CHAPTER

18

Number and Algebra

Rates and direct proportion

Speed and the rate of flow of water or other liquids are important examples of rates. We encounter many different kinds of rates in everyday life.

Simple rates provide examples of direct proportion. The distance travelled by a body moving at constant speed is directly proportional to the time it travels. Another familiar example from science is that, for a body moving with constant acceleration, the distance travelled is proportional to the square of the time travelling.

There is a huge variety of applications of proportion, and this will become evident through the many examples in this chapter.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Rates

Rates were introduced in *ICE-EM Mathematics Year 8*. They are a measure of how one quantity changes for every unit of another quantity. For example:

50 km/h means that a car travels 50 km in 1 hour.

20 L/min means 20 L of water flows in 1 minute.

30 km/L means a vehicle travels 30 km on 1L (of fuel).

In each of these examples we are describing a constant rate of change or an average rate of change.

Speed

Speed is one of the most familiar rates. It is a measure of how fast something is travelling. Many of the techniques introduced here can be applied in other rate situations.

Constant speed

If the speed of an object does not change over time, we say that the object is travelling with **constant speed**.

Three quantities are associated with questions that involve constant speed. These are distance, time and, of course, constant speed.

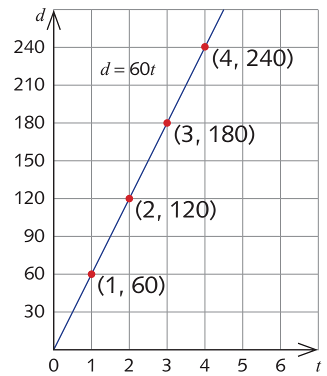
Finding the distance given a fixed speed

A car travels at a constant speed of 60 km/h.

- The car travels 60 km in 1 hour.
- The car travels 120 km in 2 hours.
- The car travels $60t$ km in t hours.

We can complete a table of values for the distance d (in kilometres) travelled by the car after t hours (see next page). The graph can be drawn by first plotting any two of the points.

t (h)	0	1	2	3	4
d (km)	0	60	120	180	240



The graph has a gradient of 60 and a d-axis intercept of 0. The gradient is the speed of the car in kilometres per hour.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 1

Maurice jogs at 6 km/h for 40 minutes.

a What is Maurice's speed in:

i metres per minute (m/min)?

ii metres per second (m/s)?

b How far does Maurice jog? Give your answers in:

i kilometres

ii metres

Solution

a i $6 \text{ km/h} = 6000 \text{ m/h} = \frac{6000}{60} = 100 \text{ m/min}$

ii $6 \text{ km/h} = 100 \text{ m/min} = \frac{100}{60} = \frac{5}{3} \text{ m/s}$

b i $\text{Distance travelled in 40 min} = 6 \times \frac{40}{60}$

$= 4 \text{ km}$

ii $\text{Distance travelled in 40 min} = 4 \times 1000$

$= 4000 \text{ m}$

Example 2

A car is travelling at 100 km/h.

a What is the formula for the distance d (in kilometres) travelled by the car in t hours?

b What is the gradient of the straight-line graph of d against t?

Solution

a In 1 hour, the car travels 100 km.
In 2 hours, the car travels 200 km.
The formula is $d = 100t$.

b The gradient of the graph of d against t is 100.

Average speed

When we drive a car or ride a bike, it is very rare for our speed to remain the same for a long period of time. Most of the time, especially in the city, we are slowing down or speeding up, so our speed is not constant. If we travel 20 km in 1 hour, we say that our **average speed** is 20 km/h.

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Example 3

A car travels 140 km in 1 hour 45 minutes. What is the average speed of the car?

Solution

$$1 \text{ hour } 45 \text{ minutes} = 1\tfrac{3}{4} \text{ hours} = \frac{7}{4} \text{ hours}$$

$$\begin{aligned}\text{Average speed} &= \frac{\text{distance travelled}}{\text{time taken}}\\ &= 140 \div \frac{7}{4}\\ &= 140 \times \frac{4}{7}\\ &= 80 \text{ km/h}\end{aligned}$$

Constant rate

Every question involving a constant rate gives rise to a straight-line graph. The gradient of the line is the constant rate.

Example 4

A cylindrical tank can hold a maximum of 40 L of water. It has 10 L of water in it to start with. Water is flowing slowly in at a rate of 5 L per minute.

a Prepare a table of values showing how much water is in the container at 1-minute intervals from 0 up to 6 minutes.

b Plot the graph of the volume V (in litres) of water in the tank against time t (in minutes) since the start.

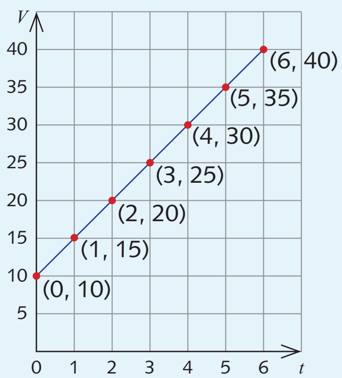
c Give the formula for V in terms of t.

Solution

a

t (min)	0	1	2	3	4	5	6
V (L)	10	15	20	25	30	35	40

b



c From the graph, the gradient is 5 and the V-axis intercept is 10 L. The formula is $V = 5t + 10$, where t takes values from 0 to 6 minutes inclusively.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 5

A man is walking home at 6 km/h. He starts at a point 18 km from his home. Draw a graph representing his trip home. State the gradient and vertical axis intercept, and give a formula that describes the trip.

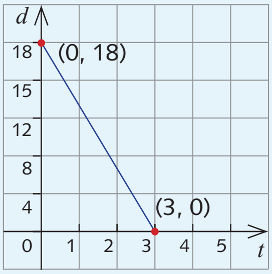
Solution

Let d km be the distance from home after travelling for t hours.

When $t = 0,\ d = 18$.

When $t = 3,\ d = 0$.

The graph can be drawn now that we have two points.



The d-axis intercept is 18.

The gradient is –6.

Thus, the formula is $d = -6t + 18,\ 0 \le t \le 3$.

Exercise 18A

1 Doug walks at 5 km/h for 45 minutes. How far does he walk? Give your answer in metres.

2 Tranh runs at 8 m/s for 10.5 seconds. How far does he run?

Example 1

3 **a** Convert 50 km/h into metres per second.

b Convert 10 m/s into kilometres per hour.

c Convert 9.5 m/s into kilometres per hour.

Example 3

4 **a** A plane travels 800 km in 1 hour 15 minutes. What is the average speed of the plane?

b A car travels 84 km in 50 minutes. What is the average speed of the car? Give your answer in kilometres per hour.

c Yusef walks the 2.1 km to the beach in 22 minutes. What is Yusef's average speed? Give your answer in metres per second.

Example 2

5 A car is travelling at 80 km/h.

a What is the formula for the distance d (in kilometres) travelled by the car in t hours?

b What is the gradient of the straight-line graph of d against t?

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Example 4

6 Maria decides to drive to the next town, which is 100 km away. She drives at 80 km/h.

a What is Maria's speed in kilometres per minute?

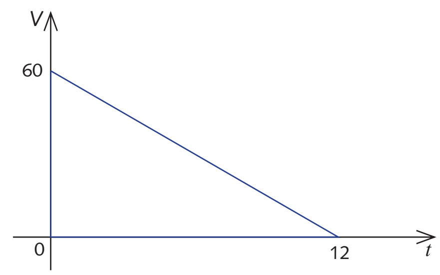
b Let t denote the number of minutes that have elapsed since Maria set out. Prepare a table of values showing how far (d km) she is from her starting point at 15-minute intervals.

c Plot the graph of d against t.

d Give the formula for d in terms of t.

e Use the formula to find how far Maria has driven after 44 minutes.

7 Water is flowing from a tank at a constant rate. The graph shows the volume of water (V litres) in the tank after t hours.



a What is the volume of water in the tank initially?

b At what rate is water flowing from the tank?

c Give the formula for V in terms of t.

d How many litres of water will be in the tank after 7 hours?

e **i** What would be the formula for V in terms of t if there were initially 120L of water in the tank and water flowed out at 6 L per hour?

ii How long would it take the tank to empty under these conditions?

Example 5

8 A man is walking to a town at 6.5 km/h. He starts at a point 20 km from the town.

a Draw a graph of distance travelled against time taken.

b State the gradient and vertical axis intercept.

c Give a formula for the distance travelled d km after t hours.

d If the man starts walking at 3 p.m. and sunset is at 5:45 p.m., will he arrive during daylight?

18B Direct proportion

In the previous section we looked at questions involving constant rates. Constant rates provide examples of direct proportion. We introduce direct proportion with a constant speed situation.

David drives from his home at a constant speed of 100 km/h. The formula for the distance d km travelled in t hours is $d = 100t$. David will go twice as far in twice the time, three times as far in three times the time and so on.

We say that d is **directly proportional** to t and the number 100 is called the **constant of proportionality**.

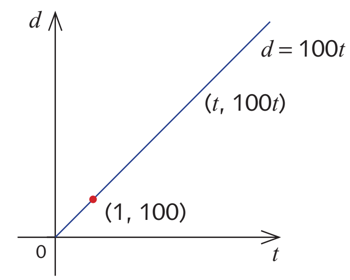
The statement 'd is directly proportional to t' is written as $d \propto t$.

The graph of d against t is a straight line passing through the origin.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

For any pair of values on the graph, (t_1, d_1), $\frac{d_1}{t_1} = 100$.

This is not just the gradient of the graph, $d = 100t$, but also the constant of proportionality in the relationship $d \propto t$.

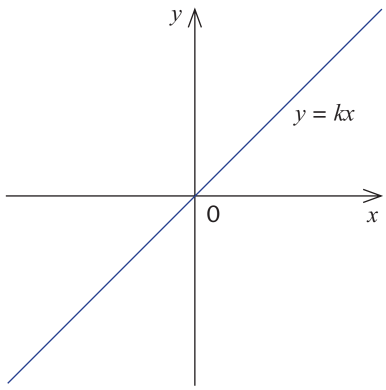


In general:

- The variable y is said to be **directly proportional** to x if $y = kx$ for some non-zero constant k.
- The constant k is called the **constant of proportionality**.
- The statement 'y is directly proportional to x' is written symbolically as

$$y \propto x$$

We know that the graph of $y = kx$ is a straight line passing through the origin. Its gradient k is the constant of proportionality. (The values that x can take are often the positive real numbers, but this is not always the case.)



And since the graph passes through the origin, only one pair of values is needed to find k.

Example 6

The cost $\$C$ of carpet 3 m wide is directly proportional to the length of carpet, ℓ metres. If 15 m of carpet cost \$1650, find:

a the formula for C in terms of ℓ

b the cost of 22 m of carpet

Solution

a It is given that: $C \propto \ell$

Therefore $C = k\ell$, for some constant k

$C = 1650$ when $\ell = 15$

so $1650 = k \times 15$

$k = 110$

Thus $C = 110\ell$

b When $\ell = 22$ $\quad C = 110 \times 22$

$= 2420$

22 metres of carpet costs \$2420.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

Example 7

In an electrical wire, the resistance (R ohms) varies directly with the length (L m) of the wire.

a If a wire 6 m long has a resistance of 5 ohms, what would be the resistance in a wire of length 4.5 m?

b How long is a wire for which the resistance is 3.8 ohms?

Solution

First, find the constant of proportionality.

$$R = kL$$

When $L = 6$ and $R = 5$

$$5 = 6k$$

So $k = \frac{5}{6}$

Thus $R = \frac{5L}{6}$

a when $L = 4.5$, $R = \frac{5 \times 4.5}{6}$

$$R = 3.75$$

The resistance of a wire of length 4.5 m is 3.75 ohms.

b When $R = 3.8$,

$$3.8 = \frac{5L}{6}$$

$$L = 4.56$$

The length of a wire of resistance 5 ohms is 4.56 m.

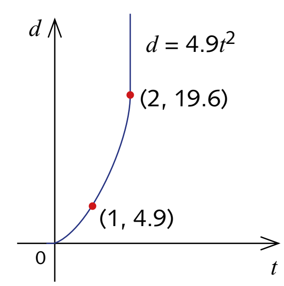
Change of variable

A metal ball is dropped from the top of a tall building and the distance it falls is recorded each second.

From physics, the formula for d metres, the distance the ball has fallen in t seconds, is given by $d = 4.9t^2$.

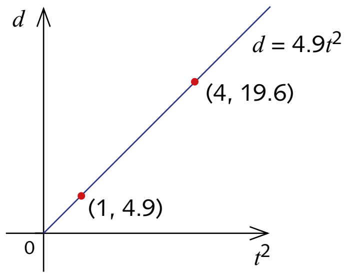
In this case, we say that d is directly proportional to the square of t.

We plot the graph of d against t. Note that, since t is positive, the graph is half a parabola.



We can also draw the graph of d against t^2.

t	0	1	2	3
t^2	0	1	4	9
d	0	4.9	19.6	44.1



ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

This is a straight line passing through the origin. The gradient of this line is 4.9.

d is directly proportional to t^2, which is written as $d \propto t^2$.

This means that for any two values t_1 and t_2 with corresponding values d_1 and d_2:

$$\frac{d_1}{t_1^2} = \frac{d_2}{t_2^2} = 4.9$$

So once again the gradient of the line is the constant of proportionality.

Example 8

From physics, the energy E microjoules (abbreviated as E mJ) of a body in motion is directly proportional to the square of its speed, v m/s. If a body travelling at a speed of 10 m/s has energy 400 mJ, find:

a the constant of proportionality

b the formula for E in terms of v

c the energy of the body when it travels at a speed of 15 m/s

d the speed if the moving body has energy 500 mJ

Solution

a Energy is directly proportional to the square of the speed:

$$E \propto v^2$$

So $\quad E = kv^2$, for some constant k

Now $\quad E = 400$ when $v = 10$

So $\quad 400 = 100k$

$$k = 4$$

b From part **a**, $E = 4v^2$

c When $v = 15$

$$\begin{aligned} E &= 4 \times 15^2 \\ &= 900 \end{aligned}$$

Therefore the body travelling at speed 15 m/s has energy of 900 mJ.

d When $E = 500$

$$\begin{aligned} 500 &= 4 \times v^2 \\ v^2 &= 125 \\ v &= \sqrt{125} \ (v > 0) \\ &= 5\sqrt{5} \\ &\approx 11.18 \text{ m/s (correct to 2 decimal places)} \end{aligned}$$

Therefore the body has energy 500 mJ when travelling at a speed of 11.18 m/s.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 9

The mass w grams of a plastic material required to mould a solid ball is directly proportional to the cube of the radius r centimetres of the ball. If 40 g of plastic is needed to make a ball of radius 2.5 cm, what size ball can be made from 200 g of the same type of plastic?

Solution

$$w \propto r^3$$
$$w = kr^3$$
$$w = 40 \text{ when } r = 2.5$$

Thus $40 = k \times (2.5)^3$

$k = 2.56$

So the formula is $w = 2.56r^3$

When $w = 200$, $200 = 2.56r^3$

$r^3 = 78.125$

$r = \sqrt[3]{78.125}$

$r \approx 4.27$ (correct to 2 decimal places)

Increase and decrease

If one quantity is proportional to another, we can investigate what happens to one of the quantities when the other is changed.

Suppose that $a \propto b$. Then $a = kb$ for a positive constant k.

If the value of b is doubled, the value of a is doubled. For example, if $b = 1$, then $a = k$. So $b = 2$ gives $a = 2k$. Similarly, if the value of b is tripled, the value of a is tripled.

Example 10

Given that $a \propto b^3$, what is the change in a when b is:

a doubled **b** halved?

Solution

Since $a \propto b^3$, $a = kb^3$ for some positive constant k.

a To see the effect of doubling b, choose $b = 1$.
(Any value can be chosen, but $b = 1$ is the easiest to deal with.)
When $b = 1$, $a = k$
When $b = 2$, $a = 8k$
When b is doubled, a is multiplied by 8.

b When $b = 1$, $a = k$
When $b = \frac{1}{2}$, $a = \frac{k}{8}$
When b is halved, a is divided by 8.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Example 11

Given that $y \propto \sqrt{x}$, what is the percentage change in:

a y when x is increased by 20%

b x when y is decreased by 30%?

Solution

Since $y \propto \sqrt{x}$, $y = k\sqrt{x}$

a When $x = 1, y = k$

If x is increased by 20%, $x = 1.2$

$$y = k\sqrt{1.2}$$
$$\approx 1.095k$$

y is approximately 109.5% of its previous value.

So y has increased by approximately 9.5%.

b If $y = k\sqrt{x}, y^2 = k^2 x$ and $x = \dfrac{y^2}{k^2}$ (making x the subject)

When $y = 1$, $x = \dfrac{1}{k^2}$

If y is decreased by 30%, $y = 0.7$ and $x = \dfrac{0.49}{k^2}$

x is 49% of its previous value.

So x has decreased by 51%.

Direct proportion

- y is **directly proportional** to x if there is a positive constant k such that $y = kx$.
- The symbol used for 'is proportional to' is $\propto$. We write $y \propto x$.
- The constant k is called the **constant of proportionality**.
- If y is directly proportional to x, the graph of y against x is a straight line through the origin. The gradient of the line is the constant of proportionality.

Exercise 18B

All variables take positive values only.

Example 6

1 **a** Given that $a \propto b$ and $b = 0.5$ when $a = 1$, find the formula for a in terms of b.

b Given that $m \propto n$ and $m = 9.6$ when $n = 3$, find the formula for m in terms of n.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

2 Consider the following table of values.

x	0	1	2	3	4	5
y	0	2	8	18	32	50

a Set up a new table of values for y and x^2.

b Plot the graph of y against x^2. What type of graph do you obtain?

c Find the gradient of the graph of y against x^2.

d Assuming that there is a simple relationship between the two variables, find a formula for y in terms of x.

3 Consider the following table of values.

p	0	1	4	9	16
q	0	3	6	9	12
$\sqrt{p}$					

a Plot the graph of q against p.

b Complete the table of values and calculate $\dfrac{q}{\sqrt{p}}$ for each pair $(q, \sqrt{p})$.

c Assuming that there is a simple relationship between the two variables, find a formula for q in terms of p.

4 Write each of the following in symbols.

a The distance d kilometres travelled by a motorist is directly proportional to t hours of travel.

b The volume V of a sphere is directly proportional to the cube of its radius r.

c The distance d kilometres to the visible horizon is directly proportional to the square root of the height h metres above sea level.

5 Write each of the following in words.

a $P \propto Q$ **b** $\ell \propto m^2$ **c** $a^2 \propto \sqrt{b}$ **d** $p^3 \propto \ell^2$

Example 7

6 a Given that $p \propto q$ and $p = 9$ when $q = 1.5$, find the formula for p in terms of q and the exact value of:

i p when $q = 4$ **ii** q when $p = 27$

b Given that $m \propto n^2$ and $m = 10$ when $n = 2$, find the formula for m in terms of n and the exact value of:

i m when $n = 5$ **ii** n when $m = 12$

7 In each of the following, find the formula connecting the pronumerals.

a $R \propto s$ and $s = 7$ when $R = 14$.

b a is directly proportional to the square root of b and $a = 3$ when $b = 4$.

c V is directly proportional to r^3 and $V = 216$ when $r = 3$.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

8 In each of the following tables $y \propto x$. Find the constant of proportionality in each case and complete the table.

a

x	0	1	2	3
y	0	7		

b

x	2	8	12	18
y	1			

c

x		3	6	15
y	24		72	

d

x	2		6	15
y		9.5	19	

9 On a particular road map, a distance of 0.5 cm on the map represents an actual distance of 8 km. What actual distance would a distance of 6.5 cm on the map represent?

10 The estimated cost \$$C$ of building a brick veneer house on a concrete slab is directly proportional to the area A of floor space in square metres. If it costs \$80 000 for 150 m^2, how much floor space could you expect for \$126 400?

Example 8

11 The mass m kilograms of a steel beam of uniform cross-section is directly proportional to its length ℓ metres. If a 6 m section of the beam has a mass of 400 kg, what will be the mass, to the nearest kilogram, of a section 5 m long?

Example 9

12 The power p kilowatts needed to run a boat varies as the cube of its speeds metres per second. If 400 kW will run a boat at 3 m/s, what power, to the nearest kilowatt, is needed to run the same boat at 5 m/s?

Example 10

13 If air resistance is neglected, the distance d metres that an object falls from rest is directly proportional to the square of the time t seconds of the fall. An object falls 9.6 m in 1.4 seconds. How far will the object fall in 2.8 seconds?

14 Given that $y \propto x^2$, what is the effect on y when x is:

a doubled **b** multiplied by 4 **c** divided by 5?

15 The surface area of a sphere, A cm^2, is directly proportional to the square of the radius, r centimetres. What is the effect on:

a the surface area when the radius is doubled

b the radius when the surface area is doubled?

16 Given that $m \propto n^4$, what is the effect on:

a m when n is doubled **b** m when n is halved

c n when m is multiplied by 16 **d** n when m is divided by 4?

Example 11

17 Given that $a \propto \sqrt{b}$, what is the effect, to 2 decimal places, on a when b is:

a increased by 21% **b** decreased by 12%?

18 Given that $p \propto \sqrt[3]{q}$, what is the effect on:

a p when q is increased by 10% **b** p when q is decreased by 10%

c q when p is increased by 20% **d** q when p is decreased by 20%?

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Review exercise

1 Andrew walks at 5 km/h for 1 hour and 45 minutes. How far does he walk? Give your answer in metres.

2 Lisbeth runs at 7.5 m/s for 12 seconds. How far does she run?

3 **a** Convert 80 km/h into metres per second.

b Convert 25 m/s into kilometres per hour.

4 **a** A plane travels 1000 km in 1 hour 20 minutes. What is the average speed of the plane?

b A car travels 125 km in 1 hour 20 minutes. What is the average speed of the car? Give your answer in kilometres per hour.

5 A car is travelling at 95 km/h.

a What is the formula for the distance d (in kilometres) travelled by the car in t hours?

b What is the gradient of the straight-line graph of d against t?

6 Write each of the following in words.

a $x \propto y$ **b** $p \propto n^2$ **c** $a \propto \sqrt{b}$ **d** $p \propto q^3$

7 **a** Given that $p \propto q$ and $p = 12$ when $q = 1.5$, find the exact value of:

i p when $q = 6$ **ii** q when $p = 81$

b Given that $a \propto b^2$ and $a = 20$ when $b = 4$, find the formula for a in terms of b and find the exact value of:

i a when $b = 5$ **ii** a when $b = 12$

8 In each of the following tables $y \propto x$. Find the constant of proportionality in each case and complete the table.

a

x	0	1	2	3
y	0	12		

b

x	2	8	12	18
y	3			

9 Given that $y \propto x^3$, what is the effect on y when x is:

a doubled? **b** multiplied by 3? **c** divided by 4?

10 Given that $a \propto b^2$, what is the effect on a when b is:

a increased by 5%? **b** decreased by 8%?

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7

Challenge exercise

1 If $a \propto c$ and $b \propto c$, prove that $a+b$, $a-b$ and $\sqrt{ab}$ are directly proportional to c.

2 It is known that $a \propto x$, $b \propto \frac{1}{x^2}$ and $y = a + b$. If $y = 30$ when $x = 2$ or $x = 3$, find an expression for y in terms of x.

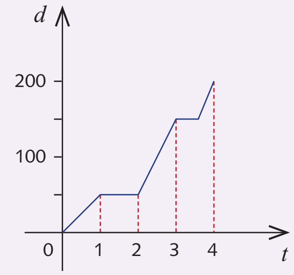
3 If x and y are positive numbers $x^2 + y^2$ varies directly as $x + y$ and $y = 2$ when $x = 2$, find the value of y when $x = \frac{4}{5}$.

4 For stones of the same quality, the value of a diamond is proportional to the square of its weight. Find the loss incurred by cutting a diamond worth $\$C$ into two pieces whose weights are in the ratio $a : b$.

5 If $a + b \propto a - b$, prove that $a^2 + b^2 \propto ab$.

6 One car travelling at 80 km/h leaves Melbourne at 8 a.m. It is followed at 10 a.m. by another car travelling on the same road at 110 km/h. At what time will the second car overtake the first?

7 A salesman travelled from town A to town B, which is a distance of 200 km. The graph shows his distance (d kilometres) from town A, t hours after noon.



From the graph, find:

a the distance travelled:

i in the first hour

ii in the third hour

b the speed at which the salesman travelled during:

i the first hour

ii the third hour

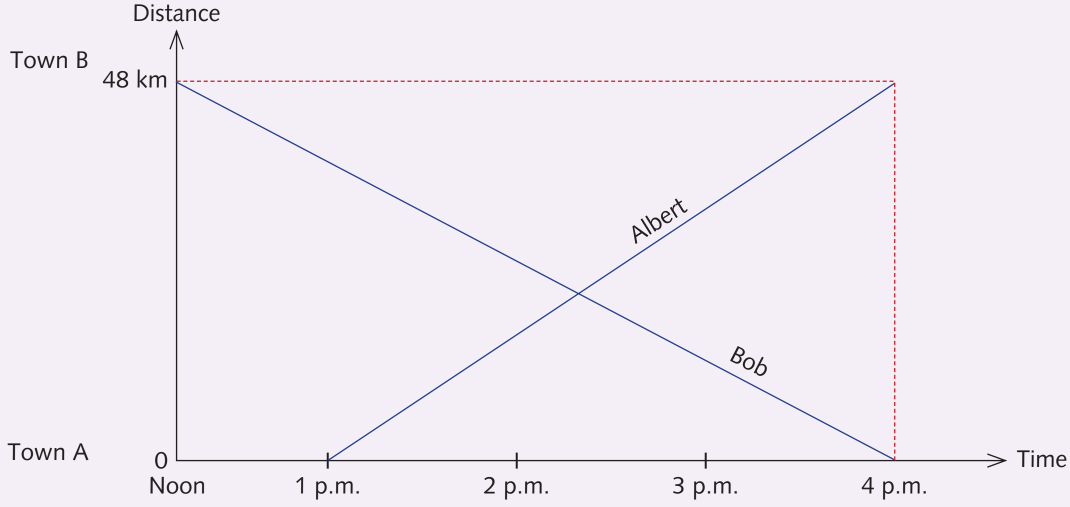
c how far from town A the salesman was after 2 hours of travelling

d at what time the salesman was first 50 km from town B

8 Two aeroplanes pass each other in flight while travelling in opposite directions. Each of the planes continues on its flight for 45 minutes, after which the planes are 840 km apart. The speed of the first aeroplane is $\frac{3}{4}$ the speed of the other aeroplane. Calculate the average speed of each aeroplane.

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7 © Brown et al. 2017 Cambridge University Press

9 The following graph shows the distance from town A against time for two cyclists, Albert and Bob. Town B is 48 km from town A.



a How far did Albert travel?

b What was Albert's speed?

c What was Bob's speed?

d After how many hours do the two cyclists pass each other?

ICE-EM Mathematics 9 3ed ISBN 978-1-108-40432-7