Chapter 8
Polish Spaces and Analytic Sets

The Borel subsets of a complete separable metric space have a number of interesting and useful characteristics. For example, if A and B are uncountable Borel subsets of complete separable metric spaces, then A and B are Borel isomorphic—that is, there is a bijection $f: A \to B$ such that f and f^{-1} are both Borel measurable. A related result says that if A is a Borel subset of a complete separable metric space, if Y is a complete separable metric space, and if $f: A \to Y$ is injective and Borel measurable, then $f(A)$ is a Borel subset of Y. If the function f is not injective, then $f(A)$ may not be a Borel set, but it will be μ-measurable for every finite Borel measure μ on Y (that is, there will be Borel subsets B_1 and B_2 of Y such that $B_1 \subseteq f(A) \subseteq B_2$ and $\mu(B_2 - B_1) = 0$).

This chapter is devoted to proving such results and to showing the context in which they arise.

8.1 Polish Spaces

A *Polish* space is a separable topological space that can be metrized using a complete metric. This section contains a number of elementary properties of Polish spaces. In Sects. 8.3 through 8.6 we will use these properties, plus the concept of an analytic set (defined in Sect. 8.2), to derive some deep and useful results about measurable sets and functions.

There are many topological spaces that are Polish, but have no complete metric that is particularly natural or simple. Furthermore, many constructions and facts of interest in measure theory depend on the existence of a complete metric, but not on the choice of a particular metric. It has thus become rather common to deal with the class of Polish spaces, rather than with the class of complete separable metric spaces.

D.L. Cohn, *Measure Theory: Second Edition*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4614-6956-8_8,

Examples 8.1.1.

(a) For each d the space $\mathbb{R}^d$, with its usual topology, is Polish.
(b) More generally, each separable Banach space, with the topology induced by its norm, is Polish.
(c) Each compact metrizable space is Polish (see Theorem D.39 and Corollary D.40). It amounts to the same thing to say that each compact Hausdorff space that has a countable base is Polish (see Proposition 7.1.13).

□

We need the following results before we look at some additional examples.

Proposition 8.1.2. *Each closed subspace, and each open subspace, of a Polish space is Polish.*

Proof. Let X be a Polish space. According to D.33, every subspace of X is separable. Hence we need only check that the closed subspaces and the open subspaces of X can be metrized by means of complete metrics.

Let d be a complete metric for X. If F is a closed subspace of X, then the restriction of d to F is a complete metric for F. Hence each closed subspace of X is Polish.

Now suppose that U is an open subspace of X. We can assume that $U \neq X$. Recall that $d(x,U^c)$, the distance between x and U^c, is defined by

$$d(x,U^c) = \inf\{d(x,z) : z \in U^c\}$$

(see D.27). It is easy to see that

$$d_0(x,y) = d(x,y) + \left| \frac{1}{d(x,U^c)} - \frac{1}{d(y,U^c)} \right|$$

defines a metric d_0 on the set U; we will check that d_0 metrizes the topology that U inherits as a subspace of X and then that U is complete under d_0.

The function $x \mapsto d(x,U^c)$ is continuous (again see D.27), from which it follows that if x and x_1, x_2, ... belong to U, then the sequence $\{x_n\}$ converges to x with respect to d if and only if it converges to x with respect to d_0. Thus d_0 metrizes the topology of U.

We turn to the completeness of U under d_0. A sequence $\{x_n\}$ that is Cauchy under d_0 is also Cauchy under d, and so converges under d to a point x of X. The point x belongs to U, since otherwise we would have $\lim_n d(x_n,U^c) = 0$, which would imply that

$$\overline{\lim_{m,n}} \, d_0(x_m,x_n) = +\infty,$$

contradicting the assumption that $\{x_n\}$ is Cauchy under d_0. It now follows that $\{x_n\}$ also converges to x under d_0, and the completeness of U under d_0 follows. □

For the next results we need to recall a technique for constructing bounded metrics. Suppose that d is a metric on a set X. It is easy to check that the formula

$$d_0(x,y) = \min(1, d(x,y)) \tag{1}$$

defines a metric on X and that $d(x,y) = d_0(x,y)$ holds whenever x and y are such that $d(x,y)$ (or $d_0(x,y)$) is less than 1. It follows that d and d_0 determine the same topology on X and that X is complete under d_0 if and only if it is complete under d.

Recall that the *disjoint union* $\sum_\alpha X_\alpha$ of an indexed collection $\{X_\alpha\}$ of topological spaces is defined by letting the underlying set $\sum_\alpha X_\alpha$ be the disjoint union[1] of the X_α's and then declaring that a subset of $\sum_\alpha X_\alpha$ is open if and only if for each α its intersection with X_α is an open subset of X_α.

Proposition 8.1.3. *The disjoint union of a finite or infinite sequence of Polish spaces is Polish.*

Proof. Let X_1, X_2, ... be Polish spaces, and let $\sum_n X_n$ be their disjoint union. For each n let D_n be a countable dense subset of X_n and let d_n be a complete metric on X_n. We can assume that $d_n(x,y) \le 1$ holds for each n and for all x and y in X_n (see Eq. (1)). Then $\sum_n D_n$ is a countable dense subset of $\sum_n X_n$, and

$$d(x,y) = \begin{cases} d_n(x,y) & \text{if } x,y \in X_n \text{ for some } n, \\ 1 & \text{if } x \in X_m \text{ and } y \in X_n, \text{ where } m \neq n \end{cases}$$

defines a complete metric that metrizes $\sum_n X_n$. □

Proposition 8.1.4. *The product of a finite or infinite sequence of Polish spaces is Polish.*

Proof. Let X_1, X_2, ... be a finite or infinite sequence of Polish spaces. We can assume that no X_n is empty. For each n let d_n be a complete metric that metrizes X_n and satisfies $d_n(x,y) \le 1$ for all x and y in X_n (see Eq. (1)). For points x and y in $\prod_n X_n$, with coordinates $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, respectively, let

$$d(x,y) = \sum_n \frac{1}{2^n} d_n(x_n, y_n).$$

It is easy to check that this defines a metric d on $\prod_n X_n$, that d metrizes the product topology on $\prod_n X_n$, and that $\prod_n X_n$ is complete under d.

---

[1]Let $\{Y_\alpha\}$ be an indexed collection of sets such that

(a) for each α the set Y_α has the same cardinality as the set X_α, and
(b) Y_{α_1} and Y_{α_2} are disjoint if $\alpha_1 \neq \alpha_2$

(for instance, one might let Y_α be $X_\alpha \times \{\alpha\}$). The *disjoint union* of the X_α's is defined to be the union of the Y_α's. (One generally thinks of the Y_α's as being identified with the corresponding X_α's.)

To prove the separability of $\prod_n X_n$, it is enough to construct a countable base for $\prod_n X_n$ (see D.10). For each n choose a countable base $\mathscr{U}_n$ for X_n (see D.32). Then the collection of subsets of $\prod_n X_n$ that have the form

$$U_1 \times \cdots \times U_N \times X_{N+1} \times X_{N+2} \times \ldots$$

for some N and some choice of sets U_n in $\mathscr{U}_n$, $n = 1, \ldots, N$, is the required base for $\prod_n X_n$. □

Proposition 8.1.5. *Let X be a Polish space. Then a subspace of X is Polish if and only if it is a G_δ in X.*

Proof. First let $\{U_n\}$ be a sequence of open subsets of X and let $Y = \cap_n U_n$. Each U_n is Polish (Proposition 8.1.2), as is the product $\prod_n U_n$ (Proposition 8.1.4). Let Δ be the subset of $\prod_n U_n$ defined by

$$\Delta = \left\{ \{u_n\} \in \prod_n U_n : u_j = u_k \text{ for all } j, k \right\}.$$

Then Δ is a closed subset of $\prod_n U_n$, and so is Polish. Furthermore Y is homeomorphic to Δ via the map that takes an element y of Y to the sequence each term of which is y. Hence Y is Polish.

We turn to the converse. So suppose that Y is a subspace of X that is Polish. Let d be a metric for the topology of X, and let d_0 be a complete metric for the topology of Y. For each n let V_n be the union of those open subsets W of X that have diameter at most $1/n$ under d and for which $W \cap Y$ is nonempty and has diameter at most $1/n$ under d_0. Since d and d_0 induce the same topology on Y, every point in Y belongs to each V_n. Let us show that

$$Y = \overline{Y} \cap (\cap_n V_n). \tag{2}$$

We just noted that $Y \subseteq V_n$ holds for each n, and so, we have $Y \subseteq \overline{Y} \cap (\cap_n V_n)$. We turn to the reverse inclusion. Suppose that $x \in \overline{Y} \cap (\cap_n V_n)$. Since $x \in \cap_n V_n$, we can choose a sequence $\{W_n\}$ of open neighborhoods of x such that for each n the sets W_n and $Y \cap W_n$ have diameters (under d and d_0, respectively) at most $1/n$. Since $x \in \overline{Y}$, our sets W_n satisfy $W_n \cap Y \neq \varnothing$ for each n. Thus we can form a sequence $\{x_n\}$ by choosing (for each n) a point x_n in $W_n \cap Y$. Our conditions on the diameters of the sets W_n under d and d_0 imply that $\{x_n\}$ converges to x with respect to d and that it is a Cauchy sequence (in Y) with respect to d_0. Thus there is a point y in Y to which $\{x_n\}$ converges under d_0. Since d and d_0 metrize the same topology on Y, it follows $\{x_n\}$ also converges to y under d and hence that $x = y \in Y$. Thus $\overline{Y} \cap (\cap_n V_n) \subseteq Y$ and the proof of (2) is complete. Since each closed subset of X (in particular, $\overline{Y}$) is a G_δ in X (see D.28), relation (2) implies that Y is a G_δ in X. □

Examples 8.1.6.

(a) Let X be a locally compact Hausdorff space that has a countable base for its topology. Its one-point compactification X^* also has a countable base (Lemma 7.1.14) and so is Polish (Example 8.1.1(c)). Proposition 8.1.2 now implies that X, as an open subset of X^*, is Polish.

(b) The space $\mathbb{N}^{\mathbb{N}}$ is, according to Proposition 8.1.4, Polish. We will often denote this space by $\mathscr{N}$. Its elements are, of course, sequences of positive integers. A typical such sequence will generally be denoted by $\{n_i\}$ or by $\mathbf{n}$ (the boldface $\mathbf{n}$ is a useful substitute for $\{n_i\}$ in complicated expressions).

For positive integers k and $n_1, \dots, n_k$ we will denote by $\mathscr{N}(n_1, \dots, n_k)$ the set of those elements $\{m_i\}$ of $\mathscr{N}$ that satisfy $m_i = n_i$ for $i = 1, \dots, k$. It is easy to check that the family of all such sets is a countable base for $\mathscr{N}$. It is also easy to check that the collection of those elements $\{m_i\}$ of $\mathscr{N}$ that are eventually constant (that is, for which there is a positive integer k such that $m_i = m_k$ holds whenever $i > k$) is a countable dense subset of $\mathscr{N}$.

(c) Next consider the space $\mathscr{I}$ of irrational numbers in the interval $(0,1)$, together with the topology it inherits from $\mathbb{R}$. The complement of $\mathscr{I}$ in $\mathbb{R}$ is an F_σ, and so $\mathscr{I}$ is a G_δ; thus Proposition 8.1.5 implies that $\mathscr{I}$ is Polish. It can be shown that $\mathscr{I}$ is homeomorphic to $\mathscr{N}$ (see Exercise 3 in Sect. 8.2).

(d) The space $\mathbb{Q}$ of rational numbers is not Polish (see Exercise 2).

(e) The space $\{0,1\}^{\mathbb{N}}$, which consists of all sequences of zeroes and ones, is Polish (Proposition 8.1.4 or Example 8.1.1(c)). It can be shown that this space is homeomorphic to the Cantor set (see Exercise 1). □

The spaces $\mathscr{N}$ and $\{0,1\}^{\mathbb{N}}$ turn out to be very important in the development of the theory of Polish spaces and analytic sets.

We turn to some basic facts about the Borel subsets of Polish spaces.

Let $(X_1, \mathscr{A}_1)$, $(X_2, \mathscr{A}_2)$, ... be measurable spaces. The *product* of these measurable spaces is the measurable space $(\prod_n X_n, \prod_n \mathscr{A}_n)$ where $\prod_n \mathscr{A}_n$ is the σ-algebra on $\prod_n X_n$ that is generated by the sets that have the form

$$A_1 \times A_2 \times \cdots \times A_N \times X_{N+1} \times X_{N+2} \times \dots \tag{3}$$

for some positive integer N and some choice of A_n in $\mathscr{A}_n$, $n = 1, \dots, N$. For each i let π_i be the projection of $\prod_n X_n$ onto X_i. Then

$$\pi_i^{-1}(A) = X_1 \times \cdots \times X_{i-1} \times A \times X_{i+1} \times \dots$$

holds for each subset A of X_i, and so π_i is measurable with respect to $\prod_n \mathscr{A}_n$ and $\mathscr{A}_i$. The set in display (3) is equal to $\cap_{i=1}^N \pi_i^{-1}(A_i)$; hence $\prod_n \mathscr{A}_n$ is the smallest σ-algebra on $\prod_n X_n$ that makes all the projections π_i measurable.

Proposition 8.1.7. *Let X_1, X_2, ... be a finite or infinite sequence of separable metrizable spaces. Then $\mathscr{B}(\prod_n X_n) = \prod_n \mathscr{B}(X_n)$.*

Proof. For each i consider the projection π_i of $\prod_n X_n$ onto X_i. Each such projection is continuous and so is measurable (Lemma 7.2.1) with respect to $\mathscr{B}(\prod_n X_n)$ and $\mathscr{B}(X_i)$. Since $\prod_n \mathscr{B}(X_n)$ is the smallest σ-algebra on $\prod_n X_n$ that makes these projections measurable (see the remarks above), it follows that $\prod_n \mathscr{B}(X_n) \subseteq \mathscr{B}(\prod_n X_n)$.

We turn to the reverse inclusion. For each n choose a countable base $\mathscr{U}_n$ for X_n (see D.32), and then let $\mathscr{U}$ be the collection of sets that have the form

$$U_1 \times \cdots \times U_N \times X_{N+1} \times \ldots$$

for some positive integer N and some choice of sets U_n in $\mathscr{U}_n$, for $n = 1, \ldots, N$. Then $\mathscr{U}$ is a countable base for $\prod_n X_n$, and $\mathscr{U} \subseteq \prod_n \mathscr{B}(X_n)$. Since each open subset of $\prod_n X_n$ is the union of a (necessarily countable) subfamily of $\mathscr{U}$, it follows that $\mathscr{B}(\prod_n X_n) \subseteq \prod_n \mathscr{B}(X_n)$. Thus $\mathscr{B}(\prod_n X_n) = \prod_n \mathscr{B}(X_n)$, and the proof is complete. □

Let X and Y be sets, and let f be a function from X to Y. The *graph* of f, denoted by $\text{gr}(f)$, is defined by

$$\text{gr}(f) = \{(x,y) \in X \times Y : y = f(x)\}.$$

Proposition 8.1.8. *Let X and Y be separable metrizable spaces, and let $f: X \to Y$ be Borel measurable. Then the graph of f is a Borel subset of $X \times Y$.*

Proof. Let $F: X \times Y \to Y \times Y$ be the map that takes (x,y) to $(f(x),y)$. The Borel measurability of f implies that if $A, B \in \mathscr{B}(Y)$, then $F^{-1}(A \times B) \in \mathscr{B}(X) \times \mathscr{B}(Y)$; hence F is measurable with respect to $\mathscr{B}(X) \times \mathscr{B}(Y)$ and $\mathscr{B}(Y) \times \mathscr{B}(Y)$ (Proposition 2.6.2) and so with respect to $\mathscr{B}(X \times Y)$ and $\mathscr{B}(Y \times Y)$ (Proposition 8.1.7). Let $\Delta = \{(y_1, y_2) \in Y \times Y : y_1 = y_2\}$. Then Δ is a closed subset of $Y \times Y$ and $\text{gr}(f) = F^{-1}(\Delta)$. It follows that $\text{gr}(f)$ is a Borel subset of $X \times Y$. □

Lemma 8.1.9. *Let $(X, \mathscr{A})$ be a measurable space, and let Y be a metrizable topological space. Then a function $f: X \to Y$ is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$ if and only if for each continuous function $g: Y \to \mathbb{R}$ the function $g \circ f$ is $\mathscr{A}$-measurable.*

Proof. If f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$, then the measurability of $g \circ f$ for each continuous g follows from the measurability of g (Lemma 7.2.1), together with Proposition 2.6.1.

Now assume that for each continuous $g: Y \to \mathbb{R}$ the function $g \circ f$ is $\mathscr{A}$-measurable, and let d be a metric that metrizes Y. Suppose that U is an open subset of Y. Then there is a continuous function $g_U: Y \to \mathbb{R}$ such that

$$U = \{y \in Y : g_U(y) > 0\}$$

(if $U \neq Y$, define g_U by $g_U(y) = d(y, U^c)$; otherwise, let g_U be the constant function 1). The set $f^{-1}(U)$ is equal to

$$\{x \in X : (g_U \circ f)(x) > 0\}$$

and so belongs to $\mathscr{A}$. Since U was an arbitrary open subset of X, the measurability of f follows (Proposition 2.6.2). □

Proposition 8.1.10. *Let $(X, \mathscr{A})$ be a measurable space, let Y be a metrizable topological space, and for each positive integer n let $f_n \colon X \to Y$ be measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$. If $\lim_n f_n(x)$ exists for each x in X, then the function $f \colon X \to Y$ given by $f(x) = \lim_n f_n(x)$ is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$.*

Proof. Note that if $g \colon Y \to \mathbb{R}$ is continuous, then $g(f(x)) = \lim_n g(f_n(x))$ holds for each x in X. The proposition is now an immediate consequence of Lemma 8.1.9 and Proposition 2.1.5. □

Proposition 8.1.11. *Let $(X, \mathscr{A})$ be a measurable space, let Y be a Polish space, and for each positive integer n let $f_n \colon X \to Y$ be measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$. Let $C = \{x \in X : \lim_n f_n(x)$ exists$\}$. Then $C \in \mathscr{A}$. Furthermore, the map $f \colon C \to Y$ defined by $f(x) = \lim_n f_n(x)$ is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$.*

Proof. Let d be a complete metric for Y. Then C is the set consisting of those x in X for which $\{f_n(x)\}$ is a Cauchy sequence in Y. For each positive integer n the set $\{(y_1, y_2) \in Y \times Y : d(y_1, y_2) < 1/n\}$ is an open subset of $Y \times Y$ and so belongs to $\mathscr{B}(Y) \times \mathscr{B}(Y)$ (Proposition 8.1.7). Thus for each i, j, and n the set $C(i, j, n)$ defined by

$$C(i,j,n) = \left\{ x \in X : d(f_i(x), f_j(x)) < \frac{1}{n} \right\}$$

belongs to $\mathscr{A}$. Since

$$C = \bigcap_n \bigcup_k \bigcap_{i \geq k} \bigcap_{j \geq k} C(i,j,n),$$

it follows that $C \in \mathscr{A}$. The measurability of f is now a consequence of Proposition 8.1.10, applied to the spaces $(C, \mathscr{A}_C)$ and Y (here $\mathscr{A}_C$ is the trace of $\mathscr{A}$ on C; see Exercise 1.5.11). □

We conclude this section with the following useful fact about measures on Polish spaces.

Proposition 8.1.12. *Every finite Borel measure on a Polish space is regular.*

Proof. Let X be a Polish space, let d be a complete metric for X, and let μ be a finite Borel measure on X. We can assume that X is not empty. Since each open subset of X is an F_σ in X (see D.28), Lemma 7.2.4 implies that each Borel subset A of X satisfies

$$\mu(A) = \inf\{\mu(U) : A \subseteq U \text{ and } U \text{ is open}\} \tag{4}$$

and

$$\mu(A) = \sup\{\mu(F) : F \subseteq A \text{ and } F \text{ is closed}\}. \tag{5}$$

We will strengthen (5) by showing that each Borel subset A of X satisfies

$$\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ and } K \text{ is compact}\}. \tag{6}$$

First consider the case where $A = X$. Let $\{x_k\}$ be a sequence whose terms form a dense subset of X, and let ε be a positive number. For each positive integer n use Proposition 1.2.5 and the fact that X is the union of the open balls $B(x_k, 1/n)$, $k = 1$, $2, \dots$, to choose a positive integer k_n such that

$$\mu\Big(\bigcup_{k=1}^{k_n} B(x_k, 1/n)\Big) > \mu(X) - \varepsilon/2^n.$$

Let $K = \cap_n \cup_{k=1}^{k_n} \overline{B(x_k, 1/n)}$. Then K is complete and totally bounded under the restriction of d to K, and so is compact (Theorem D.39). Furthermore

$$\mu(K^c) \le \sum_n \mu\Big(\Big(\bigcup_{k=1}^{k_n} B(x_k, 1/n)\Big)^c\Big) < \sum_n \varepsilon/2^n = \varepsilon,$$

and so $\mu(K) > \mu(X) - \varepsilon$. Since ε is arbitrary, (6) follows in the case where $A = X$.

Now let A be an arbitrary Borel subset of X, and let ε be a positive number. Choose a compact set K such that $\mu(K) > \mu(X) - \varepsilon$, and use (5) to choose a closed subset F of A such that $\mu(F) > \mu(A) - \varepsilon$. Then $K \cap F$ is a compact subset of A, and $\mu(K \cap F) > \mu(A) - 2\varepsilon$. Since ε is arbitrary, A must satisfy (6). Thus μ is regular.

□

Exercises

1. Show that the map that takes the sequence $\{n_k\}$ to the number $\sum_k 2n_k/3^k$ is a homeomorphism of $\{0,1\}^{\mathbb{N}}$ onto the Cantor set.
2. Show that the set $\mathbb{Q}$ of rational numbers, with the topology it inherits as a subspace of $\mathbb{R}$, is not Polish. (Hint: Use the Baire category theorem, Theorem D.37.)
3. Let $(X, \mathscr{A})$ be a measurable space, let Y be a separable metrizable space, and let $f, g \colon X \to Y$ be measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$. Show that $\{x \in X : f(x) = g(x)\}$ belongs to $\mathscr{A}$.
4. Suppose that $\{X_n\}$ is a sequence of nonempty separable metrizable spaces and that, for each n, D_n is a countable dense subset of X_n. Give (rather explicitly) a countable dense subset of $\prod_n X_n$.
5. Let X be a Polish space, let $\{U_n\}$ be a sequence of open subsets of X, and let d be a complete metric for X. Construct a complete metric for $\cap_n U_n$; show directly that it has the required properties. (Hint: Examine the proofs of Propositions 8.1.2, 8.1.4, and 8.1.5.)
6. Let $C[0,+\infty)$ be the set of all continuous real-valued functions on the interval $[0,+\infty)$.

(a) Show that the formula

$$d(f,g) = \sup\{1 \wedge |f(t) - g(t)| : t \in [0,+\infty)\}$$

defines a metric on $C[0,+\infty)$.

(b) Suppose that f and $f_1, f_2, \ldots$ belong to $C[0,+\infty)$. Show that $\{f_k\}$ converges to f with respect to the metric in part (a) if and only if it converges to f uniformly on $[0,+\infty)$.

(c) Show that $C[0,+\infty)$, when endowed with the topology determined by the metric in part (a), is not separable and hence not Polish.

7. (a) Show that the formula

$$d(f,g) = \sum_n \frac{1}{2^n} \sup\{1 \wedge |f(t) - g(t)| : t \in [0,n]\}$$

defines a metric on the set $C[0,+\infty)$ (see Exercise 6).

(b) Suppose that f and $f_1, f_2, \ldots$ belong to $C[0,+\infty)$. Show that $\{f_k\}$ converges to f with respect to the metric in part (a) if and only if it converges to f uniformly on each compact subset of $[0,+\infty)$.

(c) Show that $C[0,+\infty)$ is complete and separable under the metric defined in part (a). (Hint: See Exercise 7.1.9.)

8. Prove Proposition 8.1.10 directly, without using continuous functions.
9. Suppose that in Proposition 8.1.11 the space Y were only required to be separable and metrizable. Show by example that the set C would not need to belong to $\mathscr{A}$.
10. Show that every finite Borel measure on $\mathbb{Q}$ is regular. (Recall that $\mathbb{Q}$ is not Polish; see Exercise 2.)
11. Show by example that a finite Borel measure on a separable metrizable space can fail to be regular. (Hint: Suppose that X is a subset of $\mathbb{R}$ that satisfies $\lambda^*(X) < +\infty$ but is not Lebesgue measurable. Consider the measure on $(X, \mathscr{B}(X))$ that results when the construction of Exercise 1.5.11 is applied to Lebesgue measure.)
12. Show that every separable metrizable space is homeomorphic to a subspace of the product space $[0,1]^{\mathbb{N}}$ and that every Polish space is homeomorphic to a G_δ in $[0,1]^{\mathbb{N}}$. (Hint: Let d be a metric for the separable metrizable space X, and let $\{x_n\}$ be a sequence whose terms form a dense subset of X. Consider the map from X to $[0,1]^{\mathbb{N}}$ that takes the point x to the sequence whose n^{th} term is $\min(1, d(x,x_n))$.)
13. Let $(X, \mathscr{A})$ be a measurable space, let Y be a Polish space, let A be a subset of X that might not belong to $\mathscr{A}$, and let $\mathscr{A}_A$ be the trace of $\mathscr{A}$ on A (see Exercise 1.5.11). Show that if $f: A \to Y$ is measurable with respect to $\mathscr{A}_A$ and $\mathscr{B}(Y)$, then f has an extension $F: X \to Y$ that is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$.

14. Give a counterexample that shows that the metrizability of Y cannot be omitted in Proposition 8.1.10. (Hint: Let $(X,\mathscr{A})$ be $([0,1],\mathscr{B}([0,1]))$ and let Y be $[0,1]^{[0,1]}$ with the product topology. For each n let $f_n\colon X \to Y$ be the function that takes x to the element of Y (i.e., to the function from $[0,1]$ to $[0,1]$) given by $t \mapsto \max(0, 1 - n|t-x|)$.)

8.2 Analytic Sets

Let X be a Polish space. A subset A of X is *analytic* if there is a Polish space Z and a continuous function $f\colon Z \to X$ such that $f(Z) = A$.

We will soon see that every Borel subset of a Polish space is analytic, but that there are analytic sets that are not Borel.

Analytic sets are useful tools for the study of Borel sets and Borel measurable functions (see Sect. 8.3); they also possess measurability properties that make them useful in their own right (see Sects. 8.4 and 8.5). This section contains a few elementary properties of analytic sets, some techniques for constructing those continuous maps that will be needed later in this chapter, and a construction that provides an analytic set that is not Borel. The reader might well skip from Proposition 8.2.9 to Sect. 8.3 at a first reading, returning for the remaining results as they are needed.

Proposition 8.2.1. *Let X be a Polish space. Then each open subset, and each closed subset, of X is analytic.*

Proof. This is an immediate consequence of Proposition 8.1.2, together with the continuity of the standard injection of a subspace of X into X. □

Proposition 8.2.2. *Let X be a Polish space, and let A_1, A_2, ... be analytic subsets of X. Then $\cup_k A_k$ and $\cap_k A_k$ are analytic.*

Proof. For each k choose a Polish space Z_k and a continuous function $f_k\colon Z_k \to X$ such that $f(Z_k) = A_k$. Let Z be the disjoint union of the spaces Z_1, Z_2, ..., and define $f\colon Z \to X$ so that for each k it agrees on Z_k with f_k. Then Z is a Polish space (Proposition 8.1.3), f is a continuous function, and $f(Z) = \cup_k A_k$; hence $\cup_k A_k$ is analytic.

Next form the product space $\prod_k Z_k$, and let Δ consist of those sequences $\{z_k\}$ in $\prod_k Z_k$ such that $f_i(z_i) = f_j(z_j)$ holds for all i and j. Then Δ is a closed subspace of $\prod_k Z_k$ and so is Polish (Propositions 8.1.2 and 8.1.4). The set $\cap_k A_k$ is the image of Δ under the continuous function that takes the sequence $\{z_k\}$ to the point $f_1(z_1)$; hence it is analytic. □

It should be noted that the complement of an analytic set is not necessarily analytic. In fact, the complement of an analytic set A is analytic if and only if A is Borel (see Proposition 8.2.3 and Corollary 8.3.3).

Proposition 8.2.3. *Let X be a Polish space. Then each Borel subset of X is analytic.*

The proof will depend on the following lemma. Because of later applications, this lemma is given in a slightly stronger form than is needed here.

Lemma 8.2.4. *Let X be a Hausdorff topological space. Then $\mathscr{B}(X)$ is the smallest family of subsets of X that*

(a) *contains the open and the closed subsets of X,*
(b) *is closed under the formation of countable intersections, and*
(c) *is closed under the formation of countable disjoint unions.*

Note that closure under complementation is not one of the conditions listed in Lemma 8.2.4.

Proof. Let $\mathscr{S}$ be the smallest collection of subsets of X that satisfies conditions (a), (b), and (c) of the lemma (why does such a smallest collection exist?), and let $\mathscr{S}_0 = \{A : A \in \mathscr{S} \text{ and } A^c \in \mathscr{S}\}$. It is clear that $\mathscr{S}_0 \subseteq \mathscr{S} \subseteq \mathscr{B}(X)$. Thus if we show that $\mathscr{S}_0$ is a σ-algebra that contains each open subset of X, it will follow that $\mathscr{S}_0 = \mathscr{S} = \mathscr{B}(X)$, and the proof will be complete.

It is immediate that $\mathscr{S}_0$ contains the open subsets of X and is closed under complementation. Now suppose that $\{A_n\}$ is a sequence of sets in $\mathscr{S}_0$. Then $\cup_n A_n$ is the union of the sets

$$A_1, A_1^c \cap A_2, A_1^c \cap A_2^c \cap A_3, \ldots;$$

these sets are disjoint and belong to $\mathscr{S}$, and so $\cup_n A_n$ must also belong to $\mathscr{S}$. Furthermore $(\cup_n A_n)^c$ is the intersection of a sequence (namely $\{A_n^c\}$) of sets in $\mathscr{S}$, and so belongs to $\mathscr{S}$. Consequently $\cup_n A_n$ belongs to $\mathscr{S}_0$. It follows that $\mathscr{S}_0$ is closed under the formation of countable unions. With this we have shown that $\mathscr{S}_0$ is a σ-algebra that contains the open subsets of X, and the proof of Lemma 8.2.4 is complete. □

Proof of Proposition 8.2.3. Since the collection of analytic subsets of X satisfies conditions (a), (b), and (c) of Lemma 8.2.4 (see Propositions 8.2.1 and 8.2.2), it must include $\mathscr{B}(X)$. □

Proposition 8.2.5. *Let X_1, X_2, $\ldots$ be a finite or infinite sequence of Polish spaces, and for each k let A_k be an analytic subset of X_k. Then $\prod_k A_k$ is an analytic subset of $\prod_k X_k$.*

Proof. If some A_k is empty, then $\prod_k A_k$ is empty and so is an analytic set. Otherwise for each k choose a Polish space Z_k and a continuous function $f_k \colon Z_k \to X_k$ such that $f_k(Z_k) = A_k$. Define a function $f \colon \prod_k Z_k \to \prod_k X_k$ by $f(\{z_k\}) = \{f_k(z_k)\}$. Then $\prod_k Z_k$ is Polish, f is continuous, and $f(\prod_k Z_k) = \prod_k A_k$. Thus $\prod_k A_k$ is analytic. □

Proposition 8.2.6. *Let X and Y be Polish spaces, let A be an analytic subset of X, and let $f \colon A \to Y$ be Borel measurable (that is, measurable with respect to $\mathscr{B}(A)$ and $\mathscr{B}(Y)$). If A_1 and A_2 are analytic subsets of X and Y, respectively, then $f(A \cap A_1)$ and $f^{-1}(A_2)$ are analytic subsets of Y and X, respectively.*

Proof. Let π_Y be the projection of $X \times Y$ onto Y. Proposition 8.1.8 implies that $\mathrm{gr}(f) \in \mathscr{B}(A \times Y)$, and Lemma 7.2.2 then implies that there is a Borel subset B of $X \times Y$ such that $\mathrm{gr}(f) = B \cap (A \times Y)$. Hence $\mathrm{gr}(f) \cap (A_1 \times Y)$ is an analytic subset of $X \times Y$ (Propositions 8.2.2, 8.2.3, and 8.2.5) and so is the image of a Polish space (say Z) under a continuous map (say h). It follows that $f(A \cap A_1)$, since it is the projection of $\mathrm{gr}(f) \cap (A_1 \times Y)$ on Y, is the image of Z under the continuous map $\pi_Y \circ h$ and so is analytic. A similar argument shows that $f^{-1}(A_2)$ is analytic (note that it is the projection of $\mathrm{gr}(f) \cap (X \times A_2)$ on X). □

We turn to the construction of some continuous functions that are useful in the study of Borel and analytic sets.

Proposition 8.2.7. *Each nonempty Polish space is the image of $\mathscr{N}$ under a continuous function.*

Proof. Let X be a nonempty Polish space, and let d be a complete metric for X. We begin by constructing a family $\{C(n_1,\dots,n_k)\}$ of subsets of X, indexed by the set of all finite sequences $(n_1,\dots,n_k)$ of positive integers, in such a way that

(a) $C(n_1,\dots,n_k)$ is closed and nonempty,
(b) the diameter of $C(n_1,\dots,n_k)$ is at most $1/k$,
(c) $C(n_1,\dots,n_{k-1}) = \cup_{n_k} C(n_1,\dots,n_k)$, and
(d) $X = \cup_{n_1} C(n_1)$.

We do this by induction on k.

First, suppose that $k = 1$, and let $\{x_{n_1}\}_{n_1=1}^{\infty}$ be a sequence whose terms form a dense subset of X. For each n_1 in $\mathbb{N}$ define $C(n_1)$ to be the closed ball with center x_{n_1} and radius $1/2$. Certainly each $C(n_1)$ is closed and nonempty and has diameter at most 1. Furthermore, $X = \cup_{n_1} C(n_1)$.

Now suppose that $k > 1$ and that $C(n_1,\dots,n_{k-1})$ has already been chosen. It is easy to use a modification of the construction of the $C(n_1)$'s, now applied to $C(n_1,\dots,n_{k-1})$ rather than to X, to produce sets $C(n_1,\dots,n_k)$, $n_k = 1, 2, \dots$, that satisfy conditions (a) through (c). With this, the inductive step in our construction is complete.

We turn to the construction of a continuous function that maps $\mathscr{N}$ onto X. Let $\mathbf{n} = \{n_k\}$ be an element of $\mathscr{N}$. It follows from (a), (b), and (c) above that $C(n_1)$, $C(n_1,n_2)$, ... is a decreasing sequence of nonempty closed subsets of X whose diameters approach 0. Thus there is a unique element in the intersection of these sets (see Theorem D.35), and we can define a function $f\colon \mathscr{N} \to X$ by letting $f(\mathbf{n})$ be the unique member of $\cap_k C(n_1,\dots,n_k)$. Note that if $\mathbf{m}$ and $\mathbf{n}$ are elements of $\mathscr{N}$ such that $m_i = n_i$ holds for $i = 1, \dots, k$, then $d(f(\mathbf{m}), f(\mathbf{n})) \le 1/k$. It follows that f is continuous. Finally, (c) and (d) above imply that for each x in X there is an element $\mathbf{n} = \{n_k\}$ of $\mathscr{N}$ such that $x \in \cap_k C(n_1,\dots,n_k)$ and hence such that $x = f(\mathbf{n})$; thus f is surjective. □

Corollary 8.2.8. *Each nonempty analytic subset of a Polish space is the image of $\mathscr{N}$ under some continuous function.*

Proof. If A is the image of the Polish space Z under the continuous function f and if Z is the image of $\mathscr{N}$ under the continuous function g (Proposition 8.2.7), then A is the image of $\mathscr{N}$ under $f \circ g$. □

Proposition 8.2.9. *Let X be a Polish space. A subset A of X is analytic if and only if there is a closed subset of $\mathscr{N} \times X$ whose projection on X is A.*

Proof. The projection on X of a closed subset of $\mathscr{N} \times X$ is the image of a Polish space (see Propositions 8.1.2 and 8.1.4) under a continuous function (the projection), and so is analytic.

Now suppose that A is an analytic subset of X. If A is empty, then it is the projection of the empty subset of $\mathscr{N} \times X$. Otherwise there is a continuous function $f\colon \mathscr{N} \to X$ such that $f(\mathscr{N}) = A$ (Corollary 8.2.8). Then $\mathrm{gr}(f)$ is a closed subset of $\mathscr{N} \times X$ whose projection on X is A. □

While the preceding material is fundamental, the following results will be used only occasionally in this book. The reader who does Exercises 1 and 5 and replaces the proof for Theorem 8.3.6 given below with the one sketched in Exercise 8.3.5 can skip everything from here through Corollary 8.2.14.

We need to recall a definition and a few facts before proving Proposition 8.2.10. A topological space is *zero dimensional* if its topology has a base that consists of sets that are both open and closed. Among the zero-dimensional spaces are the space of all rational numbers, the space of all irrational numbers, and each space that has a discrete topology. Note that a subspace of a zero-dimensional space is zero dimensional, that a product of zero-dimensional spaces is zero dimensional, and that the disjoint union of a collection of zero-dimensional spaces is zero dimensional. In particular, the spaces $\mathscr{N}$ and $\{0,1\}^{\mathbb{N}}$ are products of zero-dimensional spaces, and so are zero dimensional.

Proposition 8.2.10. *Each Borel subset of a Polish space is the image under a continuous injective map of some zero-dimensional Polish space.*

Proof. We begin by showing that each Polish space is the image under a continuous injective map of some zero-dimensional Polish space. First consider the interval $[0,1]$. It is the image of the space $\{0,1\}^{\mathbb{N}}$ under the map $F\colon \{0,1\}^{\mathbb{N}} \to [0,1]$ that takes the sequence $\{x_k\}$ to the number $\sum_k (x_k/2^k)$. Each number in $[0,1)$ that has two binary expansions (that is, each number in $(0,1)$ that is of the form $m/2^n$ for some m and n) is the image under F of two elements of $\{0,1\}^{\mathbb{N}}$; the remaining members of $[0,1]$ are images of only one element of $\{0,1\}^{\mathbb{N}}$. Thus if we remove a suitable countably infinite subset from $\{0,1\}^{\mathbb{N}}$, the remaining points form a space Z such that the restriction of F to Z is a bijection of Z onto $[0,1]$. Note that F is continuous, that Z is zero dimensional (it is a subspace of the zero-dimensional space $\{0,1\}^{\mathbb{N}}$), and that Z is Polish (its complement in $\{0,1\}^{\mathbb{N}}$ is countable, and so it is a G_δ in $\{0,1\}^{\mathbb{N}}$). Hence $[0,1]$ is the image of a zero-dimensional Polish space under a continuous injective map.

It follows that $[0,1]^{\mathbb{N}}$ is the image of the zero-dimensional Polish space $Z^{\mathbb{N}}$ under a continuous injective map.

Now suppose that X is an arbitrary Polish space. Recall (see Exercise 8.1.12) that there is a homeomorphism G of X onto a G_δ in $[0,1]^{\mathbb{N}}$. Let H be a continuous injective map of $Z^{\mathbb{N}}$ onto $[0,1]^{\mathbb{N}}$. Since $G(X)$ is a G_δ in $[0,1]^{\mathbb{N}}$, it follows that $H^{-1}(G(X))$ is a G_δ in $Z^{\mathbb{N}}$, and so is Polish. Let H_0 be the restriction of H to $H^{-1}(G(X))$. Then X is the image of the zero-dimensional Polish space $H^{-1}(G(X))$ under the continuous injective map $G^{-1} \circ H_0$.

We turn to the Borel subsets of X. Let $\mathscr{F}$ consist of those Borel subsets B of X for which there is a zero-dimensional Polish space Y and a continuous injective map $f: Y \to X$ such that $f(Y) = B$. According to the first part of this proof, $\mathscr{F}$ contains the open and the closed subsets of X (see Proposition 8.1.2), and an easy modification of the proof of Proposition 8.2.2 shows that $\mathscr{F}$ is closed under the formation of countable intersections and under the formation of countable disjoint unions. Thus Lemma 8.2.4 implies that $\mathscr{F} = \mathscr{B}(X)$. □

See Theorem 8.3.7 for a rather powerful result that implies the converse of Proposition 8.2.10.

Let us make some preparations for the proof of our next major result, Proposition 8.2.13.

Lemma 8.2.11. *Let X be a zero-dimensional separable metric space, let U be an open and non-compact subset of X, and let ε be a positive number. Then U is the union of a countably infinite family of disjoint sets, each of which is nonempty, open, closed, and of diameter at most ε.*

Proof. Since U is open and not compact, there is a family $\mathscr{U}$ of open sets whose union is U, but that has no finite subfamily whose union is U. Let $\mathscr{V}$ be the collection of all subsets of X that are open, closed, of diameter at most ε, and included in some member of $\mathscr{U}$. Since X is zero dimensional, the set U is the union of the family $\mathscr{V}$. According to D.11, there is a countable subfamily $\mathscr{V}_0$ of $\mathscr{V}$ whose union is U. List the sets in $\mathscr{V}_0$ in a sequence $V_1, V_2, \dots$, and consider the nonempty sets that appear in the sequence

$$V_1,\ V_1^c \cap V_2,\ V_1^c \cap V_2^2 \cap V_3,\ \dots.$$

These sets are open, closed, disjoint, and of diameter at most ε, and their union is U. There are infinitely many of them, since otherwise there would be a finite subfamily of $\mathscr{U}$ that would cover U. □

Let X be a topological space and let A be a subset of X, possibly the entire space X. A point x of X is a *condensation point* of A if every open neighborhood of x contains uncountably many points of A.

Lemma 8.2.12. *Let X be a separable metrizable space, and let C be the set of condensation points of X. Then C is closed, and C^c is countable.*

Proof. Let $\mathscr{U}$ be a countable base for X (see D.32). Then x fails to belong to C if and only if there is a countable open set that belongs to $\mathscr{U}$ and contains x. Hence C^c is the union of a countable collection of countable open sets, and so C^c itself is countable and open. □

Proposition 8.2.13. *Let X be a Polish space, and let B be an uncountable Borel subset of X. Then there is a continuous injective map $f\colon \mathscr{N} \to X$ such that $f(\mathscr{N}) \subseteq B$ and such that $B - f(\mathscr{N})$ is countable.*

Proof. According to Proposition 8.2.10, there exist a zero-dimensional Polish space Z and a continuous injective map $g\colon Z \to X$ such that $g(Z) = B$. Thus it will suffice to construct a continuous injective map $h\colon \mathscr{N} \to Z$ such that $Z - h(\mathscr{N})$ is countable, and then to define f to be $g \circ h$.

Let Z_0 be the collection of all points in Z that are condensation points of Z. Then Z_0 is Polish (Lemma 8.2.12) and zero dimensional. Since Z_0^c is countable, every point in Z_0 is a condensation point of Z_0 (and not just a condensation point of Z).

Suppose that d is a complete metric that metrizes Z_0. For each k we construct a family of sets, indexed by $\mathbb{N}^k$, as follows. Let us begin with the case where $k = 1$. Apply Lemma 8.2.11 to the space Z_0, letting ε be 1 and letting U consist of the points that remain when one point is removed from Z_0 (this is to guarantee that U is not compact). The sets provided by Lemma 8.2.11, say $A(n_1)$, $n_1 = 1, 2, \dots$, are disjoint, nonempty, open, closed, and of diameter at most 1, each of them consists entirely of condensation points of itself, and the union of these sets is Z_0 less a single point. We can repeat this construction over and over, for each k and $n_1, \dots, n_{k-1}$ producing sets $A(n_1, \dots, n_k)$, $n_k = 1, 2, \dots$, that are disjoint, nonempty, open, closed, and of diameter at most $1/k$, and are such that $\cup_{n_k} A(n_1, \dots, n_k)$ is $A(n_1, \dots, n_{k-1})$ less a single point.

Define $h\colon \mathscr{N} \to Z$ by letting $h(\mathbf{n})$ be the unique point in $\cap_k A(n_1, \dots, n_k)$ (Theorem D.35). It is easy to check that h is continuous and injective and that $Z_0 - h(\mathscr{N})$ is the countably infinite set consisting of the points removed from Z_0 during the construction of the sets $A(n_1, \dots, n_k)$. It follows that $Z - h(\mathscr{N})$ is countable. Thus the construction of h, and so of f, is complete. □

The following is an interesting and well-known consequence of Proposition 8.2.13 (see also Exercise 1).

Corollary 8.2.14. *Each uncountable Borel subset of a Polish space includes a subset that is homeomorphic to $\{0,1\}^{\mathbb{N}}$.*

Proof. Let X be a Polish space, and let A be an uncountable Borel subset of X. Proposition 8.2.13 provides a continuous injective map $f\colon \mathscr{N} \to X$ such that $f(\mathscr{N}) \subseteq A$. If we regard $\{0,1\}^{\mathbb{N}}$ as a subspace of $\mathscr{N}$ in the natural way, then the restriction of f to $\{0,1\}^{\mathbb{N}}$ is a homeomorphism of $\{0,1\}^{\mathbb{N}}$ onto the subset $f(\{0,1\}^{\mathbb{N}})$ of A (see D.17). □

Let X be a set, and let $\mathscr{F}$ be a family of subsets of X. A subset A of $\mathscr{N} \times X$ is *universal* for $\mathscr{F}$ if the collection of sections $\{A_{\mathbf{n}} : \mathbf{n} \in \mathscr{N}\}$ is equal to $\mathscr{F}$.

Our goal now is to show that if X is Polish, then there is an analytic subset of $\mathscr{N} \times X$ that is universal for the class of analytic subsets of X. We will use such a universal set to construct an analytic set that is not a Borel set.

Lemma 8.2.15. *Let X be a separable metrizable space. Then there is an open subset of $\mathscr{N} \times X$ that is universal for the collection of open subsets of X and a closed subset of $\mathscr{N} \times X$ that is universal for the collection of closed subsets of X.*

Proof. Let $\mathscr{U}$ be a countable base for X, and let $\{U_n\}$ be an infinite sequence whose terms are the sets in $\mathscr{U}$, together with the empty set (the sequence may have repeated terms). Define a subset W of $\mathscr{N} \times X$ by

$$W = \{(\mathbf{n}, x) : x \in U_{n_k} \text{ for some } k\}$$

(recall that $\mathbf{n}$ is an abbreviation for $\{n_k\}$). For each n and each k the set $W(k,n)$ defined by

$$W(k,n) = \{\mathbf{n} \in \mathscr{N} : n_k = n\} \times U_n$$

is open, and so W, since it is equal to $\cup_k \cup_n W(k,n)$, is also open. For each $\mathbf{n}$ in $\mathscr{N}$ the section $W_{\mathbf{n}}$ is given by

$$W_{\mathbf{n}} = \bigcup_k U_{n_k};$$

hence W is universal for the collection of open subsets of X (recall the definition of the sequence $\{U_n\}$).

The complement of W is a closed subset of $\mathscr{N} \times X$ and is universal for the class of closed subsets of X. □

Proposition 8.2.16. *Let X be a Polish space. Then there is an analytic subset of $\mathscr{N} \times X$ that is universal for the collection of analytic subsets of X.*

Proof. Use Lemma 8.2.15, applied to the space $\mathscr{N} \times X$, to choose a closed subset F of $\mathscr{N} \times \mathscr{N} \times X$ that is universal for the collection of closed subsets of $\mathscr{N} \times X$. Let A be the image of F under the map $(\mathbf{m}, \mathbf{n}, x) \mapsto (\mathbf{m}, x)$. Then A is analytic, and it is easy to check that for each $\mathbf{m}$ in $\mathscr{N}$ the section $A_{\mathbf{m}}$ is the projection on X of the corresponding section $F_{\mathbf{m}}$ of F. Since F is universal for the collection of closed subsets of $\mathscr{N} \times X$, Proposition 8.2.9 implies that the analytic subsets of X are exactly the projections on X of the sections $F_{\mathbf{m}}$. Thus A is universal for the collection of analytic subsets of X. □

Corollary 8.2.17. *There is an analytic subset of $\mathscr{N}$ that is not a Borel set.*

Proof. According to Proposition 8.2.16, there is an analytic subset A of $\mathscr{N} \times \mathscr{N}$ that is universal for the collection of analytic subsets of $\mathscr{N}$. Let $S = \{\mathbf{n} \in \mathscr{N} : (\mathbf{n}, \mathbf{n}) \in A\}$. Then S is analytic, since it is the projection on $\mathscr{N}$ of the intersection of A with the diagonal $\{(\mathbf{m}, \mathbf{n}) \in \mathscr{N} \times \mathscr{N} : \mathbf{m} = \mathbf{n}\}$. Now suppose that S is a Borel set. Then S^c is a Borel set, and so is analytic (Proposition 8.2.3). Thus, since A is universal, there is an element $\mathbf{n}_0$ of $\mathscr{N}$ such that $S^c = A_{\mathbf{n}_0}$. Let us consider whether $\mathbf{n}_0$ belongs to S or to S^c. If $\mathbf{n}_0 \in S$, then by the definition of S we have $(\mathbf{n}_0, \mathbf{n}_0) \in A$ and so $\mathbf{n}_0 \in A_{\mathbf{n}_0} = S^c$, which is impossible. A similar argument shows that if $\mathbf{n}_0 \in S^c$, then $\mathbf{n}_0 \in S$. In either case we have a contradiction, and so we must reject the assumption that S is a Borel set. □

One can use Corollary 8.2.17 to show that each uncountable Polish space has an analytic subset that is not a Borel set; see Exercise 6.

Exercises

1. (a) Let A be an uncountable analytic subset of the Polish space X. Show that A has a subset that is homeomorphic to $\{0,1\}^{\mathbb{N}}$. (Hint: Let $f\colon \mathscr{N} \to X$ be a continuous function such that $f(\mathscr{N}) = A$. Choose a subset S of $\mathscr{N}$ such that the restriction of f to S is a bijection of S onto A (why does such a set exist?), and let S_0 consist of the points in S that are condensation points of S. Modify the proof of Proposition 8.2.7 so as to produce a continuous function $g\colon \{0,1\}^{\mathbb{N}} \to \mathscr{N}$ such that $f \circ g\colon \{0,1\}^{\mathbb{N}} \to X$ is injective.)
   (b) Conclude that each uncountable analytic subset of a Polish space has the cardinality of the continuum.
2. Let X be an uncountable Polish space. Show that the collection of analytic subsets of X and the collection of Borel subsets of X have the cardinality of the continuum. (Hint: Use Proposition 8.2.9 or 8.2.16.)
3. (a) Let X be a nonempty zero-dimensional Polish space such that each nonempty open subset of X is uncountable and not compact. Show that X is homeomorphic to $\mathscr{N}$. (Hint: Modify the proof of Proposition 8.2.7, and use Lemma 8.2.11.)
   (b) Conclude that the space $\mathscr{I}$ of irrational numbers in the interval $(0,1)$ is homeomorphic to $\mathscr{N}$.
4. Show that each nonempty Polish space is the image of $\mathscr{N}$ under a continuous open[2] map. (Hint: Modify the construction of the sets $C(n_1,\dots,n_k)$ in the proof of Proposition 8.2.7, replacing condition (a) with the requirement that each $C(n_1,\dots,n_k)$ be nonempty and open and adding the requirement that for each $n_1, \dots, n_k, n_{k+1}$ the closure of $C(n_1,\dots,n_{k+1})$ be included in $C(n_1,\dots,n_k)$.)
5. Show that if the phrase "zero-dimensional" is omitted from the statement of Proposition 8.2.10, then a much simpler proof can be given. (Hint: Use Lemma 8.2.4.)
6. Show that if X is an uncountable Polish space, then there is an analytic subset of X that is not a Borel set. (Hint: Use Proposition 8.2.13 and Corollary 8.2.17. One can avoid Proposition 8.2.13 by using Theorem 8.3.6.)
7. In this and the following two exercises, we study a generalization of the sequences $\mathscr{F}$, $\mathscr{F}_\sigma$, $\mathscr{F}_{\sigma\delta}$, ... and $\mathscr{G}$, $\mathscr{G}_\delta$, $\mathscr{G}_{\delta\sigma}$, ... introduced in Sect. 1.1. Suppose that X is a metrizable space. For each countable ordinal α we define collections $\mathscr{F}_\alpha(X)$ and $\mathscr{G}_\alpha(X)$ of subsets of X as follows. Let $\mathscr{F}_0(X)$ be the collection of all closed subsets of X, and let $\mathscr{G}_0(X)$ be the collection of all open subsets of X. Once $\mathscr{F}_\alpha(X)$ and $\mathscr{G}_\alpha(X)$ are defined, let $\mathscr{F}_{\alpha+1}(X)$ and $\mathscr{G}_{\alpha+1}(X)$ be given by[3]

---

[2] Suppose that X and Y are topological spaces. A function $f\colon X \to Y$ is *open* if for each open subset U of X the set $f(U)$ is an open subset of Y.

[3] Recall that each ordinal α can be written in a unique way in the form $\alpha = \beta + n$, where β is either zero or a limit ordinal and where n is finite. The ordinal α is called *even* if n is even and *odd* if n is odd.

$$\mathscr{F}_{\alpha+1}(X) = \begin{cases} (\mathscr{F}_\alpha(X))_\sigma & \text{if } \alpha \text{ is even,} \\ (\mathscr{F}_\alpha(X))_\delta & \text{if } \alpha \text{ is odd,} \end{cases}$$

and

$$\mathscr{G}_{\alpha+1}(X) = \begin{cases} (\mathscr{G}_\alpha(X))_\delta & \text{if } \alpha \text{ is even,} \\ (\mathscr{G}_\alpha(X))_\sigma & \text{if } \alpha \text{ is odd.} \end{cases}$$

Finally, if α is a limit ordinal, let $\mathscr{F}_\alpha(X)$ be $(\cup_{\beta<\alpha}\mathscr{F}_\beta(X))_\delta$ and let $\mathscr{G}_\alpha(X)$ be $(\cup_{\beta<\alpha}\mathscr{G}_\beta(X))_\sigma$.

(a) Show that for each α the sets that belong to $\mathscr{G}_\alpha(X)$ are exactly those whose complements belong to $\mathscr{F}_\alpha(X)$.
(b) Show that for each α and each A in $\mathscr{G}_\alpha(X)$ (or in $\mathscr{F}_\alpha(X)$) the set A^c belongs to $\mathscr{G}_{\alpha+1}(X)$ (or to $\mathscr{F}_{\alpha+1}(X)$).
(c) Show that $\mathscr{B}(X) = \cup_\alpha \mathscr{G}_\alpha(X) = \cup_\alpha \mathscr{F}_\alpha(X)$.
(d) Suppose that Y is also a metrizable space and that $f\colon X \to Y$ is continuous. Show that for each α and each A in $\mathscr{G}_\alpha(Y)$ (or in $\mathscr{F}_\alpha(Y)$) the set $f^{-1}(A)$ belongs to $\mathscr{G}_\alpha(X)$ (or to $\mathscr{F}_\alpha(X)$).

8. Suppose that X is an uncountable Polish space. We already know that the collection of Borel subsets of X has the cardinality of the continuum (see Exercise 2). Here you are not to use that result, but rather to use transfinite induction to show that each $\mathscr{G}_\alpha(X)$ has the cardinality of the continuum, that each $\mathscr{F}_\alpha(X)$ has the cardinality of the continuum, and that $\mathscr{B}(X)$ has the cardinality of the continuum.

9. (a) Show that if X is a Polish space, then for each countable ordinal α there is a set in $\mathscr{G}_\alpha(\mathscr{N} \times X)$ (or in $\mathscr{F}_\alpha(\mathscr{N} \times X)$) that is universal for $\mathscr{G}_\alpha(X)$ (or for $\mathscr{F}_\alpha(X)$). (Hint: Use transfinite induction. Lemma 8.2.15 provides a beginning. Next suppose that $\alpha > 0$. Let $\varphi\colon \mathscr{N} \to \mathscr{N}^{\mathbb{N}}$ be a continuous surjection, and let $\varphi_k(\mathbf{n})$, $k = 1, 2, \ldots$, be the components (in $\mathscr{N}$) of the element $\varphi(\mathbf{n})$ of $\mathscr{N}^{\mathbb{N}}$. If α is a limit ordinal, let $\{\alpha_k\}$ be an enumeration of the ordinals less than α; otherwise, let $\{\alpha_k\}$ be the sequence each of whose terms is the immediate predecessor of α. For each k choose a set A_k in $\mathscr{G}_{\alpha_k}(\mathscr{N} \times X)$ that is universal for $\mathscr{G}_{\alpha_k}(X)$; then define sets $B_1, B_2, \ldots$ by

$$B_k = \{(\mathbf{n},x) \in \mathscr{N} \times X : (\varphi_k(\mathbf{n}),x) \in A_k\}.$$

Show that the set B defined by

$$B = \begin{cases} \bigcup_k B_k & \text{if } \alpha \text{ is even,} \\ \bigcap_k B_k & \text{if } \alpha \text{ is odd,} \end{cases}$$

belongs to $\mathscr{G}_\alpha(\mathscr{N} \times X)$ and is universal for $\mathscr{G}_\alpha(X)$. Finally, use part (a) of Exercise 7.)
(b) Show that there is no set in $\mathscr{B}(\mathscr{N} \times \mathscr{N})$ that is universal for $\mathscr{B}(\mathscr{N})$. (Hint: Modify the proof of Corollary 8.2.17.)

(c) Suppose that X is an uncountable Polish space and that there is a bijection $F: \mathscr{N} \to X$ such that both F and F^{-1} are Borel measurable (such a bijection always exists; see Theorem 8.3.6). Show that there is no set in $\mathscr{B}(\mathscr{N} \times X)$ that is universal for $\mathscr{B}(X)$. Also show that no two of the sets $\mathscr{G}_0(X)$, $\mathscr{F}_0(X)$, $\dots$, $\mathscr{G}_\alpha(X)$, $\mathscr{F}_\alpha(X)$, $\dots$, $\mathscr{B}(X)$ are equal.

10. Let X be a Polish space, and let Y be a metrizable space. Show that if $A \in \mathscr{B}(X)$ and if $f: A \to Y$ is Borel measurable, then $f(A)$ is separable. (Hint: Let d be a metric for Y, and suppose that $f(A)$ is not separable. Choose a positive number ε and an uncountable subset C of $f(A)$ such that $d(x,y) \geq \varepsilon$ holds for each pair x, y of points in C; then choose a function $g: C \to A$ such that $y = f(g(y))$ holds for each y in C (check that C and g exist). Show that each subset of $g(C)$ is analytic, and then use Exercises 1 and 2 to derive a contradiction.)

8.3 The Separation Theorem and Its Consequences

This section is devoted to a fundamental technical fact about analytic sets (Theorem 8.3.1) and to some of its applications. The reader should take particular note of Theorems 8.3.6 and 8.3.7.

Let X be a Polish space, and let A_1 and A_2 be disjoint subsets of X. Then A_1 and A_2 can be *separated by Borel sets* if there are disjoint Borel subsets B_1 and B_2 of X such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$.

Theorem 8.3.1. *Let X be a Polish space, and let A_1 and A_2 be disjoint analytic subsets of X. Then A_1 and A_2 can be separated by Borel sets.*

Proof. Let us begin by showing that

(a) if C_1, C_2, $\dots$, and D are subsets of X such that for each n the sets C_n and D can be separated by Borel sets, then $\cup_n C_n$ and D can be separated by Borel sets, and
(b) if E_1, E_2, $\dots$, and F_1, F_2, $\dots$ are subsets of X such that for each m and n the sets E_m and F_n can be separated by Borel sets, then $\cup_m E_m$ and $\cup_n F_n$ can be separated by Borel sets.

First consider assertion (a). For each n choose disjoint Borel sets G_n and H_n such that $C_n \subseteq G_n$ and $D \subseteq H_n$. Then $\cup_n G_n$ and $\cap_n H_n$ are disjoint Borel sets that include $\cup_n C_n$ and D, respectively. Hence assertion (a) is proved.

Next consider assertion (b). Assertion (a) implies that for each m the sets E_m and $\cup_n F_n$ can be separated by Borel sets. Another application of assertion (a) now implies that $\cup_m E_m$ and $\cup_n F_n$ can be separated by Borel sets.

We turn to the proof of the theorem itself. So suppose that A_1 and A_2 are disjoint analytic subsets of X. Since the empty set can clearly be separated from an arbitrary subset of X by Borel sets, we can assume that neither A_1 nor A_2 is empty. Thus there are continuous functions $f, g: \mathscr{N} \to X$ such that $f(\mathscr{N}) = A_1$ and $g(\mathscr{N}) = A_2$ (Corollary 8.2.8). Suppose that A_1 and A_2 cannot be separated by Borel sets; we will derive a contradiction.

Recall (see Example 8.1.6(b)) that for positive integers k and $n_1, \ldots, n_k$ the set $\mathcal{N}(n_1,\ldots,n_k)$ is defined by

$$\mathcal{N}(n_1,\ldots,n_k) = \{\mathbf{m} \in \mathcal{N} : m_i = n_i \text{ for } i = 1, \ldots, k\}.$$

Since $A_1 = \cup_{m_1} f(\mathcal{N}(m_1))$ and $A_2 = \cup_{n_1} g(\mathcal{N}(n_1))$, assertion (b) above implies that there are positive integers m_1 and n_1 such that $f(\mathcal{N}(m_1))$ and $g(\mathcal{N}(n_1))$ cannot be separated by Borel sets. Likewise, since $f(\mathcal{N}(m_1)) = \cup_{m_2} f(\mathcal{N}(m_1,m_2))$ and $g(\mathcal{N}(n_1)) = \cup_{n_2} g(\mathcal{N}(n_1,n_2))$, there are positive integers m_2 and n_2 such that $f(\mathcal{N}(m_1,m_2))$ and $g(\mathcal{N}(n_1,n_2))$ cannot be separated by Borel sets. By continuing in this manner we can construct sequences $\mathbf{m} = \{m_i\}$ and $\mathbf{n} = \{n_i\}$ such that for each k the sets $f(\mathcal{N}(m_1,\ldots,m_k))$ and $g(\mathcal{N}(n_1,\ldots,n_k))$ cannot be separated by Borel sets. The points $f(\mathbf{m})$ and $g(\mathbf{n})$ must be equal, since otherwise they could be separated with open sets, which, by the continuity of f and g, would separate $f(\mathcal{N}(m_1,\ldots,m_k))$ and $g(\mathcal{N}(n_1,\ldots,n_k))$ for all large k. However, since $f(\mathbf{m}) \in A_1$ and $g(\mathbf{n}) \in A_2$, the equality of $f(\mathbf{m})$ and $g(\mathbf{n})$ contradicts the disjointness of A_1 and A_2. So we must conclude that A_1 and A_2 can be separated with Borel sets, and with this the proof is complete. □

Corollary 8.3.2. *Let X be a Polish space, and let A_1, A_2, ... be disjoint analytic subsets of X. Then there are disjoint Borel subsets B_1, B_2, ... of X such that $A_n \subseteq B_n$ holds for each n.*

Proof. For each positive integer n the set $\cup_{m\neq n} A_m$ is analytic, and so we can use Theorem 8.3.1 and the disjointness of A_n and $\cup_{m\neq n} A_m$ to choose a Borel set C_n such that $A_n \subseteq C_n$ and $\cup_{m\neq n} A_m \subseteq C_n^c$. Now define the Borel sets B_1, B_2, ... by letting B_n be equal to $C_n - (\cup_{m\neq n} C_m)$. □

Corollary 8.3.3. *Let X be a Polish space, and let A be a subset of X. If both A and A^c are analytic, then A is Borel.*

Proof. According to Theorem 8.3.1 there are disjoint Borel subsets B_1 and B_2 of X such that $A \subseteq B_1$ and $A^c \subseteq B_2$. It follows immediately that $A = B_1$ and $A^c = B_2$, and hence that A is Borel. □

Proposition 8.3.4. *Let X and Y be Polish spaces, let A be a Borel subset of X, and let f be a function from A to Y. Then f is Borel measurable if and only if its graph is a Borel subset of $X \times Y$.*

Proof. Proposition 8.1.8 implies that if f is Borel measurable, then $\mathrm{gr}(f)$ is a Borel subset of $A \times Y$ and hence (Lemma 7.2.2) of $X \times Y$. Now consider the converse. Suppose that $\mathrm{gr}(f)$ is a Borel subset of $X \times Y$ and that B is a Borel subset of Y. Then $\mathrm{gr}(f) \cap (X \times B)$ and $\mathrm{gr}(f) \cap (X \times B^c)$ are Borel, and hence analytic, subsets of $X \times Y$. Thus the projections of these sets on X are analytic. But these projections are $f^{-1}(B)$ and $f^{-1}(B^c)$, respectively. Furthermore the sets $f^{-1}(B)$ and $f^{-1}(B^c)$ are disjoint, and so, by Theorem 8.3.1, there are Borel sets B_1 and B_2 that separate them. It is easy to check that $f^{-1}(B)$ is equal to $A \cap B_1$ and so is a Borel set. Since B was an arbitrary Borel subset of Y, the measurability of f follows. □

Proposition 8.3.5. *Let X and Y be Polish spaces, let A be a Borel subset of X, let $f: A \to Y$ be Borel measurable, and let $B = f(A)$. If f is injective and if*[4] *$B \in \mathscr{B}(Y)$, then f^{-1} is Borel measurable.*

Proof. Note that $\text{gr}(f^{-1})$ is the image of $\text{gr}(f)$ under the homeomorphism $(x,y) \mapsto (y,x)$ of $X \times Y$ onto $Y \times X$; hence $\text{gr}(f^{-1})$ is a Borel subset of $Y \times X$ if and only if $\text{gr}(f)$ is a Borel subset of $X \times Y$. Now apply Proposition 8.3.4 twice, once to conclude that $\text{gr}(f)$ is a Borel subset of $X \times Y$ and once to conclude that f^{-1} is Borel measurable. □

Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces. A bijection $f: X \to Y$ is an *isomorphism* if f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}$ and f^{-1} is measurable with respect to $\mathscr{B}$ and $\mathscr{A}$. Equivalently, the bijection f is an isomorphism if the subsets A of X that belong to $\mathscr{A}$ are exactly those for which $f(A)$ belongs to $\mathscr{B}$. The spaces $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are *isomorphic* if there exists such an isomorphism. We will also call subsets X_0 and Y_0 of X and Y *isomorphic* if the spaces[5] $(X_0, \mathscr{A}_{X_0})$ and $(Y_0, \mathscr{B}_{Y_0})$ are isomorphic. In case $(X, \mathscr{B}(X))$ and $(Y, \mathscr{B}(Y))$ are Polish spaces, together with their Borel σ-algebras, we will often use the term *Borel isomorphism* instead of isomorphism.

The concept of a Borel isomorphism is a natural one; it is especially important because of the following easy-to-state but nontrivial result.[6]

Theorem 8.3.6. *Let A and B be Borel subsets of Polish spaces. Then A and B are Borel isomorphic if and only if they have the same cardinality. Furthermore, the cardinality of each uncountable Borel subset of a Polish space is that of the continuum.*

Proof. If A and B are isomorphic, then they certainly have the same cardinality. We turn to the converse.

Suppose that A and B have the same cardinality. If these sets are finite or countably infinite, then each of their subsets is a Borel set, and each bijection between them is an isomorphism; hence A and B are isomorphic.

Now suppose that A and B are uncountable. Note that we are simply assuming that A and B are uncountable; we are not *assuming* that they have the same cardinality. Proposition 8.2.13 says that there are continuous injective maps $f: \mathscr{N} \to A$ and $g: \mathscr{N} \to B$ such that $A - f(\mathscr{N})$ and $B - g(\mathscr{N})$ are at most countably infinite. Since they are countable, the sets $A - f(\mathscr{N})$ and $B - g(\mathscr{N})$ are Borel sets; thus $f(\mathscr{N})$ and $g(\mathscr{N})$ are also Borel sets, and (see Proposition 8.3.5) f and g are Borel isomorphisms of $\mathscr{N}$ onto $f(\mathscr{N})$ and $g(\mathscr{N})$, respectively. Thus $g \circ f^{-1}$ is a Borel isomorphism of $f(\mathscr{N})$ onto $g(\mathscr{N})$. Now let I be a countably infinite subset of $f(\mathscr{N})$, and let h be a bijection of the countably infinite set $I \cup (A - f(\mathscr{N}))$ onto

[4]We will see (Theorem 8.3.7) that the injectivity and measurability of f imply that $B \in \mathscr{B}(Y)$.

[5]Of course $\mathscr{A}_{X_0}$ and $\mathscr{B}_{Y_0}$ are the traces of $\mathscr{A}$ and $\mathscr{B}$ on X_0 and Y_0 (see Exercise 1.5.11).

[6]See Exercise 5 for a proof of Theorem 8.3.6 that does not depend on Proposition 8.2.13 or 8.3.5.

the countably infinite set $g(f^{-1}(I)) \cup (B - g(\mathscr{N}))$. It is easy to check that the map that agrees with $g \circ f^{-1}$ on $f(\mathscr{N}) - I$ and with h on $I \cup (A - f(\mathscr{N}))$ is a Borel isomorphism of A onto B.

In particular, each uncountable Borel subset of a Polish space is Borel isomorphic to $\mathbb{R}$, and so has the cardinality of the continuum. □

It follows from Theorem 8.3.6 that a Borel subset of a Polish space is Borel isomorphic to $\mathbb{R}$, to the set $\mathbb{N}$ of all positive integers, to the set $\{1, 2, \ldots, n\}$ for some positive integer n, or to $\varnothing$.

We now show that the hypothesis that $f(A)$ belongs to $\mathscr{B}(Y)$ can be removed from Proposition 8.3.5.

Theorem 8.3.7. *Let X and Y be Polish spaces, let A be a Borel subset of X, and let $f: A \to Y$ be Borel measurable and injective. Then $f(A)$ is a Borel subset of Y.*

The proof of this result will depend on the following lemma.

Lemma 8.3.8. *Let X and Y be Polish spaces, let A be a nonempty Borel subset of X, and let $f: A \to Y$ be Borel measurable and injective. Then there is a Borel measurable function $g: Y \to X$ such that $g(Y) \subseteq A$ and such that $g(f(x)) = x$ holds at each x in A.*

Proof. Let d be a metric for X, and let $\bar{x}$ be an element of A (we will hold $\bar{x}$ fixed throughout this proof). For each positive integer n we define a function $g_n: Y \to X$ as follows. Choose a finite or countably infinite partition $\{A_{n,k}\}_k$ of A into nonempty Borel subsets of diameter at most $1/n$, and in each $A_{n,k}$ choose a point $x_{n,k}$. The sets $f(A_{n,k})$, $k = 1, 2, \ldots$, are disjoint and analytic (Proposition 8.2.6), and so we can choose disjoint Borel sets $B_{n,k}$, $k = 1, 2, \ldots$, such that $f(A_{n,k}) \subseteq B_{n,k}$ holds for each k (Corollary 8.3.2). Now define $g_n: Y \to X$ by letting $g_n(y) = x_{n,k}$ if $y \in B_{n,k}$ and letting $g_n(y) = \bar{x}$ if $y \notin (\cup_k B_{n,k})$. It is easy to check that each g_n is Borel measurable. Define $g: Y \to A$ by letting $g(y) = \lim_n g_n(y)$ if the limit exists and belongs to A and letting $g(y) = \bar{x}$ otherwise. Proposition 8.1.11 implies that g is Borel measurable. If $x \in A$, then $d(x, g_n(f(x))) \leq 1/n$ holds for each n, and so $g(f(x)) = x$. Thus g is the required function. □

Proof of Theorem 8.3.7. We can certainly assume that A is not empty. According to Lemma 8.3.8 there is a Borel measurable function $g: Y \to X$ such that $g(Y) \subseteq A$ and such that $g(f(x)) = x$ holds at each x in A. It is easy to check that

$$f(A) = \{y \in Y : f(g(y)) = y\}.$$

Thus Exercise 8.1.3, applied to the functions $y \mapsto f(g(y))$ and $y \mapsto y$, implies that $f(A) \in \mathscr{B}(Y)$. □

Exercises

1. Let X and Y be Polish spaces, and let $f: X \to Y$ be a function whose graph is an analytic subset of $X \times Y$. Show that f is Borel measurable.
2. Let X and Y be uncountable Polish spaces. Show that the cardinality of the collection of Borel measurable functions from X to Y is that of the continuum.
3. Show that there is a Lebesgue measurable function $f: \mathbb{R} \to \mathbb{R}$ such that no real-valued (as opposed to $[-\infty, +\infty]$-valued) Borel measurable function f_1 satisfies $f(x) \le f_1(x)$ at each x in $\mathbb{R}$. Thus the $[-\infty, +\infty]$-valued functions f_0 and f_1 in Proposition 2.2.5 cannot necessarily be replaced with real-valued functions, even if the function f is real-valued. (Hint: Let K be the Cantor set. According to the preceding exercise, we can choose a bijection $x \mapsto g_x$ of K onto the set of real-valued Borel functions on K. Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} g_x(x) + 1 & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases}$$

and check that f meets the requirements above.)
4. Let X be a Polish space, let μ be a Borel measure on X such that $\mu(X) = 1$, and let λ be Lebesgue measure on the Borel subsets of $[0,1]$. Show that there is a Borel measurable function $f: [0,1] \to X$ such that $\mu = \lambda f^{-1}$. (Hint: This is easy if X is finite or countably infinite. Otherwise use Theorem 8.3.6, together with either Exercise 2.6.6 or Proposition 10.1.15.)
5. Give an alternate proof of the isomorphism theorem for Borel sets (Theorem 8.3.6) by supplying the details missing from the following outline. (This proof depends neither on the separation theorem and its consequences nor on Proposition 8.2.13.)

   (a) Show that every Borel subset of a Polish space is Borel isomorphic to a Borel subset of $\{0,1\}^{\mathbb{N}}$. (Hint: Begin by showing that the interval $[0,1]$ is Borel isomorphic to a Borel subset of $\{0,1\}^{\mathbb{N}}$ (consider binary expansions). From this conclude that $[0,1]^{\mathbb{N}}$ is Borel isomorphic to a Borel subset of $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ and hence to a Borel subset of $\{0,1\}^{\mathbb{N}}$. Finally, use Exercise 8.1.12.)
   (b) Show that each uncountable Borel subset of a Polish space has a Borel subset that is Borel isomorphic to $\{0,1\}^{\mathbb{N}}$. (Hint: Use Corollary 8.2.14 or, to avoid Proposition 8.2.13, Exercise 8.2.1.)
   (c) (A version of the Schröder–Bernstein theorem for Borel sets—see item A.7 in Appendix A.) Suppose that X and Y are Polish spaces, that A and B are Borel subsets of X and Y, respectively, that A is Borel isomorphic to a Borel subset of B, and that B is Borel isomorphic to a Borel subset of A. Show that A and B are Borel isomorphic to one another. (Hint: Let f and g be Borel isomorphisms of A and B onto Borel subsets of B and A, respectively. Define sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ inductively by $A_0 = A$, $B_0 = B$,

$A_{n+1} = g(B_n)$, and $B_{n+1} = f(A_n)$. Show that

$$h(x) = \begin{cases} f(x) & \text{if } x \in \cap_0^\infty A_n \text{ or } x \in \cup_0^\infty (A_{2n} - A_{2n+1}), \\ g^{-1}(x) & \text{if } x \in \cup_0^\infty (A_{2n+1} - A_{2n+2}) \end{cases}$$

gives a Borel isomorphism $h\colon A \to B$. See the proof of Proposition G.2 in Appendix G for a more detailed description of the construction of the function h.)

8.4 The Measurability of Analytic Sets

Let $(X, \mathscr{A})$ be a measurable space, and let μ be a measure on $(X, \mathscr{A})$. Recall that in Sect. 1.5 we defined the completion of $\mathscr{A}$ under μ to be the collection $\mathscr{A}_\mu$ of subsets A of X for which there are sets E and F that belong to $\mathscr{A}$ and satisfy the relations $E \subseteq A \subseteq F$ and $\mu(F - E) = 0$. The sets in $\mathscr{A}_\mu$ are often called *μ-measurable*.

We also defined the outer measure $\mu^*(A)$ and the inner measure $\mu_*(A)$ of an arbitrary subset A of X by

$$\mu^*(A) = \inf\{\mu(B) : A \subseteq B \text{ and } B \in \mathscr{A}\} \tag{1}$$

and

$$\mu_*(A) = \sup\{\mu(B) : B \subseteq A \text{ and } B \in \mathscr{A}\}. \tag{2}$$

We saw that a set A such that $\mu^*(A) < +\infty$ belongs to $\mathscr{A}_\mu$ if and only if $\mu_*(A) = \mu^*(A)$, that $\mathscr{A}_\mu$ is a σ-algebra on X, and that the restriction of μ^* (or of μ_*) to $\mathscr{A}_\mu$ is a measure on $\mathscr{A}_\mu$, which is called the completion of μ and is denoted by $\overline{\mu}$. It is easy to see that $\overline{\mu}$ is the only measure on $\mathscr{A}_\mu$ that agrees on $\mathscr{A}$ with μ.

We can now state the main result of this section.

Theorem 8.4.1. *Let X be a Polish space, and let μ be a finite Borel measure on X. Then every analytic subset of X is μ-measurable.*

For the proof we need the following lemma.

Lemma 8.4.2. *Let $(X, \mathscr{A})$ be a measurable space, let μ be a finite measure on $(X, \mathscr{A})$, and let μ^* be defined by Eq. (1). If $\{A_n\}$ is an increasing sequence of subsets of X, then*

$$\mu^*\Big(\bigcup_n A_n\Big) = \lim_n \mu^*(A_n).$$

Proof. The monotonicity of μ^* implies that the limit $\lim_n \mu^*(A_n)$ exists and satisfies $\lim_n \mu^*(A_n) \le \mu^*(\cup_n A_n)$. We need to verify the reverse inequality. Let ε be a positive number, and for each positive integer n use (1) to choose a set B_n that belongs to $\mathscr{A}$, includes A_n, and satisfies $\mu(B_n) \le \mu^*(A_n) + \varepsilon$. By replacing B_n

with $\cap_{j=n}^{\infty} B_j$, we can assume that the sequence $\{B_n\}$ is increasing. Proposition 1.2.5 implies that $\mu(\cup_n B_n) = \lim_n \mu(B_n)$, and so we have

$$\mu^*\Big(\bigcup_n A_n\Big) \leq \mu\Big(\bigcup_n B_n\Big) = \lim_n \mu(B_n) \leq \lim_n \mu^*(A_n) + \varepsilon.$$

Since ε is arbitrary, the proof of the lemma is complete. □

Proof of Theorem 8.4.1. Let A be an analytic subset of X. We will show that A is μ-measurable by showing that $\mu_*(A) = \mu^*(A)$, and we will do this by producing, for an arbitrary positive ε, a compact subset K of A such that $\mu(K) \geq \mu^*(A) - \varepsilon$.

We can certainly assume that A is nonempty. Thus we can choose a continuous function $f\colon \mathscr{N} \to X$ such that $f(\mathscr{N}) = A$ (Corollary 8.2.8). We need some notation. For positive integers k and $n_1, \ldots, n_k$ let $\mathscr{L}(n_1, \ldots, n_k)$ be the set of those elements $\mathbf{m}$ of $\mathscr{N}$ that satisfy $m_i \leq n_i$ for $i = 1, \ldots, k$. We will construct an element $\mathbf{n} = \{n_i\}$ of $\mathscr{N}$ such that

$$\mu^*(f(\mathscr{L}(n_1, \ldots, n_k))) > \mu^*(A) - \varepsilon \tag{3}$$

holds for each k. We begin by choosing the first term n_1 of the sequence $\mathbf{n}$. Note that $\{\mathscr{L}(n_1)\}_{n_1=1}^{\infty}$ is an increasing sequence of sets whose union is $\mathscr{N}$, and so $\{f(\mathscr{L}(n_1))\}_{n_1=1}^{\infty}$ is an increasing sequence of sets whose union is A. Thus $\mu^*(A) = \lim_{n_1} \mu^*(f(\mathscr{L}(n_1)))$ (Lemma 8.4.2), and so we can pick a positive integer n_1 such that $\mu^*(f(\mathscr{L}(n_1))) > \mu^*(A) - \varepsilon$. Since $\mathscr{L}(n_1) = \cup_{n_2} \mathscr{L}(n_1, n_2)$, a similar argument produces a positive integer n_2 such that $\mu^*(f(\mathscr{L}(n_1, n_2))) > \mu^*(A) - \varepsilon$. Continuing in this way we obtain a sequence $\mathbf{n} = \{n_k\}$ of positive integers such that (3) holds for each k. Now let $L = \cap_k \mathscr{L}(n_1, \ldots, n_k)$. Then L is equal to

$$\{\mathbf{m} \in \mathscr{N} : m_i \leq n_i \text{ for each } i\}$$

and so is compact (see D.20 or D.42); it follows that the set K defined by $K = f(L)$ is a compact subset of A. We will show that $\mu(K) \geq \mu^*(A) - \varepsilon$.

Let us begin by showing that

$$K = \bigcap_k f(\mathscr{L}_k)^-, \tag{4}$$

where for each k we have abbreviated $\mathscr{L}(n_1, \ldots, n_k)$ by $\mathscr{L}_k$. Since it is clear that $K \subseteq \cap_k f(\mathscr{L}_k)^-$, we turn to the reverse inclusion. Let d be a metric for the topology of X. Suppose that x is a member of $\cap_k f(\mathscr{L}_k)^-$. For each k we can choose an element $\mathbf{m}_k$ of $\mathscr{L}_k$ such that $d(f(\mathbf{m}_k), x) \leq 1/k$. Note that for each i the ith components of the terms of $\{\mathbf{m}_k\}$ form a bounded subset of $\mathbb{N}$; hence the terms of $\{\mathbf{m}_k\}$ form a relatively compact[7] subset of $\mathscr{N}$, and we can choose a convergent subsequence of $\{\mathbf{m}_k\}$. Let $\mathbf{m}$ be the limit of this subsequence. It is easy to check that $\mathbf{m} \in \cap_k \mathscr{L}_k$ and that $f(\mathbf{m}) = x$. Hence $\cap_k f(\mathscr{L}_k)^- \subseteq K$, and (4) is proved.

[7] A subset of a Hausdorff space is *relatively compact* if its closure is compact.

For each k the set $f(\mathscr{L}_k)^-$ is closed and includes $f(\mathscr{L}_k)$; hence (see (3))

$$\mu(f(\mathscr{L}_k)^-) \geq \mu^*(f(\mathscr{L}_k)) > \mu^*(A) - \varepsilon. \tag{5}$$

Furthermore the sequence $\{f(\mathscr{L}_k)^-\}$ is decreasing, and so (4), (5), and Proposition 1.2.5 imply that

$$\mu(K) = \lim_k \mu(f(\mathscr{L}_k)^-) \geq \mu^*(A) - \varepsilon. \tag{6}$$

Thus $\mu_*(A) \geq \mu^*(A) - \varepsilon$ and so, since ε was arbitrary, the proof is complete. □

Let $(X, \mathscr{A})$ be a measurable space. A subset of X is *universally measurable* (with respect to $(X, \mathscr{A})$) if it is μ-measurable for every finite measure μ on $(X, \mathscr{A})$. Let $\mathscr{A}_*$ be the family of all universally measurable subsets of X. Then $\mathscr{A}_* = \cap_\mu \mathscr{A}_\mu$, where μ ranges over the family of finite measures on $(X, \mathscr{A})$; hence $\mathscr{A}_*$ is a σ-algebra. It is easy to check that for each finite measure μ on $(X, \mathscr{A})$ there is a unique measure on $(X, \mathscr{A}_*)$ that agrees on $\mathscr{A}$ with μ.

Now assume that X is a Polish space. The *universally measurable* subsets of X are those that are universally measurable with respect to $(X, \mathscr{B}(X))$.

Theorem 8.4.1 can now be reformulated as follows.

Corollary 8.4.3. *Every analytic subset of a Polish space is universally measurable.*

Proof. This corollary is simply a restatement of Theorem 8.4.1. □

Let X be an uncountable Polish space. Corollary 8.4.3 implies that the σ-algebra of universally measurable subsets of X includes the σ-algebra generated by the analytic subsets of X. These σ-algebras contain the complements of the analytic sets, and so contain some nonanalytic sets; thus the collection of universally measurable subsets of X is larger than the collection of analytic subsets of X, which in turn is larger than $\mathscr{B}(X)$.

Suppose that X and Y are Polish spaces. Note that if C is a Borel (or even analytic) subset of $X \times Y$, then the projection of C on X is analytic and so is universally measurable. This fact has the following useful generalization, in which the space X is not required to be Polish.

Proposition 8.4.4. *Let $(X, \mathscr{A})$ be a measurable space, let Y be a Polish space, and let C be a subset of $X \times Y$ that belongs to the product σ-algebra $\mathscr{A} \times \mathscr{B}(Y)$. Then the projection of C on X is universally measurable with respect to $(X, \mathscr{A})$.*

The proof depends on the following two lemmas; they will allow us to replace X with a suitable Polish space.

Lemma 8.4.5. *Let $(X, \mathscr{A})$, Y, and C be as in Proposition 8.4.4, and let $Z = \{0,1\}^{\mathbb{N}}$. Then there exist a function $h\colon X \to Z$ and a subset D of $Z \times Y$ such that*

(a) *h is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Z)$,*
(b) *$D \in \mathscr{B}(Z \times Y)$, and*
(c) *$C = H^{-1}(D)$, where $H\colon X \times Y \to Z \times Y$ is the map that takes (x,y) to $(h(x),y)$.*

Proof. Recall that $\mathscr{A} \times \mathscr{B}(Y)$ is generated by the family of all rectangles $A \times B$ such that $A \in \mathscr{A}$ and $B \in \mathscr{B}(Y)$. It follows from Exercise 1.1.7 that this family of rectangles has a countable subfamily $\mathscr{C}$ such that $C \in \sigma(\mathscr{C})$. Let $A_1 \times B_1$, $A_2 \times B_2$, ... be the rectangles belonging to $\mathscr{C}$, and define $h\colon X \to Z$ by letting $h(x)$ be the sequence $\{\chi_{A_n}(x)\}$. Since the subsets $E_1, E_2, \ldots$ of Z defined by

$$E_k = \{\{n_i\} \in Z : n_k = 1\}$$

generate $\mathscr{B}(Z)$ (see Proposition 8.1.7) and since $h^{-1}(E_k) = A_k$ holds for each k, Proposition 2.6.2 implies that h is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Z)$. Define $H\colon X \times Y \to Z \times Y$ by $H(x,y) = (h(x), y)$, and let

$$\mathscr{F} = \{H^{-1}(D) : D \in \mathscr{B}(Z \times Y)\}.$$

Then $\mathscr{F}$ is a σ-algebra on $X \times Y$ that contains each $A_i \times B_i$. Hence $\sigma(\mathscr{C}) \subseteq \mathscr{F}$, and so $C \in \mathscr{F}$. With this the lemma is proved. □

Lemma 8.4.6. *Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces, and let $f\colon X \to Y$ be measurable with respect to $\mathscr{A}$ and $\mathscr{B}$. Then f is measurable with respect to the σ-algebras $\mathscr{A}_*$ and $\mathscr{B}_*$ of universally measurable sets.*

Proof. Suppose that $B_* \in \mathscr{B}_*$. We need to show that $f^{-1}(B_*) \in \mathscr{A}_*$. Let μ be a finite measure on $\mathscr{A}$. Recall that μf^{-1} is the measure on $\mathscr{B}$ defined by $\mu f^{-1}(B) = \mu(f^{-1}(B))$. Since B_* belongs to $\mathscr{B}_*$, it belongs to $\mathscr{B}_{\mu f^{-1}}$, and so there are sets B_0 and B_1 in $\mathscr{B}$ that satisfy $B_0 \subseteq B_* \subseteq B_1$ and $\mu f^{-1}(B_1 - B_0) = 0$. Then the sets $f^{-1}(B_0)$ and $f^{-1}(B_1)$ belong to $\mathscr{A}$ and satisfy $f^{-1}(B_0) \subseteq f^{-1}(B_*) \subseteq f^{-1}(B_1)$ and $\mu(f^{-1}(B_1) - f^{-1}(B_0)) = 0$. Hence $f^{-1}(B_*) \in \mathscr{A}_\mu$. Since μ was arbitrary, we conclude that $f^{-1}(B_*)$ belongs to $\mathscr{A}_*$, and the proof is complete. □

Proof of Proposition 8.4.4. Let $(X, \mathscr{A})$, Y, and C be as in the statement of Proposition 8.4.4, and construct h, H, and D as in Lemma 8.4.5. Let π_X be the projection of $X \times Y$ onto X, and let π_Z be the projection of $Z \times Y$ onto Z. Corollary 8.4.3 implies that $\pi_Z(D)$ is a universally measurable subset of Z, and so Lemma 8.4.6 implies that $h^{-1}(\pi_Z(D))$ is a universally measurable subset of X. Thus, in view of the easily verified relation

$$\pi_X(C) = \pi_X(H^{-1}(D)) = h^{-1}(\pi_Z(D)),$$

$\pi_X(C)$ is universally measurable. □

Exercises

1. Let $(X, \mathscr{A})$ be a measurable space.

   (a) Show that a function $f\colon X \to [-\infty, +\infty]$ is $\mathscr{A}_*$-measurable if and only if for each finite measure μ on $(X, \mathscr{A})$ there are $\mathscr{A}$-measurable functions

$f_0, f_1 \colon X \to [-\infty, +\infty]$ that satisfy $f_0 \le f \le f_1$ everywhere on X and are equal to one another μ-almost everywhere on X. (Hint: See Proposition 2.2.5.)

(b) Show that if $f \colon X \to [-\infty, +\infty]$ is $\mathscr{A}_*$-measurable and if the functions f_0 and f_1 in part (a) can be chosen independently of μ, then f is $\mathscr{A}$-measurable.

2. Let $(X, \mathscr{A})$ be a measurable space.

(a) Show that $(\mathscr{A}_*)_* = \mathscr{A}_*$.

(b) Show that if μ is a finite measure on $(X, \mathscr{A})$, then $(\mathscr{A}_\mu)_* = \mathscr{A}_\mu$.

3. Show that there is a Lebesgue measurable subset of $\mathbb{R}$ that is not universally measurable.
4. Show that each uncountable Polish space has a subset that is not universally measurable. (Hint: Use Theorem 8.3.6.)
5. Show by example that Lemma 8.4.2 would not be valid if μ^* were allowed to be an arbitrary outer measure on X.
6. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces, and let K be a kernel from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$ such that $\sup\{K(x, Y) : x \in X\}$ is finite (see Exercise 2.4.7). For each x in X let $B \mapsto \overline{K}(x, B)$ be the restriction to $\mathscr{B}_*$ of the completion of the measure $B \mapsto K(x, B)$. Finally, for each finite measure μ on $(X, \mathscr{A})$ let μK be the measure on $(Y, \mathscr{B})$ defined by $(\mu K)(B) = \int K(x, B)\, \mu(dx)$ (see part (a) of Exercise 2.4.7).

(a) Show that $(x, B) \mapsto \overline{K}(x, B)$ is a kernel from $(X, \mathscr{A}_*)$ to $(Y, \mathscr{B}_*)$. (Hint: Use Exercise 1. Let B belong to $\mathscr{B}_*$, and let μ be a finite measure on $(X, \mathscr{A})$. Choose sets B_0 and B_1 that belong to $\mathscr{B}$ and satisfy the conditions $B_0 \subseteq B \subseteq B_1$ and $(\mu K)(B_1 - B_0) = 0$; then consider the functions $x \mapsto K(x, B_0)$ and $x \mapsto K(x, B_1)$.)

(b) Suppose that μ is a finite measure on $(X, \mathscr{A})$ and that $\overline{\mu}$ and $\overline{\mu K}$ are the restrictions to $\mathscr{A}_*$ and $\mathscr{B}_*$ of the completions of μ and μK. Show that $\overline{\mu K} = \overline{\mu}\,\overline{K}$ (that is, show that

$$\overline{\mu K}(B) = \int \overline{K}(x, B)\, \overline{\mu}(dx)$$

holds for each B in $\mathscr{B}_*$.)

7. Let X be a Hausdorff space. A *capacity* on X is a function $I \colon \mathscr{P}(X) \to [-\infty, +\infty]$ such that

(i) if $A \subseteq B \subseteq X$, then $I(A) \le I(B)$,
(ii) each increasing sequence $\{A_n\}$ of subsets of X satisfies $I(\cup_n A_n) = \lim_n I(A_n)$, and
(iii) each decreasing sequence $\{K_n\}$ of compact subsets of X satisfies $I(\cap_n K_n) = \lim_n I(K_n)$.

A subset A of X is *I-capacitable* if

$$I(A) = \sup\{I(K) : K \subseteq A \text{ and } K \text{ is compact}\}.$$

Show that if the Hausdorff space X is Polish and if I is a capacity on X, then every relatively compact analytic subset of X is I-capacitable. (Hint: Modify the proof of Theorem 8.4.1.)

8. (a) Show that the space $\mathscr{I}$ of irrational numbers in the interval $(0,1)$ is not σ-compact.
   (b) Let X be a Polish space that is not σ-compact, and define $I\colon \mathscr{P}(X) \to [-\infty,+\infty]$ by letting $I(A)$ be 0 if A is included in some σ-compact set and letting $I(A)$ be 1 otherwise. Show that
      (i) I is a capacity on X, and
      (ii) there is an analytic subset of X that is not I-capacitable.

8.5 Cross Sections

Let X and Y be Polish spaces, let A be a Borel or analytic subset of $X \times Y$, and let A_0 be the projection of A on X. It is sometimes useful to have a measurable function from A_0 to Y whose graph is a subset of A. Of course, the axiom of choice guarantees the existence of a function from A_0 to Y whose graph is a subset of A, but it asserts nothing about the measurability of that function. We will see below, however, that the theory of analytic sets allows one to construct such a function in a way that makes it measurable with respect to the σ-algebra of universally measurable subsets of X.

One should note that this construction does not always produce a Borel measurable function. In fact, there is a Borel subset A of $[0,1] \times [0,1]$ such that

(a) the image of A under the projection $(x,y) \mapsto x$ is all of $[0,1]$, and
(b) there is no Borel function from $[0,1]$ to $[0,1]$ whose graph is a subset of A

(see Blackwell [10] or Novikoff [94]).

We will need a few more facts about $\mathscr{N}$ for our proof of Theorem 8.5.3. Let $\leq$ be lexicographic order on $\mathscr{N}$. In other words, we define a relation $<$ on $\mathscr{N}$ by declaring that $\mathbf{m} < \mathbf{n}$ holds if

(a) $\mathbf{m} \neq \mathbf{n}$ and
(b) $m_{i_0} < n_{i_0}$, where i_0 is the smallest of those positive integers i for which $m_i \neq n_i$;

then we declare that $\mathbf{m} \leq \mathbf{n}$ means that $\mathbf{m} < \mathbf{n}$ or $\mathbf{m} = \mathbf{n}$. It is easy to check that $\leq$ is a linear order on $\mathscr{N}$.

Recall also (see Example 8.1.6(b)) that $\mathscr{N}(n_1,\dots,n_k)$ is the set of all elements of $\mathscr{N}$ whose first k elements are $n_1, \dots, n_k$.

Lemma 8.5.1. *Each nonempty closed subset of $\mathscr{N}$ has a smallest element.*

Proof. Let C be a nonempty closed subset of $\mathscr{N}$. We define a sequence $\{n_j\}$ of positive integers as follows. Let

$$n_1 = \inf\{k \in \mathbb{N} : k = m_1 \text{ for some } \mathbf{m} \text{ in } C\}.$$

Next suppose that $n_1, \ldots, n_j$ have been chosen, and let

$$n_{j+1} = \inf\{k \in \mathbb{N} : k = m_{j+1} \text{ for some } \mathbf{m} \text{ in } C \cap \mathscr{N}(n_1,\ldots,n_j)\}.$$

It is easy to check that the sequence $\mathbf{n} = \{n_j\}$ produced by continuing in this way is the required element of C. □

Lemma 8.5.2. *Each subset of $\mathscr{N}$ that has the form*

$$\{\mathbf{m} \in \mathscr{N} : \mathbf{m} < \mathbf{n}\} \tag{1}$$

for some $\mathbf{n}$ *in $\mathscr{N}$ is open. The collection of all subsets of $\mathscr{N}$ of the form* (1) *generates $\mathscr{B}(\mathscr{N})$.*

Proof. Note that $\{\mathbf{m} \in \mathscr{N} : \mathbf{m} < \mathbf{n}\}$ is equal to $\bigcup_{k=1}^{\infty} \bigcup_{j<n_k} \mathscr{N}(n_1,\ldots,n_{k-1},j)$, and so, as the union of a collection of open sets, is open.

Let $\mathscr{F}$ be the σ-algebra generated by the sets of the form (1). Since each set of the form (1) is open, $\mathscr{F}$ is included in $\mathscr{B}(\mathscr{N})$. On the other hand, it is easy to check that for each k and each $n_1, \ldots, n_k$ the set $\mathscr{N}(n_1,\ldots,n_k)$ is the intersection of

$$\{\mathbf{m} \in \mathscr{N} : \mathbf{m} < (n_1,n_2,\ldots,n_{k-1},n_k+1,1,1,\ldots)\}$$

with the complement of

$$\{\mathbf{m} \in \mathscr{N} : \mathbf{m} < (n_1,n_2,\ldots,n_{k-1},n_k,1,1,\ldots)\}$$

and so belongs to $\mathscr{F}$. Since the sets $\mathscr{N}(n_1,\ldots,n_k)$ form a countable base for $\mathscr{N}$ (see Example 8.1.6(b)), they generate $\mathscr{B}(\mathscr{N})$, and it follows that $\mathscr{B}(\mathscr{N}) \subseteq \mathscr{F}$. Thus $\mathscr{B}(\mathscr{N}) = \mathscr{F}$. □

For the following theorem we will, as usual, let $\mathscr{B}(X)_*$ denote the σ-algebra of universally measurable subsets of the Polish space X; we will also let $\mathscr{A}(X)$ denote the σ-algebra generated by the analytic subsets of X.

Theorem 8.5.3. *Let X and Y be Polish spaces, let A be an analytic subset of $X \times Y$, and let A_0 be the projection of A on X. Then there is a function $f: A_0 \to Y$ such that*

(a) *the graph of f is a subset of A, and*
(b) *f is measurable with respect to $\mathscr{A}(X)$ and $\mathscr{B}(Y)$ and with respect to $\mathscr{B}(X)_*$ and $\mathscr{B}(Y)$.*

Proof. We can assume that A is not empty, and so we can choose a continuous function $g: \mathscr{N} \to X \times Y$ such that $g(\mathscr{N}) = A$ (Corollary 8.2.8). Let π_X and π_Y be the projections of $X \times Y$ onto X and Y, respectively. Then $\pi_X \circ g: \mathscr{N} \to X$ is continuous, and $(\pi_X \circ g)(\mathscr{N}) = \pi_X(A) = A_0$. Hence if $x \in A_0$, then $(\pi_X \circ g)^{-1}(\{x\})$ is a nonempty closed subset of $\mathscr{N}$, and so has a smallest member (Lemma 8.5.1). Define $h: A_0 \to \mathscr{N}$ by letting $h(x)$ be this smallest member of $(\pi_X \circ g)^{-1}(\{x\})$. Let $f = \pi_Y \circ g \circ h$. It is easy to check that f is a function from A_0 to Y whose graph is included in A. Since g and π_Y are continuous and since $\mathscr{A}(X) \subseteq \mathscr{B}(X)_*$ (Corollary 8.4.3), the measurability of f will follow if we prove that h is measurable with respect to $\mathscr{A}(X)$ and $\mathscr{B}(\mathscr{N})$.

Note that if for each $\mathbf{n}$ in $\mathscr{N}$ we let

$$U_{\mathbf{n}} = \{\mathbf{m} \in \mathscr{N} : \mathbf{m} < \mathbf{n}\},$$

then $h^{-1}(U_{\mathbf{n}})$ is equal to $(\pi_X \circ g)(U_{\mathbf{n}})$, and so, as the image of the open set $U_{\mathbf{n}}$ under the continuous map $\pi_X \circ g$, is analytic. Since the sets $U_{\mathbf{n}}$ generate $\mathscr{B}(\mathscr{N})$ (Lemma 8.5.2), the measurability of h with respect to $\mathscr{A}(X)$ and $\mathscr{B}(\mathscr{N})$ follows (Proposition 2.6.2). Thus f is measurable, and the proof is complete. □

Theorem 8.5.3 implies the following result, in which X is no longer required to be Polish. Recall that if $(X, \mathscr{A})$ is an arbitrary measurable space, then $\mathscr{A}_*$ is the σ-algebra of sets that are universally measurable with respect to $(X, \mathscr{A})$.

Corollary 8.5.4. *Let $(X, \mathscr{A})$ be a measurable space, let Y be a Polish space, let C be a subset of $X \times Y$ that belongs to the σ-algebra $\mathscr{A} \times \mathscr{B}(Y)$, and let C_0 be the projection of C on X. Then there is a function $f: C_0 \to Y$ such that*

(a) *the graph of f is a subset of C, and*
(b) *f is measurable with respect to $\mathscr{A}_*$ and $\mathscr{B}(Y)$.*

Proof. Let Z, h, H, and D be as in Lemma 8.4.5, and let D_0 be the projection of D on Z. Note that $C_0 = h^{-1}(D_0)$. According to Theorem 8.5.3 there is a function $f_0: D_0 \to Y$ that is measurable with respect to $\mathscr{B}(Z)_*$ and $\mathscr{B}(Y)$ and whose graph is a subset of D. Define $f: C_0 \to Y$ by $f(x) = f_0(h(x))$. The fact that $C = H^{-1}(D)$ implies that the graph of f is included in C, and Lemma 8.4.6 implies that h is measurable with respect to $\mathscr{A}_*$ and $\mathscr{B}(Z)_*$ and hence that f is measurable with respect to $\mathscr{A}_*$ and $\mathscr{B}(Y)$. □

Exercises

1. Show by example that the Polish space Y in Proposition 8.4.4 and Corollary 8.5.4 cannot be replaced with an arbitrary measurable space $(Y, \mathscr{B})$. (Hint: Let $(X, \mathscr{A})$ be $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, let Y be a subset of $\mathbb{R}$ that is not Lebesgue measurable, and let $\mathscr{B}$ be the trace of $\mathscr{B}(\mathbb{R})$ on Y. For Proposition 8.4.4 consider the subset $\{(x,y) : x = y\}$ of $X \times Y$.)
2. Let $(X, \mathscr{A})$ be a measurable space, let Y be a Polish space, and let C be a subset of $X \times Y$ such that

   (i) for each x in X the section C_x is closed and nonempty, and
   (ii) for each open subset U of Y the set $\{x \in X : C_x \cap U \neq \varnothing\}$ belongs to $\mathscr{A}$.

   Show that there is a function $f: X \to Y$ such that
   (a) f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$, and

(b) the graph of f is included in C.
(Hint: Let d be a complete metric for Y, and let D be a countable dense subset of Y. Choose a sequence $\{f_n\}$ of $\mathscr{A}$-measurable functions from X to D such that $d(f_n(x), C_x) < 1/2^n$ and $d(f_n(x), f_{n+1}(x)) < 1/2^n$ hold for all n and x; then define f by $f(x) = \lim_n f_n(x)$.)

3. Let $(X, \mathscr{A})$, Y, and C be as in Exercise 2. Show that there is a sequence $\{f_n\}$ of functions from X to Y such that
(a) each f_n is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$, and
(b) for each x in X the section C_x is the closure of the set $\{f_n(x) : n \in \mathbb{N}\}$.
(Hint: Let $\{U_n\}$ be an enumeration of the nonempty sets in some countable base for Y. Define sets X_1, X_2, ... by letting X_n be the set of x's for which $C_x \cap U_n$ is not empty; then use Exercise 2 to construct for each n a measurable function $g_n\colon X_n \to U_n$ whose graph is included in $C \cap (X_n \times U_n)$. Construct the f_n's by extending the g_n's to X in a suitable way.)
4. Let X be a Polish space, let $(Y, \mathscr{A})$ be a measurable space, and let $f\colon X \to Y$ be a function such that

(i) if $y \in Y$, then $f^{-1}(\{y\})$ is a nonempty closed subset of X, and
(ii) if U is an open subset of X, then $f(U)$ belongs to $\mathscr{A}$.

Use Exercise 2 to show that there is a function $g\colon Y \to X$ that is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(X)$ and that satisfies $y = f(g(y))$ for each y in Y.
5. Use Exercise 8.2.4, together with ideas from the proof of Theorem 8.5.3, to give an alternate construction of the function g in Exercise 4.

8.6 Standard, Analytic, Lusin, and Souslin Spaces

A measurable space $(X, \mathscr{A})$ is *standard* if there is a Polish space Z such that $(X, \mathscr{A})$ is isomorphic to $(Z, \mathscr{B}(Z))$, and is *analytic* if there is a Polish space Z and an analytic subset A of Z such that $(X, \mathscr{A})$ is isomorphic to $(A, \mathscr{B}(A))$ (recall that $\mathscr{B}(A)$ is the Borel σ-algebra of the subspace A, and so, according to Lemma 7.2.2, is the family of subsets of A that have the form $A \cap B$ for some Borel subset B of Z).

Of course, the earlier sections of this chapter contain a number of properties of standard and analytic measurable spaces. (For example, Theorem 8.3.6 implies that if $(X, \mathscr{A})$ is a standard measurable space, then either X is countable and $\mathscr{A}$ contains all the subsets of X or else $(X, \mathscr{A})$ is isomorphic to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.) This section contains a few more such properties, plus some techniques for verifying that a measurable space is standard or analytic.

We need to define a few more terms. Let $(X, \mathscr{A})$ be a measurable space. A subfamily $\mathscr{C}$ of $\mathscr{A}$ *generates* $\mathscr{A}$ if $\sigma(\mathscr{C}) = \mathscr{A}$. The σ-algebra $\mathscr{A}$, or the measurable space $(X, \mathscr{A})$, is *countably generated* if $\mathscr{A}$ has a countable subfamily that generates it. A family $\mathscr{C}$ of subsets of X *separates the points* of X if for each pair x, y of distinct points in X there is a member of $\mathscr{C}$ that contains exactly one of x and y. The space $(X, \mathscr{A})$, or the σ-algebra $\mathscr{A}$, is *separated* if $\mathscr{A}$ separates the points of X, and is *countably separated* if $\mathscr{A}$ has a countable subfamily that separates the points of X.

See Exercises 1, 2, and 4 for some information about the relationships among the concepts just defined.

Let us begin with a couple of general facts about analytic measurable spaces (Lemma 8.6.1 and Proposition 8.6.2), and then turn to ways of recognizing the analytic and standard spaces among the countably generated spaces (Propositions 8.6.5 and 8.6.6).

Lemma 8.6.1. *Let $(X,\mathscr{A})$ be an analytic measurable space, let Y be a Polish space, and let $f\colon X\to Y$ be measurable with respect to $\mathscr{A}$ and $\mathscr{B}(Y)$. Then the images under f of the sets in $\mathscr{A}$ are analytic.*

Proof. Since $(X,\mathscr{A})$ is an analytic measurable space, we can choose a Polish space Z, an analytic subset A_0 of Z, and an isomorphism g of $(A_0,\mathscr{B}(A_0))$ onto $(X,\mathscr{A})$. Suppose that $A\in\mathscr{A}$. Then $g^{-1}(A)\in\mathscr{B}(A_0)$, and so there is a set B in $\mathscr{B}(Z)$ such that $g^{-1}(A)=B\cap A_0$ (Lemma 7.2.2). Consequently $f(A)$, as the image of $g^{-1}(A)$ under the measurable map $f\circ g$, is analytic (Proposition 8.2.6). □

Proposition 8.6.2. *Each bijective measurable map between analytic measurable spaces is an isomorphism.*

Proof. Suppose that $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ are analytic measurable spaces and that $f\colon X\to Y$ is a measurable bijection. We need to show that if $A\in\mathscr{A}$, then $f(A)\in\mathscr{B}$. Since $(Y,\mathscr{B})$ is analytic, there is a Polish space Z, an analytic subset A_0 of Z, and an isomorphism g of $(Y,\mathscr{B})$ onto $(A_0,\mathscr{B}(A_0))$. Of course g is measurable with respect to $\mathscr{B}$ and $\mathscr{B}(Z)$ (Lemma 7.2.2). Now suppose that $A\in\mathscr{A}$. The measurability of $g\circ f$ with respect to $\mathscr{A}$ and $\mathscr{B}(Z)$ implies that $g(f(A))$ and $g(f(A^c))$ are analytic subsets of Z (Lemma 8.6.1), while the injectivity of $g\circ f$ implies that $g(f(A))$ and $g(f(A^c))$ are disjoint; hence the separation theorem for analytic sets (Theorem 8.3.1) provides a Borel subset B of Z such that $g(f(A))\subseteq B$ and $g(f(A^c))\subseteq B^c$. It is easy to check that $f(A)$ is equal to $g^{-1}(B)$, and so belongs to $\mathscr{B}$. Since A was an arbitrary set in $\mathscr{A}$, the measurability of f^{-1} follows. □

We need the following elementary construction for our proof of Proposition 8.6.5.

Lemma 8.6.3. *Let $(X,\mathscr{A})$ be a countably generated measurable space, and suppose that the sets A_1, A_2, … generate $\mathscr{A}$. Define $F\colon X\to\{0,1\}^{\mathbb{N}}$ by letting F take x to the sequence $\{\chi_{A_n}(x)\}$. Then*

$$\mathscr{A}=\{B\subseteq X : B=F^{-1}(C)\ \textit{for some } C \textit{ in } \mathscr{B}(\{0,1\}^{\mathbb{N}})\}. \tag{1}$$

Proof. Let us denote the set on the right-hand side of (1) by $\mathscr{A}_F$. Since the sets E_1, E_2, … defined by

$$E_k=\{\{n_i\}\in\{0,1\}^{\mathbb{N}} : n_k=1\}$$

generate $\mathscr{B}(\{0,1\}^{\mathbb{N}})$ (Proposition 8.1.7) and since $A_k=F^{-1}(E_k)$ holds for each k, Proposition 2.6.2 implies that F is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(\{0,1\}^{\mathbb{N}})$, and hence that $\mathscr{A}_F \subseteq \mathscr{A}$. On the other hand, $\mathscr{A}_F$ is a σ-algebra on X that contains each A_k, and hence includes the σ-algebra these sets generate, namely $\mathscr{A}$. Thus $\mathscr{A} = \mathscr{A}_F$. □

Corollary 8.6.4. *Let $(X, \mathscr{A})$ be a separated and countably generated measurable space. Then there is a subset A of $\{0,1\}^{\mathbb{N}}$ such that $(X, \mathscr{A})$ is isomorphic to $(A, \mathscr{B}(A))$.*

Proof. Use Lemma 8.6.3 to construct a map $F\colon X \to \{0,1\}^{\mathbb{N}}$ such that

$$\mathscr{A} = \{B \subseteq X : B = F^{-1}(C) \text{ for some } C \text{ in } \mathscr{B}(\{0,1\}^{\mathbb{N}})\}. \tag{2}$$

Let $A = F(X)$. Since $\mathscr{A}$ was assumed to separate the points of X, (2) implies first that F is injective and then that F is an isomorphism between $(X, \mathscr{A})$ and $(A, \mathscr{B}(A))$ (note that if $B = F^{-1}(C)$, then $F(B) = C \cap A$; also see Lemma 7.2.2). □

Proposition 8.6.5. *Let $(X, \mathscr{A})$ be an analytic measurable space, let $(Y, \mathscr{B})$ be a separated and countably generated measurable space, and let $f\colon X \to Y$ be surjective and measurable. Then $(Y, \mathscr{B})$ is analytic.*

Proof. Use Corollary 8.6.4 to construct a function $F\colon Y \to \{0,1\}^{\mathbb{N}}$ that induces an isomorphism of $(Y, \mathscr{B})$ onto $(F(Y), \mathscr{B}(F(Y)))$. Lemma 8.6.1, applied to the map $F \circ f$, implies that $F(Y)$ is an analytic subset of $\{0,1\}^{\mathbb{N}}$. Thus $(Y, \mathscr{B})$, since it is isomorphic to $(F(Y), \mathscr{B}(F(Y)))$, is an analytic space. □

Proposition 8.6.6. *Let $(X, \mathscr{A})$ be a standard measurable space, let $(Y, \mathscr{B})$ be a separated and countably generated measurable space, and let $f\colon X \to Y$ be bijective and measurable. Then $(Y, \mathscr{B})$ is standard.*

Proof. Proposition 8.6.5 implies that $(Y, \mathscr{B})$ is analytic, and Proposition 8.6.2 then implies that $(Y, \mathscr{B})$ is isomorphic to $(X, \mathscr{A})$. Since $(X, \mathscr{A})$ is standard, $(Y, \mathscr{B})$ must also be standard. □

We turn to an important result due to Blackwell and to some of its consequences. For this we need to define the atoms of a σ-algebra. Let $(X, \mathscr{A})$ be a measurable space, and let x be an element of X. The *atom* of $\mathscr{A}$ determined by x is the intersection of those sets that belong to $\mathscr{A}$ and contain x. Note that a point y belongs to the atom determined by x if and only if x and y belong to exactly the same sets in $\mathscr{A}$. It is easy to check that the atoms of $\mathscr{A}$ form a partition of X, that an atom of $\mathscr{A}$ does not necessarily belong to $\mathscr{A}$ (see Exercise 5), and that an atom of $\mathscr{A}$ can contain more than one point (see Exercise 3).

Theorem 8.6.7 (Blackwell). *Let $(X, \mathscr{A})$ be an analytic measurable space, and let $\mathscr{A}_0$ be a countably generated sub-σ-algebra of $\mathscr{A}$. Then a subset of X belongs to $\mathscr{A}_0$ if and only if it belongs to $\mathscr{A}$ and is the union of a family of atoms of $\mathscr{A}_0$.*

Proof. Certainly every set that belongs to $\mathscr{A}_0$ also belongs to $\mathscr{A}$ and is the union of a family of atoms of $\mathscr{A}_0$. We need to prove the converse.

Use Lemma 8.6.3 to choose a function $F\colon X \to \{0,1\}^{\mathbb{N}}$ such that

$$\mathscr{A}_0 = \{B \subseteq X : B = F^{-1}(C) \text{ for some } C \text{ in } \mathscr{B}(\{0,1\}^{\mathbb{N}})\}. \tag{3}$$

Note that F is measurable with respect to $\mathscr{A}_0$ and $\mathscr{B}(\{0,1\}^{\mathbb{N}})$ and so with respect to $\mathscr{A}$ and $\mathscr{B}(\{0,1\}^{\mathbb{N}})$, and that the atoms of $\mathscr{A}_0$ are the nonempty subsets of X that are inverse images under F of one-point subsets of $\{0,1\}^{\mathbb{N}}$. Now suppose that A belongs to $\mathscr{A}$ and is the union of a family of atoms of $\mathscr{A}_0$. Then $F(A)$ and $F(A^c)$ are disjoint analytic subsets of $\{0,1\}^{\mathbb{N}}$ (use Lemma 8.6.1 and the assumption that A is the union of a collection of atoms of $\mathscr{A}_0$). Hence the separation theorem for analytic sets provides a Borel subset C of $\{0,1\}^{\mathbb{N}}$ such that $F(A) \subseteq C$ and $F(A^c) \subseteq C^c$. Then A is equal to $F^{-1}(C)$ and so in view of (3) belongs to $\mathscr{A}_0$. With this the proof is complete. □

Corollary 8.6.8. *Let $(X, \mathscr{A})$ be an analytic measurable space, and let $\mathscr{A}_0$ be a separated and countably generated sub-σ-algebra of $\mathscr{A}$. Then $\mathscr{A}_0 = \mathscr{A}$.*

Proof. Since $\mathscr{A}_0$ is separated, each of its atoms contains only one point, and so each subset of X is the union of a family of atoms of $\mathscr{A}_0$. Thus Theorem 8.6.7 implies that a subset of X belongs to $\mathscr{A}_0$ if and only if it belongs to $\mathscr{A}$. □

The following strengthened versions of Propositions 8.6.5 and 8.6.6 now follow. They will be useful for the study of Lusin and Souslin spaces later in this section.

Corollary 8.6.9. *Let $(X, \mathscr{A})$ be an analytic measurable space, let $(Y, \mathscr{B})$ be a countably separated measurable space, and let $f : X \to Y$ be surjective and measurable. Then $(Y, \mathscr{B})$ is analytic.*

Proof. We begin by showing that $\mathscr{B}$ is countably generated. Choose a countable subfamily $\mathscr{C}$ of $\mathscr{B}$ that separates the points of Y. We will show that $\mathscr{B}$ is equal to the countably generated σ-algebra $\sigma(\mathscr{C})$. Let B be an arbitrary element of $\mathscr{B}$, and let $\mathscr{B}_0 = \sigma(\mathscr{C} \cup \{B\})$. Then $\mathscr{B}_0$ is separated and countably generated, and f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}_0$; hence $(Y, \mathscr{B}_0)$ is analytic (Proposition 8.6.5). Furthermore, $\sigma(\mathscr{C})$ is a separated and countably generated sub-σ-algebra of $\mathscr{B}_0$, and so Corollary 8.6.8 implies that $\sigma(\mathscr{C}) = \mathscr{B}_0$. Thus $B \in \sigma(\mathscr{C})$. Since B was an arbitrary member of $\mathscr{B}$, it follows that $\mathscr{B} = \sigma(\mathscr{C})$.

Now that we have proved that $\mathscr{B}$ is countably generated, we can use Proposition 8.6.5 to conclude that $(Y, \mathscr{B})$ is analytic. □

Corollary 8.6.10. *Let $(X, \mathscr{A})$ be a standard measurable space, let $(Y, \mathscr{B})$ be a countably separated measurable space, and let $f : X \to Y$ be bijective and measurable. Then $(Y, \mathscr{B})$ is standard.*

Proof. This follows from Corollary 8.6.9 in the same way that Proposition 8.6.6 follows from Proposition 8.6.5. □

Let us turn to the study of some not necessarily metrizable topological spaces that are closely related to the Polish spaces. A *Lusin space* is a Hausdorff space that is the image of a Polish space under a continuous bijection, and a *Souslin space* is a Hausdorff space that is the image of a Polish space under a continuous surjection. Of course, every Lusin space is a Souslin space.

Examples 8.6.11.

(a) It is clear that the Souslin subspaces of a Polish space X are exactly the analytic subsets of X. Proposition 8.2.10 (or Exercise 8.2.5) and Theorem 8.3.7 imply that the Lusin subspaces of a Polish space X are exactly the Borel subsets of X.
(b) Now suppose that X is a Polish space, and let X_0 be constructed by replacing the topology of X with a weaker Hausdorff topology. The function $f\colon X \to X_0$ defined by $f(x) = x$ is continuous, and so X_0 is a Lusin space. In particular, if X is a separable Banach space, then X with its weak topology[8] is a Lusin space. Likewise, if the dual X^* of the Banach space X is separable, then X^* with its weak* topology is a Lusin space. Furthermore, if the Banach space X is infinite dimensional, then the weak topology on X and the weak* topology on X^* are not metrizable.[9] Thus non-metrizable Lusin spaces arise in a natural way. □

The rest of this section is devoted to some basic measure-theoretic facts about Lusin and Souslin spaces. We will prove that if X is a Lusin space, then $(X, \mathscr{B}(X))$ is a standard measurable space, that if X is a Souslin space, then $(X, \mathscr{B}(X))$ is an analytic measurable space, and that if X is a Souslin space, then every finite Borel measure on X is regular.

Lemma 8.6.12. *If X is a Souslin space, then $\mathscr{B}(X)$ is countably separated.*

Proof. Choose a Polish space Z and a continuous surjection $f\colon Z \to X$. Define $F\colon Z \times Z \to X \times X$ by $F(z_1, z_2) = (f(z_1), f(z_2))$. Let Δ be the subset of $X \times X$ defined by

$$\Delta = \{(x_1, x_2) \in X \times X : x_1 = x_2\},$$

and let $\mathscr{U}$ be the collection of those open rectangles in $X \times X$ that are included in the complement of Δ. Then F is continuous, Δ is closed, and $\Delta^c = \cup\mathscr{U}$. Hence

$$F^{-1}(\Delta^c) = \bigcup\{F^{-1}(U) : U \in \mathscr{U}\},$$

[8] This example assumes more Banach space theory than is developed in this book.

[9] Suppose that X is an infinite-dimensional Banach space. If the weak topology on X is metrizable, then there is an infinite sequence $\{f_i\}$ in X^* such that each f in X^* is a linear combination of f_1, …, f_n for some n (choose $\{f_i\}$ so that for each weakly open neighborhood U of 0 there is a positive integer n and a positive number ε such that $x \in U$ holds whenever x satisfies $|f_i(x)| < \varepsilon$ for $i = 1$, …, n; then use Lemma V.3.10 in [42]). Thus X^* is the union of a sequence of finite-dimensional subspaces of X^*. Since this is impossible (use Exercise 3.5.6, Corollary IV.3.2 in [42], and the Baire category theorem), we have a contradiction, and the weak topology on X is not metrizable. A similar argument shows that the weak* topology on X^* is not metrizable.

and D.11, applied to $\{F^{-1}(U) : U \in \mathscr{U}\}$, implies that there is a countable subcollection $\mathscr{U}_0$ of $\mathscr{U}$ such that

$$F^{-1}(\Delta^c) = \bigcup\{F^{-1}(U) : U \in \mathscr{U}_0\}.$$

This and the surjectivity of F imply that

$$\Delta^c = \bigcup \mathscr{U}_0.$$

Thus for each pair x, y of distinct points in X there is a set $V \times W$ in $\mathscr{U}_0$ such that $(x,y) \in V \times W$ and hence (recall that $(V \times W) \cap \Delta = \varnothing$) such that $x \in V$, $y \in W$, and $V \cap W = \varnothing$. Consequently the sides of the rectangles in $\mathscr{U}_0$ form a countable subfamily of $\mathscr{B}(X)$ that separates the points of X. □

Proposition 8.6.13. *If X is a Souslin space, then $(X, \mathscr{B}(X))$ is an analytic measurable space, while if X is a Lusin space, then $(X, \mathscr{B}(X))$ is a standard measurable space.*

Proof. Let X be a Souslin space, and choose a Polish space Z and a continuous surjection $f : Z \to X$. Since f is Borel measurable and $\mathscr{B}(X)$ is countably separated (Lemma 8.6.12), Corollary 8.6.9 implies that $(X, \mathscr{B}(X))$ is analytic. A similar argument, based on Lemma 8.6.12 and Corollary 8.6.10, shows that if X is a Lusin space, then $(X, \mathscr{B}(X))$ is standard. □

Theorem 8.6.14. *Every finite Borel measure on a Souslin space is regular.*

Proof. Let X be a Souslin space, and let μ be a finite Borel measure on X. We will show that

$$\mu(B) = \sup\{\mu(K) : K \subseteq B \text{ and } K \text{ is compact}\} \tag{4}$$

holds for each B in $\mathscr{B}(X)$. This gives the inner regularity of μ. It also implies the outer regularity of μ, since for each B in $\mathscr{B}(X)$ we can use (4), applied to B^c, to approximate B^c from below by compact sets and hence to approximate B from above by open sets.

So suppose that B belongs to $\mathscr{B}(X)$. We can assume that B is not empty. Let us begin by producing a continuous function $f : \mathscr{N} \to X$ such that $f(\mathscr{N}) = B$. For this choose a Polish space Z and a continuous surjection $g : Z \to X$, note that $g^{-1}(B)$ is a Borel and hence analytic subset of Z, choose a continuous function $h : \mathscr{N} \to Z$ such that $h(\mathscr{N}) = g^{-1}(B)$, and let $f = g \circ h$.

For each positive number ε we can use the constructions in the proof of Theorem 8.4.1 to choose sets $\mathscr{L}(n_1, \dots, n_k)$ (to be abbreviated as $\mathscr{L}_k$) such that $\mu^*(f(\mathscr{L}_k)) > \mu(B) - \varepsilon$ holds for each k. Arguments used in the proof of Theorem 8.4.1 show that the sets L and K defined by $L = \cap_k \mathscr{L}_k$ and $K = f(L)$ are compact, that $\mu(\cap_k f(\mathscr{L}_k)^-) \geq \mu(B) - \varepsilon$, and that $K \subseteq \cap_k f(\mathscr{L}_k)^-$.

Our earlier proof of the reverse inclusion works only if X is metrizable; hence it must be replaced. Suppose that $x \in \cap_k f(\mathscr{L}_k)^-$ and that U is an open neighborhood of x. For each k choose an element $\mathbf{m}_k$ of $\mathscr{L}_k$ such that $f(\mathbf{m}_k) \in U$. As before, the sequence $\{\mathbf{m}_k\}$ has a convergent subsequence, say with limit $\mathbf{m}$. Then $\mathbf{m} \in L$, and the continuity of f implies that $f(\mathbf{m}) \in U^-$ and hence that U^- meets K. Since this is valid for each open neighborhood U of x, it follows that $x \in K$ (otherwise, since K is compact, Proposition 7.1.2 would provide disjoint open sets U_0 and V_0 such that $x \in U_0$ and $K \subseteq V_0$, and U_0 would be an open neighborhood of x such that $\overline{U_0} \cap K = \varnothing$). Since x was an arbitrary element of $\cap_k f(\mathscr{L}_k)^-$, it follows that $\cap_k f(\mathscr{L}_k)^- \subseteq K$ and hence that $K = \cap_k f(\mathscr{L}_k)^-$. With this we have constructed a compact subset K of B such that $\mu(K) \geq \mu(B) - \varepsilon$, and (4) is proved. □

Exercises

1. Let $(X, \mathscr{A})$ be a measurable space. Show that if $\mathscr{A}$ is separated and countably generated, then $\mathscr{A}$ is countably separated.
2. Give a σ-algebra on $\mathbb{R}$ that is included in $\mathscr{B}(\mathbb{R})$ and is separated but not countably separated.
3. Let $(X, \mathscr{A})$ be a measurable space. Show that each atom of $\mathscr{A}$ contains only one point if and only if $\mathscr{A}$ separates the points of X.
4. Give an example of a measurable space that is countably separated but not countably generated.
5. Let $X = \{0,1\}^{\mathbb{R}}$ and let $\mathscr{A}$ be the smallest σ-algebra on X that makes each coordinate projection of X onto $\{0,1\}$ measurable (of course, $\{0,1\}$ is to have the σ-algebra consisting of all of its subsets).
   (a) Show that for each A in $\mathscr{A}$ there is a countable subset S of $\mathbb{R}$ such that if $x \in A$, if $y \in X$, and if $x(s) = y(s)$ holds at each s in S, then $y \in A$. (Hint: See Exercise 1.1.7.)
   (b) Show that the atoms of $\mathscr{A}$ do not belong to $\mathscr{A}$.
6. Show by example that the hypothesis that $\mathscr{A}_0$ is countably generated cannot be removed from Theorem 8.6.7.
7. Show by example that the hypothesis that $(X, \mathscr{A})$ is analytic cannot be removed from Theorem 8.6.7. (Hint: Let $X = \mathbb{R}$, let A be a subset of $\mathbb{R}$ that is not Borel, and let $\mathscr{A} = \sigma(\mathscr{B}(\mathbb{R}) \cup \{A\})$.)
8. Let X be a Souslin space. Show that if $\mathscr{U}$ is a collection of open subsets of X, then there is a countable subcollection $\mathscr{U}_0$ of $\mathscr{U}$ such that $\cup \mathscr{U}_0 = \cup \mathscr{U}$. (Hint: Study the proof of Lemma 8.6.12.)
9. Show that if X and Y are Souslin spaces, then $\mathscr{B}(X \times Y) = \mathscr{B}(X) \times \mathscr{B}(Y)$. (Hint: Apply Exercise 8 to the space $X \times Y$.)
10. Show that each compact Souslin space is metrizable.

Notes

The classical theory of analytic sets was developed by the Polish and Russian schools of mathematics between the First and Second World Wars. See, for example, Kuratowski [77]. In the mid-1950s Blackwell [11] noted that the theory of analytic sets can be applied profitably to probability theory, while Mackey [86] noted that it is useful for the study of group representations; their work has done much to stimulate interest in the subject. Rather thorough recent treatments of analytic sets have been given by Kechris [68] and Srivastava [112]. See also [29,62,83,101,107].

Analytic and Borel subsets of non-separable spaces have been studied by A.H. Stone and his students. See Stone [113] for a survey and for further references.

Exercise 8.1.14 is due to Dudley [39].

The reader who wants to see additional (and more explicit) examples of analytic sets that are not Borel should see Mazurkiewicz [88] and Dellacherie [35]. For example, Mazurkiewicz shows that the subset A of $C[0,1]$ consisting of the differentiable functions (that is, of the continuous functions on $[0,1]$ that are differentiable at each point in $[0,1]$) is the complement of an analytic set, but is not itself analytic (thus A^c is analytic but not Borel).

The proof of Theorem 8.3.6 given in the text was suggested by Kuratowski and Mostowski [78], while that in Exercise 8.3.5 is taken from Parthasarathy [96]. The proof given here for Theorem 8.3.7 is due to Dellacherie [36].

Theorems 8.4.1 and 8.5.3 are classical. That they imply Proposition 8.4.4 and Corollary 8.5.4 has been noticed (independently) by a number of people. See Castaing and Valadier [26] and Wagner [121] (and of course [62,68,101,112]) for further information and references. The concepts of capacity and capacitability are due to Choquet [28].

The results in the first part of Sect. 8.6 are due to Blackwell [11] and Mackey [86]. Bourbaki (see Chapter IX of [17]) introduced the terms Lusin space and Souslin space for metrizable spaces that are images of Polish spaces under continuous bijections and surjections; Cartier [25] noted that the assumption of metrizability is not needed.