The numbers are 8 and 1.

Hence
$$\begin{aligned} 2x^2 + 9x + 4 &= 2x^2 + 8x + x + 4 \\ &= 2x(x + 4) + 1(x + 4) \\ &= (2x + 1)(x + 4) \end{aligned}$$

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Example 5

Factorise:

a $6x^2 - 19x + 10$ **b** $3x^2 - 5x - 28$

Solution

a Multiply 6 by 10 to obtain 60. Look for two numbers whose product is 60 and whose sum is −19. The numbers are −15 and −4.

Hence
$$\begin{aligned} 6x^2 - 19x + 10 &= 6x^2 - 15x - 4x + 10 \\ &= 3x(2x - 5) - 2(2x - 5) \\ &= (2x - 5)(3x - 2) \end{aligned}$$

b Multiply 3 by −28 to obtain −84. Look for two numbers whose product is −84 and whose sum is −5. The numbers are −12 and 7.

$$\begin{aligned} 3x^2 - 5x - 28 &= 3x^2 - 12x + 7x - 28 \\ &= 3x(x - 4) + 7(x - 4) \\ &= (x - 4)(3x + 7) \end{aligned}$$

There are other methods for factorising quadratics in which the coefficient of x^2 is not 1, but this method is recommended.

Exercise 15B

Example 4

1 Factorise the quadratic expressions.

a $2x^2 + 7x + 5$ **b** $2x^2 + 11x + 5$ **c** $2x^2 + 13x + 21$

d $2x^2 + 17x + 21$ **e** $3x^2 + 14x + 8$ **f** $3x^2 + 10x + 8$

g $4x^2 + 16x + 15$ **h** $4x^2 + 23x + 15$ **i** $2x^2 + 31x + 15$

j $2x^2 + 17x + 15$ **k** $4x^2 + 32x + 15$ **l** $6x^2 + 19x + 10$

m $12x^2 + 32x + 5$ **n** $12x^2 + 16x + 5$ **o** $8x^2 + 34x + 21$

Example 5

2 Factorise the quadratic expressions.

a $3x^2 - 14x + 8$ **b** $3x^2 - 10x + 8$ **c** $4x^2 - 21x + 5$

d $4x^2 - 9x + 5$ **e** $6x^2 - 11x + 3$ **f** $6x^2 - 7x + 2$

g $4x^2 - 12x + 5$ **h** $4x^2 - 16x + 15$ **i** $6x^2 - 19x + 10$

j $6x^2 - 25x + 4$ **k** $6x^2 - 11x + 5$ **l** $6x^2 - 17x + 10$

m $4x^2 - 39x + 27$ **n** $4x^2 - 24x + 27$ **o** $10x^2 - 7x + 1$

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3 Factorise the quadratic expressions.

a $2x^2 - 9x - 5$ **b** $2x^2 - 3x - 5$ **c** $2x^2 + 9x - 5$

d $2x^2 + 9x - 18$ **e** $3x^2 + x - 10$ **f** $3x^2 - 13x - 10$

g $4x^2 - 4x - 15$ **h** $4x^2 + 17x - 15$ **i** $4x^2 - 7x - 15$

j $4x^2 - 28x - 15$ **k** $12x^2 + 17x - 5$ **l** $12x^2 + 11x - 5$

m $6x^2 - 11x - 10$ **n** $10x^2 - 31x - 14$ **o** $20x^2 - 7x - 3$

4 Find the missing factor and check by expansion.

a $x^2 + 7x + 10 = (x + 5)\ldots\ldots$ **b** $x^2 - 9 = (x - 3)\ldots\ldots$

c $x^2 + 8x + 16 = (x + 4)\ldots\ldots$ **d** $a^2 + 2a + ab + 2b = (a + 2)\ldots\ldots$

e $9x^2 - 16y^2 = (3x + 4y)\ldots\ldots$ **f** $2x^2 - 7x - 4 = (2x + 1)\ldots\ldots$

g $6x^2 + 13x + 6 = (3x + 2)\ldots\ldots$ **h** $6x^2 - 17x - 3 = (6x + 1)\ldots\ldots$

i $6x^2 + 35x - 6 = (x + 6)\ldots\ldots$ **j** $4x^2 + 12x + 9 = (2x + 3)\ldots\ldots$

5 Copy and complete.

a $x^2 + 7x + \ldots = (x + 5)(x + \ldots)$ **b** $x^2 - 6x \ldots = (x - 4)(x \ldots)$

c $x^2 + \ldots x + 15 = (x + 3)(x + \ldots)$ **d** $x^2 \ldots x - 24 = (x - 4)(x \ldots)$

6 **a** If $ax^2 + bx + c = (px + q)(rx + s)$, show that $a = pr$, $b = ps + qr$ and $c = qs$.

b Hence show that if $ax^2 + bx + c$ can be factorised, then b can be written as the sum of two numbers whose product is ac. (This shows why our method of factorisation works.)

15C Simplifying, multiplying and dividing algebraic fractions

We can sometimes use factorisation techniques to simplify algebraic fractions. In this section, we show how some quotients and products of quotients can be simplified by first factorising and then cancelling common factors.

Example 6

Simplify:

a $\dfrac{x^2 + xy}{x^2 - y^2}$ **b** $\dfrac{8n^2 - 50m^2}{4n^3 + 10mn^2}$

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Solution

a $\dfrac{x^2 + xy}{x^2 - y^2} = \dfrac{x(x+y)}{(x+y)(x-y)}$

$= \dfrac{x}{x-y}$

b $\dfrac{8n^2 - 50m^2}{4n^3 + 10mn^2} = \dfrac{2(4n^2 - 25m^2)}{2n^2(2n+5m)}$

$= \dfrac{(2n+5m)(2n-5m)}{n^2(2n+5m)}$

$= \dfrac{2n-5m}{n^2}$

Example 7

Simplify:

a $\dfrac{x^2 - x - 12}{x^2 - 12x + 32}$ **b** $\dfrac{2x^2 + 9x - 18}{2x^2 - 11x + 12}$

Solution

a $\dfrac{x^2 - x - 12}{x^2 - 12x + 32} = \dfrac{(x-4)(x+3)}{(x-4)(x-8)} = \dfrac{(x+3)}{(x-8)}$

b $2x^2 + 9x - 18 = 2x^2 + 12x - 3x - 18$ $(2 \times (-18) = -36 = 12 \times (-3);$ $12 + (-3) = 9)$

$= 2x(x+6) - 3(x+6)$

$= (x+6)(2x-3)$

and $2x^2 - 11x + 12 = 2x^2 - 8x - 3x + 12$ $(2 \times 12 = 24 = (-8) \times (-3);$ $-8 + (-3) = -11)$

$= 2x(x-4) - 3(x-4)$

$= (x-4)(2x-3)$

so $\dfrac{2x^2 + 9x - 18}{2x^2 - 11x + 12} = \dfrac{(x+6)(2x-3)}{(x-4)(2x-3)}$

$= \dfrac{x+6}{x-4}$

Example 8

Simplify:

a $\dfrac{x^2 - 3x - 10}{x^2 - 5x} \times \dfrac{2x^2 - 2x}{x^2 - x - 6}$ **b** $\dfrac{4x^2 - 1}{4x^2 + 4x + 1} \div \dfrac{4x^2 - 4x + 1}{2x^2 - 3x + 1}$

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Solution

a $$\frac{x^2-3x-10}{x^2-5x}\times\frac{2x^2-2x}{x^2-x-6}=\frac{(x-5)(x+2)}{x(x-5)}\times\frac{2x(x-1)}{(x-3)(x+2)}$$
$$=\frac{2(x-1)}{(x-3)}$$

b $$\frac{4x^2-1}{4x^2+4x+1}\div\frac{4x^2-4x+1}{2x^2-3x+1}=\frac{4x^2-1}{4x^2+4x+1}\times\frac{2x^2-3x+1}{4x^2-4x+1}$$
$$=\frac{(2x+1)(2x-1)}{(2x+1)(2x+1)}\times\frac{(2x-1)(x-1)}{(2x-1)(2x-1)}$$
$$=\frac{x-1}{2x+1}$$

Exercise 15C

Example 6

1 Simplify:

a $\frac{n-m}{m-n}$ **b** $\frac{k^2-k\ell}{\ell-k}$ **c** $\frac{6x^2+6xy}{y^2-x^2}$

d $\frac{18k^2-8\ell^2}{6k-4\ell}$ **e** $\frac{12p^2-27q^2}{6p^2-9pq}$ **f** $\frac{pq}{p^2q^2+pq}$

g $\frac{m^2+mn}{n^2+nm}$ **h** $\frac{s^2-t^2}{t-s}$ **i** $\frac{3x-3y}{xy-x^2}$

j $\frac{3p^3-3pq^2}{p^2+pq}$ **k** $\frac{x^3-xy^2}{2x+2y}$ **l** $\frac{m^3-m^2n}{n-m}$

Example 7a

2 Simplify:

a $\frac{x^2-1}{x^2-x-2}$ **b** $\frac{x^2-4}{x^2-3x-10}$ **c** $\frac{x^2+4x-5}{x^2-2x+1}$

d $\frac{x^2+x-12}{x^2-3x}$ **e** $\frac{x^2-x-20}{x^2+x-12}$ **f** $\frac{x^2+6x+9}{x^2+x-6}$

g $\frac{2x^2-18}{3x^2+3x-18}$ **h** $\frac{x^2-5x+6}{x^2-4x+4}$ **i** $\frac{3x^2+3x-36}{x^2+8x+16}$

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Example 7b

3 Simplify:

a $\dfrac{(2x-3)(x+1)}{(2x-3)(x+2)}$ **b** $\dfrac{(5x+6)(x+1)}{(2x-1)(5x+6)}$ **c** $\dfrac{(7x+1)(x-3)}{(x-3)(7x+1)}$

d $\dfrac{2x^2-3x+1}{x^2-2x+1}$ **e** $\dfrac{4x^2-1}{4x-2}$ **f** $\dfrac{5x^2+9x-2}{x^2-4}$

g $\dfrac{2x^2+3x-2}{2x^2-7x+3}$ **h** $\dfrac{2x^2-5x+3}{2x^2-x-3}$ **i** $\dfrac{2x^2-7x+3}{2x^2-5x-3}$

Example 8a

4 Simplify:

a $\dfrac{3x^2}{2x+1}\times\dfrac{4x^2-1}{x^3+5x^2}$ **b** $\dfrac{x^2-x-2}{x^2+3x}\times\dfrac{5x}{x-2}$

c $\dfrac{2x^2+3x-2}{x^2+2x}\times\dfrac{x^2-4x}{x^2-5x-4}$ **d** $\dfrac{3x^2-10x+3}{3x^2-7x+2}\times\dfrac{3x^2-6x}{4x^2-11x-3}$

Example 8b

5 Simplify:

a $\dfrac{3x^2-3x}{x+1}\div\dfrac{1-x}{x^2+x}$ **b** $\dfrac{x^2-4}{2x^2-6x}\div\dfrac{6-x-x^2}{9-x^2}$

c $\dfrac{2x^2+5x-3}{x^2-1}\div\dfrac{x^2-x-12}{x^2-3x-4}$ **d** $\dfrac{16x^2+8x+1}{8x^2+14x+3}\div\dfrac{4x+1}{4x^2+4x-3}$

e $\dfrac{2x^2-3x-2}{x^2+3x}\div\dfrac{2x^2+x}{x^2+2x-3}\div\dfrac{x^2-3x+2}{x^2}$

6 Simplify:

a $\dfrac{6x^2+x-2}{10x^2-9x+2}\times\dfrac{10x^2+x-2}{6x^2+7x+2}$ **b** $\dfrac{x^2-9}{2x^2-7x+3}\div\dfrac{2x^2-3x+1}{x^2-1}$

c $\dfrac{x^2}{2x^2-7x+3}\times\dfrac{2x^2-11x+15}{2x^2-5x}$ **d** $\dfrac{2x^2-x-3}{x^2-1}\div\dfrac{2x^2-5x+3}{x^2+2x+1}$

e $\dfrac{p^2-q^2}{2p^2+p}\times\dfrac{2pq+q}{p^2-pq}\div(p+q)$

f $\dfrac{2a^2-32}{a^2+7a+12}\div\dfrac{4a^2-4a-48}{12a^2+15a}\div\dfrac{(a-1)(4a+5)}{a}$

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15D Adding and subtracting algebraic fractions

Recall how we add two fractions with different denominators:

$$\begin{aligned}\frac{3}{4}+\frac{7}{10}&=\frac{15}{70}+\frac{49}{70} \quad \text{(Express fractions with a common denominator.)}\\&=\frac{64}{70}\\&=\frac{32}{35}\end{aligned}$$

Notice that we use the lowest common multiple of 14 and 10, which is 70, as the common denominator.

We use the same procedures to add and subtract algebraic fractions.

Example 9

Express as a single fraction.

a $\frac{2x}{5}+\frac{x}{3}$ **b** $-\frac{m}{3}+\frac{2m}{9}$

Solution

a $$\begin{aligned}\frac{2x}{5}+\frac{x}{3}&=\frac{6x}{15}+\frac{5x}{15}\\&=\frac{11x}{15}\end{aligned}$$

b $$\begin{aligned}-\frac{m}{3}+\frac{2m}{9}&=-\frac{3m}{9}+\frac{2m}{9}\\&=-\frac{m}{9}\end{aligned}$$

When the denominator is an algebraic expression, the procedure is the same.

Example 10

Express as a single fraction.

a $\frac{4}{x}+\frac{2}{x}$ **b** $\frac{4}{x}+\frac{2}{3x}$ **c** $\frac{5}{x^2}+\frac{4}{7x}$ **d** $\frac{4}{x-1}+\frac{2}{x+1}$

Solution

a $$\frac{4}{x}+\frac{2}{x}=\frac{6}{x}$$

b $$\begin{aligned}\frac{4}{x}+\frac{2}{3x}&=\frac{12}{3x}+\frac{2}{3x}\\&=\frac{14}{3x}\end{aligned}$$

c $$\begin{aligned}\frac{5}{x^2}+\frac{4}{7x}&=\frac{35}{7x^2}+\frac{4x}{7x^2}\\&=\frac{4x+35}{7x^2}\end{aligned}$$

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d $\dfrac{4}{x-1} + \dfrac{2}{x+1} = \dfrac{4(x+1)}{(x-1)(x+1)} + \dfrac{2(x-1)}{(x+1)(x-1)}$

$= \dfrac{4(x+1) + 2(x-1)}{(x+1)(x-1)}$

$= \dfrac{4x + 4 + 2x - 2}{(x+1)(x-1)}$

$= \dfrac{6x+2}{(x+1)(x-1)}$ or $\dfrac{6x+2}{x^2-1}$

Either answer is acceptable.

Example 11

Simplify $\dfrac{4}{x-1} + \dfrac{2}{1-x}$.

Solution

Note that $\dfrac{2}{1-x} = \dfrac{-2}{x-1}$

Hence $\dfrac{4}{x-1} + \dfrac{2}{1-x} = \dfrac{4}{x-1} + \dfrac{-2}{x-1}$

$= \dfrac{2}{x-1}$

Example 12

Simplify $\dfrac{5}{(x-3)(x+2)} - \dfrac{4}{(x+2)(x-1)}$.

Solution

The common denominator of the two expressions is $(x-3)(x+2)(x-1)$.

$$\frac{5}{(x-3)(x+2)} - \frac{4}{(x+2)(x-1)} = \frac{5(x-1) - 4(x-3)}{(x-3)(x+2)(x-1)}$$

$$= \frac{5x - 5 - 4x + 12}{(x-3)(x+2)(x-1)}$$

$$= \frac{x+7}{(x-3)(x+2)(x-1)}$$

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Example 13

Simplify $\dfrac{4}{x^2+5x+6}+\dfrac{3}{x^2+4x+3}$.

Solution

Note that $x^2+5x+6=(x+3)(x+2)$ and $x^2+4x+3=(x+3)(x+1)$.

The common denominator of the two fractions is $(x+1)(x+2)(x+3)$.

$$\begin{aligned}\text{So }\frac{4}{x^2+5x+6}+\frac{3}{x^2+4x+3}&=\frac{4}{(x+3)(x+2)}+\frac{3}{(x+3)(x+1)}\\&=\frac{4(x+1)}{(x+3)(x+2)(x+1)}+\frac{3(x+2)}{(x+3)(x+2)(x+1)}\\&=\frac{4x+4+3x+6}{(x+3)(x+2)(x+1)}\\&=\frac{7x+10}{(x+3)(x+2)(x+1)}\end{aligned}$$

Exercise 15D

Example 9

1 Express as a single fraction.

a $\dfrac{3x}{8}+\dfrac{x}{8}$ **b** $\dfrac{4x}{5}-\dfrac{2x}{5}$ **c** $\dfrac{2x}{7}-\dfrac{5x}{7}$ **d** $\dfrac{4x}{5}+\dfrac{x}{2}$

e $\dfrac{6x}{5}-\dfrac{x}{4}$ **f** $\dfrac{2x}{7}-\dfrac{2x}{3}$ **g** $\dfrac{2x}{3}+\dfrac{x}{2}-\dfrac{3x}{4}$ **h** $\dfrac{3x}{5}-\dfrac{2x}{3}$

Example 10a, b

2 Express as a single fraction.

a $\dfrac{4}{x}+\dfrac{3}{x}$ **b** $\dfrac{5}{x}+\dfrac{6}{x}$ **c** $\dfrac{3}{x}+\dfrac{4}{2x}$

d $\dfrac{5}{x^2}+\dfrac{3}{x}$ **e** $\dfrac{5}{x}-\dfrac{3}{x}$ **f** $\dfrac{7}{x^2}-\dfrac{3}{x}$

g $\dfrac{4}{x}+\dfrac{3}{2x}$ **h** $\dfrac{7}{2x}-\dfrac{4}{3x}$ **i** $\dfrac{11}{3x}-\dfrac{5}{2x}$

Example 10c, 11

3 Simplify:

a $\dfrac{2}{x+1}+\dfrac{1}{x+2}$ **b** $\dfrac{3}{x+4}+\dfrac{2}{x-1}$ **c** $\dfrac{5}{x+3}+\dfrac{4}{x-2}$

d $\dfrac{3}{x+1}-\dfrac{2}{x+3}$ **e** $\dfrac{7}{x+2}-\dfrac{4}{x-1}$ **f** $\dfrac{3}{2x-1}-\dfrac{1}{x+2}$

g $\dfrac{2}{2a+1}+\dfrac{3}{a-4}$ **h** $\dfrac{4}{b+3}-\dfrac{2}{2b-1}$ **i** $\dfrac{4}{b-1}-\dfrac{3}{1-2b}$

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4 Simplify:

a $\frac{1}{x-5}+\frac{3}{x-5}$ **b** $\frac{4}{a+5}+\frac{2}{a-5}$ **c** $\frac{x}{x-3}+\frac{1}{x+4}$

d $\frac{2x}{(x-2)^2}+\frac{2}{x-2}$ **e** $\frac{1}{a-5}-\frac{2}{a-5}$ **f** $\frac{2a}{a-4}+\frac{3a}{a+4}$

g $\frac{1}{(x-7)^2}-\frac{2}{x-7}$ **h** $\frac{2}{(a-1)^2}-\frac{2a}{(a-1)^2}$ **i** $\frac{2x+3}{x-4}-\frac{2x-4}{x-4}$

Example 12, 13

5 Simplify:

a $\frac{1}{(x+1)(x+2)}+\frac{2}{(x+2)(x+3)}$ **b** $\frac{2}{(x-1)(x+3)}+\frac{1}{(x-1)(x-2)}$

c $\frac{2}{(x-3)(x-2)}-\frac{4}{(x-1)(x-2)}$ **d** $\frac{4}{(x+1)(x-3)}-\frac{3}{x+1}$

e $\frac{3}{x^2-x-2}+\frac{4}{x^2-3x+2}$ **f** $\frac{5}{x^2-1}+\frac{3}{x^2-2x-3}$

g $\frac{3}{(x+1)(x-2)}+\frac{1}{(x+3)(2-x)}$ **h** $\frac{4}{x^2+2x-3}-\frac{1}{1-x^2}$

i $\frac{3}{x(x+2)}+\frac{4}{(x+2)(x-1)}$ **j** $\frac{1}{(x-2)(x+4)}-\frac{3}{(x+4)(x-3)}$

k $\frac{3}{(x+1)(x+3)}+\frac{2}{x(x+3)}$ **l** $\frac{4}{(x+1)(x+2)}+\frac{3}{x+1}$

m $\frac{1}{x^2-2x-8}+\frac{3}{x^2+3x+2}$ **n** $\frac{2}{x^2+2x}+\frac{3}{x^2-4}$

o $\frac{2}{x(3-x)}+\frac{4}{(x+1)(x-3)}$ **p** $\frac{3}{4-x^2}+\frac{4}{x(x-2)}$

6 Simplify:

a $\frac{2}{x+1}+\frac{3}{x}$ **b** $\frac{3}{2m-n}+\frac{5}{3m+n}$ **c** $\frac{m}{m-n}-\frac{m}{m+n}$

d $\frac{x}{3x+1}-\frac{5}{x^2}$ **e** $\frac{x-1}{x+1}-\frac{x-2}{x+2}$ **f** $\frac{3k+1}{k^2-1}+\frac{3k-1}{k+1}$

g $\frac{5}{2p+1}+\frac{1}{p}$ **h** $\frac{1}{1+q}+\frac{1}{1-q}$ **i** $\frac{7}{2k+\ell}+\frac{5}{k-2\ell}$

j $\frac{2p}{p+1}+\frac{3p}{p-1}$ **k** $\frac{p+2}{p-1}+\frac{p-1}{p+2}$ **l** $\frac{7m+n}{2m-n}+\frac{3m-n}{2m+n}$

m $\frac{1}{x+2}+\frac{2}{x-2}+\frac{4}{x^2-4}$ **n** $\frac{1}{x+3}+\frac{6}{x^2-9}+\frac{1}{3-x}$

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o $\frac{3}{x+3} + \frac{2}{x-4} - \frac{5x-6}{x^2 - x - 12}$

p $\frac{2}{k+5} + \frac{1}{k-4} - \frac{9}{k^2 + k - 20}$

q $\frac{1}{x+1} + \frac{1}{x-1} + \frac{2x-2}{1-x^2}$

r $\frac{4}{2+x} - \frac{3}{2-x} - \frac{7x}{x^2 - 4}$

s $\frac{5}{3+x} + \frac{2}{x-3} - \frac{6-6x}{9-x^2}$

t $\frac{2\ell}{(1-2\ell)^2} - \frac{1-\ell}{1-5\ell+6\ell^2} + \frac{2}{1-3\ell}$

u $\frac{p}{p+q} + \frac{q}{p-q} + \frac{p^2+q^2}{q^2-p^2}$

v $\frac{1}{p^2-4p+3} - \frac{1}{p^2-3p+2} - \frac{1}{p^2-5p+6}$

15E Completing the square

In this section a general technique for factorising quadratics is introduced.

What number must be added to $x^2 + 6x$ to make a perfect square?

It is 9, which is the square of half of the coefficient of x. We get

$$x^2 + 6x + 9 = (x+3)^2$$

This idea is the basis for an important technique called **completing the square**.

The key step is to add and subtract the square of half the coefficient of x. For example, to complete the square of $x^2 + 10x - 6$, we add and subtract 25, which is the square of half of 10.

Focus on $x^2 + 10x$. The related perfect square is $x^2 + 10x + 25$.

$$\begin{aligned} x^2 + 10x - 6 &= x^2 + 10x + 25 - 25 - 6 \quad \text{(Add and subtract 25.)} \\ &= (x^2 + 10x + 25) - 31 \\ &= (x+5)^2 - 31 \end{aligned}$$

Completing the square

- Add the square of half of the coefficient of x and then take it away.
- Write the first part of the quadratic as a perfect square.

For example, $x^2 - 10x - 6 = (x^2 - 10x + 25) - 25 - 6$
$= (x-5)^2 - 31$

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Example 14

Complete the square.

a $x^2 + 4x + 2$ **b** $x^2 + 7x - 4$ **c** $x^2 - 12x - 10$

Solution

a $x^2 + 4x + 2 = (x^2 + 4x + 4) - 4 + 2$

$= (x + 2)^2 - 2$

b $x^2 + 7x - 4 = \left(x^2 + 7x + \frac{49}{4}\right) - \frac{49}{4} - 4$

$= \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} - \frac{16}{4}$

$= \left(x + \frac{7}{2}\right)^2 - \frac{65}{4}$

c $x^2 - 12x - 10 = (x^2 - 12x + 36) - 36 - 10$

$= (x - 6)^2 - 46$

Example 15

Factorise by completing the square.

a $x^2 - 2x - 8$ **b** $x^2 - 4x - 3$ **c** $x^2 + 7x + 1$

Solution

a $x^2 - 2x - 8 = (x^2 - 2x + 1) - 1 - 8$

$= (x - 1)^2 - 9$ ($9 = 3^2$. Use the 'difference of two squares' identity.)

$= (x - 1 - 3)(x - 1 + 3)$

$= (x - 4)(x + 2)$

b $x^2 - 4x - 3 = (x^2 - 4x + 4) - 4 - 3$

$= (x - 2)^2 - 7$

$= (x - 2)^2 - (\sqrt{7})^2$

$= (x - 2 - \sqrt{7})(x - 2 + \sqrt{7})$

c $x^2 + 7x + 1 = x^2 + 7x + \frac{49}{4} - \frac{49}{4} + 1$

$= \left(x + \frac{7}{2}\right)^2 - \frac{45}{4}$

$= \left(x + \frac{7}{2}\right)^2 - \left(\frac{\sqrt{45}}{2}\right)^2$

$= \left(x + \frac{7}{2} - \frac{3\sqrt{5}}{2}\right)\left(x + \frac{7}{2} + \frac{3\sqrt{5}}{2}\right)$

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Exercise 15E

1 Copy and complete the following.

a $x^2 + 6x + \ldots = (x + 3)^2$

b $x^2 + 12x + \ldots = (x + 6)^2$

c $x^2 + 5x + \ldots = \left(x + \frac{5}{2}\right)^2$

d $x^2 + 7x + \ldots = \left(x + \frac{7}{2}\right)^2$

e $x^2 - 12x + \ldots = (x - 6)^2$

f $x^2 - 100x + \ldots = (x - \ldots)^2$

g $x^2 - \ldots x + \ldots = (x - 5)^2$

h $x^2 - 80x + \ldots = (x - \ldots)^2$

Example 14

2 Complete the square.

a $x^2 + 2x - 5$

b $x^2 + 2x + 7$

c $x^2 + 4x + 1$

d $x^2 + 6x + 2$

e $x^2 + 6x - 3$

f $x^2 - 6x + 6$

g $x^2 - 8x - 5$

h $x^2 + 8x + 25$

i $x^2 + 12x - 11$

j $x^2 + 3x - 2$

k $x^2 + 11x + 7$

l $x^2 - 10x - 3$

m $x^2 + 4x + 10$

n $x^2 - 8x + 20$

o $x^2 - 6x + 9$

p $x^2 + 5x + 8$

q $x^2 + 10x - 4$

r $x^2 + 5x - 10$

Example 15

3 Factorise by first completing the square and then using the difference of two squares identity.

a $x^2 + 4x + 3$

b $x^2 + 5x + 4$

c $x^2 + 5x + 6$

d $x^2 + 6x + 5$

e $x^2 + 6x - 3$

f $x^2 - 10x - 3$

g $x^2 + 12x - 5$

h $x^2 - 12x + 5$

i $x^2 + 8x - 5$

j $x^2 - 10x - 6$

k $x^2 - 11x + 1$

l $x^2 + 9x - 8$

m $x^2 - 7x + 2$

n $x^2 - 7x - 2$

o $x^2 + 13x - 5$

Review exercise

1 Factorise:

a $9x - 63$

b $24x^2 + 8$

c $-36x + 42x^2$

d $8y - 16y^2$

e $-64b - 32$

f $21m^2 + 7mn$

2 Factorise:

a $a(a + 5) + 4(a + 5)$

b $2p(6p - 1) - 7q(6p - 1)$

c $4x(3y + 5) - (3y + 5)$

d $2m + 9 + 2n(2m + 9)$

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3 Factorise:

a $a^2 - 121$ **b** $9x^2 - 36$ **c** $64m^2 - 169n^2$

d $7x^2 - 63$ **e** $x^2 - 36$ **f** $x^2 - 16y^2$

g $3m^2 - 27$ **h** $6x^2 - 24$ **i** $4b^2 - 100$

4 Factorise the quadratic expressions.

a $x^2 + 5x - 24$ **b** $x^2 - 15x + 36$ **c** $x^2 - 5x - 14$

d $4a^2 + 24a + 20$ **e** $3r^2 - 6r - 24$ **f** $5a^2 - 60a + 180$

g $5a^2 - 10a + 5$ **h** $-2p^2 + 16p - 32$ **i** $-b^2 - 3b + 4$

5 Factorise:

a $bx + 3b + 4x + 12$ **b** $x^2 - 5x - 6x + 30$

c $6x - 12xy - 4 + 8y$ **d** $x^3 - 3x + 5x^2 - 15$

6 Factorise the quadratic expression.

a $4x^2 + 25x + 6$ **b** $6x^2 - 7x + 2$ **c** $12x^2 - 17x - 5$

d $2t^2 + 5t - 3$ **e** $10a^2 - a - 2$ **f** $28x^2 - 85x + 63$

g $39x^2 + 131x - 44$ **h** $-96x^2 + 76x - 15$ **i** $15x^2 + 4x - 35$

7 Find the missing factor.

a $x^2 + 9x + 20 = (x + 4)\ldots\ldots$ **b** $x^2 - 16 = (x - 4)\ldots\ldots$

c $x^2 + 10x + 25 = (x + 5)\ldots\ldots$ **d** $m^2 + 3m + mn + 3n = (m + 3)\ldots\ldots$

e $25a^2 - 9b^2 = (5a + 3b)\ldots\ldots$ **f** $3x^2 - 10x - 8 = (3x + 2)\ldots\ldots$

8 Simplify:

a $\dfrac{(x+5)(x+3)}{(x+5)}$ **b** $\dfrac{x^2 - x - 6}{x^2 - 8x + 15}$ **c** $\dfrac{x^2 + 7x + 10}{x^2 + 6x + 8}$ **d** $\dfrac{(x+3)(3x+1)}{x^2 + 6x + 9}$

e $\dfrac{3x^2 - 7x + 2}{x^2 - 4x + 4}$ **f** $\dfrac{9x^2 - 4}{12x - 8}$ **g** $\dfrac{8x^2 + 5x - 3}{x^2 - 1}$ **h** $\dfrac{3x^2 + x - 2}{3x^2 + 4x - 4}$

9 Simplify:

a $\dfrac{a^2 + ab}{a^2 - ab} \times \dfrac{ab^2 + b^2}{a^3 + a^2b}$ **b** $\dfrac{a^2}{a^2 - 4} \times \dfrac{a^2 - 5a + 6}{3a - a^2}$

c $\dfrac{m^2 - m - 6}{m^2 - 9} \times \dfrac{m^2}{m^2 + 2m}$ **d** $\dfrac{3x^2 + x - 2}{x^2 - 1} \times \dfrac{x^2 - 2x + 1}{x^2 - x - 2}$

e $\dfrac{4x^2 + x - 3}{x^2 + 2x} \div \dfrac{x^2 - 4x - 5}{x^2 - 3x - 10}$ **f** $\dfrac{2x^2 + x - 3}{x^2 + 2x} \div \dfrac{x^2 - 4x + 3}{x^2} \div \dfrac{2x^2 + 3x}{x^2 - x - 6}$

g $\dfrac{x^2 - 3x - 4}{x^2 - 4x} \div \dfrac{x^2 - 4x + 4}{x^2 - 4}$

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10 Simplify:

a $\frac{3x}{10} + \frac{x}{10}$

b $\frac{7x}{6} - \frac{3x}{4}$

c $\frac{x}{2} + \frac{3x}{4} - \frac{2x}{3}$

d $\frac{3}{x+1} + \frac{1}{x+3}$

e $\frac{5}{x+2} + \frac{4}{x-1}$

f $\frac{5}{x-2} + \frac{1}{1-3x}$

g $\frac{2}{x+4} + \frac{3}{x-4}$

h $\frac{2}{(x-3)^2} + \frac{4}{x-3}$

i $\frac{3x}{(x-5)^2} + \frac{1}{x-5}$

11 Simplify:

a $\frac{1}{(x-2)(x+1)} + \frac{2}{(x-2)(x-3)}$

b $\frac{3}{(x+3)(x+1)} + \frac{6}{x+3}$

c $\frac{2}{x^2+x-2} + \frac{5}{x^2-3x+2}$

d $\frac{3}{x^2-4} + \frac{6}{x^2-2x-8}$

e $\frac{2}{(x+7)(x-3)} + \frac{3}{(x+4)(3-x)}$

f $\frac{6}{x(x+3)} + \frac{2}{(x+3)(x-4)}$

12 Simplify:

a $\frac{b}{a-b} - \frac{a}{a+b}$

b $\frac{3x+5}{x^2-9} + \frac{3x-1}{x+3}$

c $\frac{3}{4p+q} + \frac{3}{p-2q}$

d $\frac{2}{x+5} + \frac{1}{x-5} + \frac{5}{x^2-25}$

e $\frac{2}{x+4} + \frac{3}{x-5} - \frac{4x-1}{x^2-x-20}$

f $\frac{a}{a-b} + \frac{b}{a+b} - \frac{a^2+b^2}{a^2-b^2}$

g $\frac{1}{a^2-6a+8} - \frac{1}{a^2-5a+6} - \frac{1}{a^2-7a+12}$

13 Complete the square.

a x^2+4x-4 **b** x^2-6x+7 **c** x^2-8x-6 **d** x^2+3x-1

14 Factorise:

a $15x^2+5x-10$

b $12x^2+23x+10$

c $12x^2-7x-10$

d $9x^2-36x+35$

e $3x^2-26x+55$

f $3x^2+16x-35$

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Challenge exercise

1 Simplify:

a $\dfrac{1}{c-d}+\dfrac{1}{d-c}$

b $\dfrac{p}{p-q}+\dfrac{q}{q-p}$

c $\dfrac{1}{a+\frac{1}{b}}+\dfrac{1}{b+\frac{1}{a}}-\dfrac{1}{\frac{1}{2}a+\frac{1}{2}b}$

d $\dfrac{b+c}{b-c}+\dfrac{b-c}{b+c}+\dfrac{4bc}{c^2-b^2}$

e $\dfrac{b+c}{(b-a)(c-a)}+\dfrac{c+a}{(c-b)(a-b)}+\dfrac{a+b}{(a-c)(b-c)}$

2 Use the method of completing the square to show that:

a $x^2+4x+15 \geq 11$

b $x^2+2x+15 \geq 14$

3 **a** Show that the area, A cm^2, of a rectangle of perimeter 20 cm is given by the formula $A = w(10-w)$, where w cm is the width.

b Complete the square for the quadratic expression $10w - w^2$ and hence show $A \leq 25$.

c What value of w makes the area equal to 25 m^2?

d Complete: Of all the rectangles with perimeter 20 m, the one with the largest area is ...

4 By considering $\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)$, prove that 'the sum of a positive number and its reciprocal is greater than or equal to 2'.

5 For positive numbers a and b:

a Show that $\dfrac{a+b}{2} \geq \sqrt{ab}$.

b When are $\dfrac{a+b}{2}$ and $\sqrt{ab}$ equal?

c Deduce that $\dfrac{a}{b}+\dfrac{b}{a} \geq 2$.

d Show that $\dfrac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \dfrac{a+b}{2}$.

Mean

Each of these numbers has a name. If a and b are positive then:

- $\dfrac{a+b}{2}$ is called the **arithmetic mean** of a and b.
- $\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$ is called the **harmonic mean** of a and b.
- $\sqrt{ab}$ is called the **geometric mean** of a and b.

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6 Prove that a positive number plus one-quarter of its reciprocal is always greater than 1.

7 If $(a - b)^2 + 10ab = 120$, find the maximum possible value of ab.

8 If a and b are positive numbers, prove that $a^3 + b^3 \geq a^2b + ab^2$.

9 **a** Find all integer solutions of the equation $xy + 2x + 3y = 8$ by first writing it in the form $(x + a)(y + b) = c$.

b Find all integer solutions of $xy - 5x + 2y = 21$.

10 By factorising the left-hand side of the equation, find all integer solutions of the equation $2x^2 + xy - 15y^2 = 51$.

11 Given that $x^2 + y^2 = 28$ and $xy = 14$, find the value of $x^2 - y^2$.

12 Prove that for any numbers a, b, c, d:

a $2abcd \leq a^2b^2 + c^2d^2$

b $6abcd \leq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2$

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