to show that if f belongs to $\mathscr{K}(G)$ and satisfies $\nu * \mu_f = 0$, then $\int \check{f}\, d\nu = 0$; then use Theorem 7.3.6.)

11. Show that $L^1(G)$ has an identity if and only if the topology of G is discrete. (Hint: Use Exercise 6 in Sect. 9.2 and Exercises 7 and 10.)
12. Let G be $\mathbb{R}^2$, with the usual group operation but with the topology defined in Exercise 7.2.4. Show that
    (a) G is a locally compact group,
    (b) $\{0\} \times \mathbb{R}$ is an open, closed, and σ-compact subgroup of G, and
    (c) there is a function $f: G \to \mathbb{R}$ that is not Borel measurable, but for which each section f_x is Borel measurable. (Hint: See Exercises 8.2.7 and 8.2.9.)

    This explains the footnote in the proof of Theorem 9.4.8.

Notes

The history of Haar measure is summarized in the notes at the ends of Sections 15 and 16 of Hewitt and Ross [58].

The reader can find a more extensive introduction to topological groups in Pontryagin [98] or in Hewitt and Ross [58].

The proof given here for the existence of Haar measure (which is a modification of Halmos's modification of Weil's [126] proof) depends on the axiom of choice. Proofs that do not depend on this axiom have been given by Cartan [24] and Bredon [19]. Cartan's proof is given by Hewitt and Ross [58] and by Nachbin [93]. Hewitt and Ross and Nachbin also give calculations of Haar measure for a number of groups.

Chapter 9
Haar Measure

We saw in Chap. 1 that Lebesgue measure on $\mathbb{R}^d$ is translation invariant, in the sense that $\lambda(A+x) = \lambda(A)$ holds for each A in $\mathscr{B}(\mathbb{R}^d)$ and each x in $\mathbb{R}^d$. Furthermore, we saw that Lebesgue measure is essentially the only such Borel measure on $\mathbb{R}^d$: if μ is a nonzero Borel measure on $\mathbb{R}^d$ that is finite on the compact subsets of $\mathbb{R}^d$ and satisfies $\mu(A+x) = \mu(A)$ for each A in $\mathscr{B}(\mathbb{R}^d)$ and each x in $\mathbb{R}^d$, then there is a positive number c such that $\mu(A) = c\lambda(A)$ holds for every Borel subset A of $\mathbb{R}^d$.

It turns out that very similar results hold for every locally compact group (see Sect. 9.1 for the definition of such groups); the role of Lebesgue measure is played by what is called Haar measure. This chapter is devoted to an introduction to Haar measure.

Section 9.1 contains some basic definitions and facts about topological groups. Section 9.2 contains a proof of the existence and uniqueness of Haar measure, and Sect. 9.3 contains additional basic properties of Haar measures. In Sect. 9.4 we construct two algebras, $L^1(G)$ and $M(G)$, which are fundamental for the study of harmonic analysis on a locally compact group G.

9.1 Topological Groups

A *topological group* is a set G that has the structure of a group (say with group operation $(x,y) \mapsto xy$) and of a topological space and is such that the operations $(x,y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous. Note that $(x,y) \mapsto xy$ is a function from the product space $G \times G$ to G and that we are requiring that it be continuous with respect to the product topology on $G \times G$; thus xy must be "jointly continuous" in x and y and not merely continuous in x with y held fixed and continuous in y with x held fixed (see Exercise 3). A *locally compact topological group*, or simply a *locally compact group*, is a topological group whose topology is locally compact and Hausdorff. A *compact group* is a topological group whose topology is compact and Hausdorff.

D.L. Cohn, *Measure Theory: Second Edition*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4614-6956-8_9,

Examples 9.1.1.

(a) The set $\mathbb{R}$, with its usual topology and with addition as the group operation, is a locally compact group.
(b) Likewise, $\mathbb{R}^d$, $\mathbb{Z}$, and $\mathbb{Z}^d$ are locally compact groups.
(c) The set $\mathbb{R}^*$ of nonzero real numbers, with the topology it inherits as a subspace of $\mathbb{R}$ and with multiplication as the group operation, is a locally compact group.
(d) Let $\mathbb{T}$ be the set consisting of those complex numbers z that satisfy $|z| = 1$. Then $\mathbb{T}$, with the topology it inherits as a subspace of $\mathbb{C}$ and with multiplication as the group operation, is a compact group.
(e) The set $\mathbb{Q}$ of rational numbers, with the topology it inherits as a subspace of $\mathbb{R}$ and with addition as the group operation, is a topological group; it is not locally compact.
(f) An arbitrary group G, with the topology that makes every subset of G open, is a locally compact group; it is compact if and only if G is finite. □

See Exercises 9–11 for some additional examples.

Let X be a topological space, and let x belong to X. Recall that a family $\mathscr{U}$ of subsets of X is a *base for the family of neighborhoods of* x if

(a) each member of $\mathscr{U}$ is an open neighborhood of x, and
(b) for each open neighborhood V of x there is a set that belongs to $\mathscr{U}$ and is included in V.

Let G be a group. If a is an element of G and if B is a subset of G, then the products aB and Ba are defined by

$$aB = \{ab : b \in B\}$$

and

$$Ba = \{ba : b \in B\}.$$

Likewise, if B and C are subsets of G, then BC and B^{-1} are defined by

$$BC = \{bc : b \in B \text{ and } c \in C\},$$

and

$$B^{-1} = \{b^{-1} : b \in B\}.$$

The set B is *symmetric* if $B = B^{-1}$. Thus B is symmetric if and only if the condition $x \in B$ is equivalent to the condition $x^{-1} \in B$.

Proposition 9.1.2. *Let G be a topological group, let e be the identity element of G, and let a be an arbitrary element of G.*

(a) *The functions $x \mapsto ax$, $x \mapsto xa$, and $x \mapsto x^{-1}$ are homeomorphisms of G onto G.*
(b) *If $\mathscr{U}$ is a base for the family of neighborhoods of e, then $\{aU : U \in \mathscr{U}\}$ and $\{Ua : U \in \mathscr{U}\}$ are bases for the family of neighborhoods of a.*

$$\lim_n \|f * \varphi_n - f\|_1 = \lim_n \|\varphi_n * f - f\|_1 = 0 \tag{6}$$

holds for each f in $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$. Such a sequence is called an *approximate identity*. (Hint: Let $\{U_n\}$ be a decreasing sequence of open neighborhoods of e such that each open neighborhood of e includes some U_n. For each n let φ_n be a nonnegative function that belongs to $\mathscr{K}(X)$, vanishes outside U_n, and satisfies the relations $\varphi_n = (\varphi_n)\check{}$ and $\int \varphi_n \, d\mu = 1$. In verifying (6) it might be convenient to begin with the case where $f \in \mathscr{K}(G)$.)

(b) Now omit the assumption that G has a countable base for its topology. Show that there is a net[5] $\{\varphi_\alpha\}_{\alpha \in A}$ of nonnegative functions in $\mathscr{K}(G)$ such that $\int \varphi_\alpha \, d\mu = 1$ holds for each α and such that $\lim_\alpha \|f * \varphi_\alpha - f\|_1 = \lim_\alpha \|\varphi_\alpha * f - f\|_1 = 0$ holds for each f in $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$. (Hint: Let the directed set A be the collection of all open neighborhoods of e, and declare that $U \le V$ holds if and only if $V \subseteq U$.)

7. Show that δ_e, the point mass concentrated at e, is an identity for the algebra $M(G)$.
8. Show that G is commutative if and only if convolution is a commutative operation on $M(G)$.
9. Suppose that $\nu \in M(G)$, that $f \in \mathscr{L}^1(G, \mathscr{B}(G), \mu)$, and that μ_f is the finite signed or complex regular Borel measure defined by $\mu_f(A) = \int_A f \, d\mu$ (see Proposition 7.3.8). Define functions g and h on G by

$$g(t) = \begin{cases} \int f(s^{-1}t)\,\nu(ds) & \text{if } s \mapsto f(s^{-1}t) \text{ is } |\nu|\text{-integrable,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h(s) = \begin{cases} \int f(st^{-1})\Delta(t^{-1})\,\nu(dt) & \text{if } t \mapsto f(st^{-1})\Delta(t^{-1}) \text{ is } |\nu|\text{-integrable,} \\ 0 & \text{otherwise.} \end{cases}$$

Show that g and h belong to $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$ and that $(\nu * \mu_f)(A) = \int_A g \, d\mu$ and $(\mu_f * \nu)(A) = \int_A h \, d\mu$ hold for each A in $\mathscr{B}(G)$.
10. Let ν, f, and μ_f be as in Exercise 9. Show that $\nu * \mu_f = 0$ holds for each f in $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$ if and only if $\nu = 0$. (Hint: Use Exercise 9 and Corollary 9.1.6

[5] Recall that a *directed set* is a partially ordered set A (say ordered by $\le$) such that for each α and β in A, there is an element γ of A that satisfies $\alpha \le \gamma$ and $\beta \le \gamma$. A *net* is a family indexed by a directed set. A net $\{x_\alpha\}_{\alpha \in A}$ in a topological space X is said to *converge* to a point x of X if for each open neighborhood U of x there is an element α_0 of A such that $x_\alpha \in U$ holds whenever α satisfies $\alpha \ge \alpha_0$. Thus $\lim_\alpha \|f * \varphi_\alpha - f\|_1 = 0$ holds if and only if for each positive ε there is an element α_0 of A such that $\|f * \varphi_\alpha - f\|_1 < \varepsilon$ holds whenever α satisfies $\alpha \ge \alpha_0$. See Kelley [69] for an extended treatment of nets.

σ-finite (recall that C, as a coset of H, is σ-compact), we can choose a bounded Borel measurable function g_C on C such that $F_C(\langle f\rangle) = \int f g_C \, d\mu_C$ holds for each f in $\mathscr{L}^1(C, \mathscr{B}(C), \mu_C)$ (see Theorem 4.5.1). By modifying g_C on a μ_C-null set if necessary, we can assume that $|g_C(x)| \le \|F_C\| \le \|F\|$ holds at each x in C. Now choose a sequence $\{g_n\}$ of continuous functions on G such that

(a) $|g_n(x)| \le \|F\|$ holds at each x in G, and
(b) for each C in $\mathscr{H}$ the sequence $\{g_n\}$ converges to g_C μ-almost everywhere on C

(construct the functions g_n on each C separately, using Lusin's theorem (Theorem 7.4.4) and the σ-compactness of the sets in $\mathscr{H}$; see also D.6). Finally, define[4] g by $g = \limsup_n g_n$ (in case we are dealing with complex-valued functions, define the real and imaginary parts of g separately). Then g is a bounded Borel function, and the relation $F(\langle f\rangle) = \int f g \, d\mu$ holds for each f in $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$. Thus T is surjective, and the proof is complete. □

Exercises

Note: In the following exercises G is a locally compact group with identity element e, and μ is a left Haar measure on G.

1. Show that if f and g belong to $\mathscr{K}(G)$, then $f * g$ belongs to $\mathscr{K}(G)$.
2. Show that if f and g belong to $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$ and if $\{f_n\}$ and $\{g_n\}$ are sequences in $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$ such that $\lim_n \|f_n - f\|_1 = 0$ and $\lim_n \|g_n - g\|_1 = 0$, then $\lim_n \|f_n * g_n - f * g\|_1 = 0$.
3. Suppose that f and g belong to $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$. Show that in the definition of $f * g$ the expression $f(s)g(s^{-1}t)$ can be replaced
   (a) with $f(ts)g(s^{-1})$,
   (b) with $f(s^{-1})g(st)\Delta(s^{-1})$, and
   (c) with $f(ts^{-1})g(s)\Delta(s^{-1})$.
4. Show that if f_1, f_2, g_1, and g_2 belong to $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$, if $f_1 = f_2$ μ-almost everywhere, and if $g_1 = g_2$ μ-almost everywhere, then $f_1 * g_1 = f_2 * g_2$ *everywhere*.
5. Show that G is commutative if and only if convolution is a commutative operation on $L^1(G)$. (Hint: To show that the commutativity of $L^1(G)$ implies that of G, consider $f * g$ and $g * f$ for suitable nonnegative functions f and g in $\mathscr{K}(G)$.)
6. (a) Suppose that the locally compact group G has a countable base for its topology. Show that there is a sequence $\{\varphi_n\}$ of nonnegative functions in $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$ (or even in $\mathscr{K}(G)$) such that $\int \varphi_n \, d\mu = 1$ holds for each n and such that

[4]The function g cannot be defined simply be requiring that its restriction to each C in $\mathscr{H}$ be g_C; see Exercise 12.

(c) *If K and L are compact subsets of G, then aK, Ka, KL, and K^{-1} are compact subsets of G.*

Proof. The definition of a topological group, together with the continuity of the maps $x \mapsto (x,a)$ and $x \mapsto (a,x)$, implies the continuity of the functions in part (a). Since these functions have continuous inverses (namely the functions that take x to $a^{-1}x$, to xa^{-1}, and to x^{-1}), they are homeomorphisms of G onto G.

Part (b) is an immediate consequence of part (a).

Part (c) follows from the fact that the image of a compact set under a continuous map is compact (as usual, the compactness of the subset $K \times L$ of $G \times G$ is given by Tychonoff's theorem, Theorem D.20). □

Proposition 9.1.3. *Let G be a topological group, let e be the identity element of G, and let U be an open neighborhood of e.*

(a) *There is an open neighborhood V of e such that $VV \subseteq U$.*
(b) *There is a symmetric open neighborhood of e that is included in U.*

Proof. Since the map $(x,y) \mapsto xy$ is continuous, the set W defined by $W = \{(x,y) : xy \in U\}$ is an open neighborhood of (e,e) in $G \times G$, and so there are open neighborhoods V_1 and V_2 of e that satisfy $V_1 \times V_2 \subseteq W$. The set V defined by $V = V_1 \cap V_2$ is then an open neighborhood of e that satisfies $VV \subseteq U$.

We turn to part (b). The continuity of the map $x \mapsto x^{-1}$ implies that if U is an open neighborhood of e, then U^{-1} is also an open neighborhood of e. Thus $U \cap U^{-1}$ is a symmetric open neighborhood of e that is included in U. □

Proposition 9.1.4. *Let G be a topological group, let K be a compact subset of G, and let U be an open subset of G that includes K. Then there are open neighborhoods V_R and V_L of e such that $KV_R \subseteq U$ and $V_LK \subseteq U$.*

Proof. For each x in K choose open neighborhoods W_x and V_x of e such that $xW_x \subseteq U$ and $V_xV_x \subseteq W_x$ (see Propositions 9.1.2 and 9.1.3). Then $\{xV_x\}_{x\in K}$ is an open cover of the compact set K, and so there is a finite collection $x_1, \ldots, x_n$ of points in K such that the sets $x_iV_{x_i}$, $i = 1, \ldots, n$, cover K. Let $V_R = \cap_{i=1}^n V_{x_i}$. If $x \in K$, then there is an index i such that $x \in x_iV_{x_i}$, and so

$$xV_R \subseteq x_iV_{x_i}V_{x_i} \subseteq x_iW_{x_i} \subseteq U.$$

Since x was an arbitrary element of K, it follows that $KV_R \subseteq U$. The construction of V_L is similar. □

Let G be a topological group, and let f be a real- or complex-valued function on G. Then f is *left uniformly continuous* if for each positive number ε there is an open neighborhood U of e such that $|f(x) - f(y)| < \varepsilon$ holds whenever x and y belong to G and satisfy $y \in xU$. Likewise, f is *right uniformly continuous* if for each positive number ε there is an open neighborhood U of e such that $|f(x) - f(y)| < \varepsilon$ holds whenever x and y belong to G and satisfy $y \in Ux$. Note that we can replace the neighborhoods of e appearing in this definition with smaller symmetric neighborhoods of e (Proposition 9.1.3) and that for such symmetric neighborhoods U the condition $x \in yU$ is equivalent to the condition $y \in xU$ and the condition $x \in Uy$ is equivalent to the condition $y \in Ux$. Thus x and y do in fact enter our definition symmetrically.

Proposition 9.1.5. *Let G be a locally compact group. Then each function in $\mathscr{K}(G)$ is left uniformly continuous and right uniformly continuous.*

Proof. Let f belong to $\mathscr{K}(G)$, and let K be the support of f. Suppose that ε is a positive number. For each x in K choose first an open neighborhood U_x of e such that $|f(x) - f(y)| < \varepsilon/2$ holds whenever y belongs to xU_x and then an open neighborhood V_x of e such that $V_xV_x \subseteq U_x$ (see Propositions 9.1.2 and 9.1.3). The family $\{xV_x\}_{x\in K}$ is an open cover of the compact set K, and so there is a finite collection $x_1, \dots, x_n$ of points in K such that the sets $x_iV_{x_i}$, $i = 1, \dots, n$, cover K. Let V be a symmetric open neighborhood of e that is included in $\cap_{i=1}^n V_{x_i}$ (Proposition 9.1.3). We will show that if x and y belong to G and satisfy $y \in xV$, then $|f(x) - f(y)| < \varepsilon$.

This inequality certainly holds if neither x nor y belongs to K (for then $f(x) = f(y) = 0$). Now suppose that $x \in K$ and $y \in xV$. Then there is an index i such that $x \in x_iV_{x_i}$ and hence such that x and y belong to $x_iU_{x_i}$ (note that $x \in x_iV_{x_i} \subseteq x_iU_{x_i}$ and $y \in xV \subseteq x_iV_{x_i}V_{x_i} \subseteq x_iU_{x_i}$). It follows that $|f(x) - f(x_i)| < \varepsilon/2$ and $|f(y) - f(x_i)| < \varepsilon/2$ and hence that $|f(x) - f(y)| < \varepsilon$. The remaining case to deal with is where $y \in K$ and $y \in xV$. Since V is symmetric, this is exactly the case where $y \in K$ and $x \in yV$, and the details we just looked at (with x and y interchanged) handle this. The left uniform continuity of f follows. The right uniform continuity of f can be proved in a similar way. □

Corollary 9.1.6. *Let G be a locally compact group, let μ be a regular Borel measure on G, and let f belong to $\mathscr{K}(G)$. Then the functions $x \mapsto \int f(xy)\,\mu(dy)$ and $x \mapsto \int f(yx)\,\mu(dy)$ are continuous.*

Proof. We will check the continuity of $x \mapsto \int f(yx)\,\mu(dy)$ at an arbitrary point x_0 in G; the proof for $x \mapsto \int f(xy)\,\mu(dy)$ is similar.

Let K be the support of f, and let W be an open neighborhood of x_0 whose closure is compact. It is easy to check that for each x in W the function $y \mapsto f(yx)$ is continuous and vanishes outside the compact set $K(W^-)^{-1}$. Suppose that ε is a positive number, choose a positive number ε' such that $\varepsilon'\mu(K(W^-)^{-1}) < \varepsilon$, and use the left uniform continuity of f (Proposition 9.1.5) to choose an open neighborhood V of e such that $|f(s) - f(t)| < \varepsilon'$ holds whenever s and t belong to G and satisfy $s \in tV$. Then for each x in $W \cap x_0V$ and each y in G we have $yx \in yx_0V$, and so

$$\begin{aligned}\left|\int f(yx)\,\mu(dy) - \int f(yx_0)\,\mu(dy)\right| &\le \int |f(yx) - f(yx_0)|\,\mu(dy) \\ &\le \varepsilon'\mu(K(W^-)^{-1}) \le \varepsilon.\end{aligned}$$

Since ε is arbitrary, the proof is complete. □

The next two results will be used only in Sect. 9.4.

Proposition 9.4.7. *Let G be a locally compact group. Then*

(a) *$M_a(G)$ is an ideal in the algebra $M(G)$,*
(b) *if μ is a left Haar measure on G, then the map $f \mapsto \nu_f$ (where ν_f is defined by $\nu_f(A) = \int_A f\,d\mu$) induces a norm-preserving algebra homomorphism of $L^1(G, \mathscr{B}(G), \mu)$ into $M(G)$, and*
(c) *the image of $L^1(G, \mathscr{B}(G), \mu)$ under this homomorphism is $M_a(G)$.*

Proof. It is clear that $M_a(G)$ is a linear subspace of $M(G)$. Suppose that μ is a left Haar measure on G, that $\nu_1 \in M(G)$, and that $\nu_2 \in M_a(G)$. Let A be a Borel subset of G that satisfies $\mu(A) = 0$. The translation invariance of μ implies that $\mu(x^{-1}A) = 0$ holds for each x in G; since $\nu_2 \ll \mu$, the relation $\nu_2(x^{-1}A) = 0$ also holds for each x in G. The definition of $\nu_1 * \nu_2$ now implies that $(\nu_1 * \nu_2)(A) = 0$. Hence $\nu_1 * \nu_2 \in M_a(G)$. The proof that $\nu_2 * \nu_1 \in M_a(G)$ is similar (use Corollary 9.3.7 to conclude that if $\mu(A) = 0$, then $\mu(Ay^{-1}) = 0$ holds for each y in G). Thus $M_a(G)$ is an ideal in $M(G)$.

We already know that the map $f \mapsto \nu_f$ induces a norm-preserving linear map whose image is $M_a(G)$ (Proposition 7.3.10). The calculation

$$\begin{aligned}
\nu_{f*g}(A) &= \int \chi_A(t) \int f(s) g(s^{-1}t)\,\mu(ds)\,\mu(dt) \\
&= \int\int \chi_A(st) f(s) g(t)\,\mu(dt)\,\mu(ds) \\
&= \int\int \chi_A(st)\,\nu_g(dt)\,\nu_f(ds) \\
&= (\nu_f * \nu_g)(A)
\end{aligned}$$

shows that it preserves convolutions. □

Proposition 9.4.7 provides a “coordinate-free” description of $L^1(G, \mathscr{B}(G), \mu)$: it is isomorphic to the algebra $M_a(G)$, whose definition depends only on the existence of Haar measures and not on the choice of a particular left or right Haar measure.

Let us close this section by returning to the map T constructed in Sect. 3.5 (see also Theorem 4.5.1, Example 4.5.2, Theorem 7.5.4, and the remarks following the proof of Theorem 7.5.4).

Theorem 9.4.8. *Let G be a locally compact group, and let μ be a regular Borel measure on G. Then the map T constructed in Sect. 3.5 is an isometric isomorphism of $L^\infty(G, \mathscr{B}(G), \mu)$ onto the dual of $L^1(G, \mathscr{B}(G), \mu)$.*

Proof. According to Proposition 3.5.5 we need only show that T is surjective. So suppose that F belongs to $(L^1(G, \mathscr{B}(G), \mu))^*$.

Let H be a subgroup of G that is open and σ-compact (see Proposition 9.1.8), and let $\mathscr{H}$ be the family of left cosets of H. For each C in $\mathscr{H}$ let $\mathscr{B}(C)$ be the σ-algebra of Borel subsets of C, let μ_C be the restriction of μ to $\mathscr{B}(C)$, and let F_C be the linear functional on $L^1(C, \mathscr{B}(C), \mu_C)$ defined by $F_C(\langle f \rangle) = F(\langle f' \rangle)$ (here f' is the function on G that agrees with f on C and vanishes on C^c). Since μ_C is

It is easy to check that if μ and ν belong to $M(G)$ and if f is a bounded Borel function on G, then

$$\int f\,d(\mu * \nu) = \int\int f(xy)\,\mu(dx)\,\nu(dy) = \int\int f(xy)\,\nu(dy)\,\mu(dx) \tag{5}$$

(first check (5) for characteristic functions, and then use the linearity of the integral and the dominated convergence theorem).

Proposition 9.4.6. *Let G be a locally compact group. Then $M(G)$, with convolution as multiplication, is a Banach algebra.*

Proof. Let ν_1, ν_2, and ν_3 belong to $M(G)$. Then each Borel subset A of G satisfies

$$\begin{aligned}(\nu_1 * (\nu_2 * \nu_3))(A) &= \int (\nu_2 * \nu_3)(x^{-1}A)\,\nu_1(dx)\\ &= \int\int \nu_3(y^{-1}x^{-1}A)\,\nu_2(dy)\,\nu_1(dx)\end{aligned}$$

and

$$\begin{aligned}((\nu_1 * \nu_2) * \nu_3)(A) &= \int \nu_3(u^{-1}A)\,(\nu_1 * \nu_2)(du)\\ &= \int\int \nu_3((xy)^{-1}A)\,\nu_2(dy)\,\nu_1(dx)\end{aligned}$$

(in the last step of this calculation we used (5)). The associativity of convolution follows.

We turn to the inequality $\|\mu * \nu\| \le \|\mu\|\|\nu\|$. Let $\{A_i\}_1^n$ be a finite partition of G into Borel sets. Then Exercise 4.2.8 implies that

$$\begin{aligned}\sum_i |(\mu * \nu)(A_i)| &= \sum_i \left| \int \mu(A_i y^{-1})\,\nu(dy) \right|\\ &\le \int \sum_i |\mu(A_i y^{-1})|\,|\nu|(dy) \le \int \|\mu\|d|\nu| = \|\mu\|\|\nu\|.\end{aligned}$$

Since the partition $\{A_i\}$ was arbitrary, the inequality $\|\mu * \nu\| \le \|\mu\|\|\nu\|$ follows. The remaining conditions in the definition of a Banach algebra are clearly satisfied. □

Let us consider the relationship between the convolution of functions and the convolution of measures. Corollary 9.3.7 implies that an element of $M(G)$ is absolutely continuous with respect to the left Haar measures on G if and only if it is absolutely continuous with respect to the right Haar measures on G. Thus we can define $M_a(G)$ to be the collection of elements of $M(G)$ that are absolutely continuous with respect to some (and hence every) Haar measure on G.

Recall that an *ideal* in an algebra A is a linear subspace I of A such that $u \cdot v$ and $v \cdot u$ belong to I whenever u belongs to I and v belongs to A.

Proposition 9.1.7. *Let G be a topological group, and let H be an open subgroup of G. Then H is closed.*

Proof. The complement of H is the union of the left cosets xH, where x ranges through the complement of H. Proposition 9.1.2 implies that each of these cosets is open. It follows that the complement of H is open and hence that H itself is closed. □

Proposition 9.1.8. *Let G be a locally compact group. Then there is a subgroup H of G that is open, closed, and σ-compact.*

Proof. Since G is locally compact, we can choose an open neighborhood U of e whose closure is compact. Use Proposition 9.1.3 to choose a symmetric open neighborhood V of e that is included in U. Of course V^- is compact. Define sets V^n, $n = 1, 2, \ldots$, inductively by means of the equations $V^1 = V$ and $V^n = V^{n-1}V$, and then define H by $H = \cup_n V^n$. If $x \in V^m$ and $y \in V^n$, then $xy \in V^{m+n}$ and $x^{-1} \in V^m$ (recall that V is symmetric); hence H is a subgroup of G. It is clear that H is open and so also closed (see Exercise 4 and Proposition 9.1.7). Since V^- is compact and H is closed, the closure of each V^n is compact and included in H; the σ-compactness of H follows. □

Exercises

1. Suppose that G is a group and a topological space. Show that G is a topological group if and only if the map $(x,y) \mapsto xy^{-1}$ from $G \times G$ to G is continuous.
2. Let G be $\mathbb{R}$, with addition as the group operation and with the weakest topology that makes each interval of the form $(a,b]$ open. Show that $(x,y) \mapsto x+y$ is continuous, but that $x \mapsto -x$ is not continuous. Thus G is not a topological group.
3. Let G be $\mathbb{R}$, with addition as the group operation and with the topology for which the open sets are those that either are empty or have a countable complement (check that these sets do form a topology on G). Show that
   (a) $x \mapsto -x$ is continuous,
   (b) $(x,y) \mapsto x+y$ is continuous in x when y is held fixed and continuous in y when x is held fixed, and
   (c) $(x,y) \mapsto x+y$ is not continuous.

   Thus G is not a topological group.
4. Let G be a topological group, let U be an open subset of G, and let A be an arbitrary subset of G. Show that AU and UA are open subsets of G. (Hint: Note that $AU = \cup_{a \in A} aU$.)
5. Show that if G_1 and G_2 are topological groups, then $G_1 \times G_2$, with the product topology and with the operation defined by $(x_1,x_2)(y_1,y_2) = (x_1y_1, x_2y_2)$, is a topological group.

6. Let G be a topological group. Show that the following conditions are equivalent:

   (i) The topology of G is Hausdorff.
   (ii) For each a in G the set $\{a\}$ is closed.
   (iii) For some a in G the set $\{a\}$ is closed.

7. Find all closed subgroups of $\mathbb{R}$. In other words, find all subgroups of the additive group $\mathbb{R}$ that are closed in the usual topology for $\mathbb{R}$.
8. Let G be a Hausdorff topological group, and let E and F be subsets of G.

   (a) Show that if E is compact and F is closed, then EF is closed.
   (b) Show by example that if E and F are closed (but not compact), then EF can fail to be closed. (Hint: Such examples can be found in the case where $G = \mathbb{R}$.)

9. Let G consist of the 2 by 2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where a is a positive real number and b is an arbitrary real number. Show that G, with the operation of matrix multiplication and with the topology it inherits as a subspace of $\mathbb{R}^4$, is a locally compact group.
10. Let $GL(d,\mathbb{R})$ be the collection of all invertible d by d matrices with real entries. Show that $GL(d,\mathbb{R})$, with the operation of matrix multiplication and with the topology it inherits as a subspace of $\mathbb{R}^{d^2}$, is a locally compact group (it is called the *general linear group*). (Hint: See Lemma 6.1.2, and recall how Cramer's rule for the solution of systems of linear equations gives an explicit formula for the inverse of a matrix.)
11. Let $O(d)$ be the collection of all orthogonal[1] d by d matrices. Show that $O(d)$, with the operation of matrix multiplication and with the topology it inherits as a subspace of $\mathbb{R}^{d^2}$, is a compact group (it is called the *orthogonal group*).
12. Let G be the locally compact group introduced in Exercise 9. Construct a real-valued function on G that is right uniformly continuous, but not left uniformly continuous. (Hint: Consider

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \varphi(b),$$

    where φ is a suitable function from $\mathbb{R}$ to $\mathbb{R}$.)
13. Derive Proposition 9.1.5 from Proposition 9.1.4. (Hint: Suppose that f belongs to $\mathscr{K}(G)$. Consider the group $G \times G$ and the sets K and U defined by $K = \{(x,x) : x \in \text{supp}(f)\}$ and $U = \{(x,y) : |f(x) - f(y)| < \varepsilon\}$.)

[1]Recall that a square matrix with real entries is *orthogonal* if the product of it with its transpose is the identity matrix.

are dense in $L^1(G, \mathscr{B}(G), \mu)$ (Proposition 7.4.3), the associative law follows (see Exercise 2). □

Let us turn to the convolution of measures. We begin with the following lemma.

Lemma 9.4.5. *Let G be a locally compact group. If μ and ν are finite positive regular Borel measures on G and if $\mu \times \nu$ is the regular Borel product of μ and ν, then the formula*

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(x,y) \in G \times G : xy \in A\})$$

defines a regular Borel measure on G. Furthermore,

$$(\mu * \nu)(A) = \int \nu(x^{-1}A)\,\mu(dx) = \int \mu(Ay^{-1})\,\nu(dy) \tag{2}$$

holds for each A in $\mathscr{B}(G)$.

Note that Corollary 7.6.6 implies that the functions appearing on the right side of (2) are Borel measurable.

Proof. Let $F\colon G \times G \to G$ be the group operation (in other words, define F by $F(x,y) = xy$). Then $\mu * \nu$ is given by the equation $(\mu * \nu)(A) = (\mu \times \nu)(F^{-1}(A))$, and so is a measure on $\mathscr{B}(G)$ (see Sect. 2.6). Corollary 7.6.6 implies that each A in $\mathscr{B}(G)$ satisfies (2). We need to check the regularity of $\mu * \nu$.

We begin by checking that an arbitrary Borel subset A of G satisfies

$$(\mu * \nu)(A) = \sup\{(\mu * \nu)(K) : K \subseteq A \text{ and } K \text{ is compact}\}. \tag{3}$$

Suppose that ε is a positive number, that K_0 is a compact subset of $F^{-1}(A)$ such that $(\mu \times \nu)(K_0) > (\mu \times \nu)(F^{-1}(A)) - \varepsilon$ (see Proposition 7.2.6), and that $K = F(K_0)$. Then K is a compact subset of A such that $F^{-1}(K) \supseteq K_0$ and hence such that $(\mu * \nu)(K) > (\mu * \nu)(A) - \varepsilon$. Since ε is arbitrary, (3) follows. In particular, $\mu * \nu$ is inner regular. Since for each A in $\mathscr{B}(G)$ we can use (3), applied to A^c, to approximate A^c from below by compact sets and hence to approximate A from above by open sets, the outer regularity of $\mu * \nu$ follows. □

Recall that $M_r(G,\mathbb{R})$ is the Banach space of all finite signed regular Borel measures on G (the norm of μ is the total variation of μ). Likewise, $M_r(G,\mathbb{C})$ is the Banach space of all complex regular Borel measures on G. Here we will denote each of those spaces by $M(G)$.

Let μ and ν belong to $M(G)$. We define their *convolution* $\mu * \nu$ by

$$(\mu * \nu)(A) = \int \nu(x^{-1}A)\,\mu(dx) = \int \mu(Ay^{-1})\,\nu(dy). \tag{4}$$

It follows from the preceding lemma and the Jordan decomposition theorem that the two integrals appearing in (4) exist and are equal, and that $\mu * \nu$ is regular. Thus $\mu * \nu \in M(G)$.

(see also Exercise 4). Thus convolution on $\mathscr{L}^1(G,\mathscr{B}(G),\mu)$ induces an operation on $L^1(G,\mathscr{B}(G),\mu)$; this operation is also denoted by $*$ and called *convolution*.

We will show that $L^1(G,\mathscr{B}(G),\mu)$, with convolution as multiplication, is a Banach algebra. (This Banach algebra is often denoted by $L^1(G)$.) Recall that an *algebra* is a vector space A on which there is defined an operation $\cdot$ (called multiplication) for which the identities

$$\begin{aligned} u\cdot(v\cdot w) &= (u\cdot v)\cdot w, \\ u\cdot(v+w) &= u\cdot v+u\cdot w, \\ (u+v)\cdot w &= u\cdot w+v\cdot w, \text{ and} \\ \alpha(u\cdot v) &= (\alpha u)\cdot v = u\cdot(\alpha v) \end{aligned}$$

hold for all u, v, and w in A and all scalars α. A *Banach algebra* is an algebra for which

(a) the underlying vector space has the structure of a Banach space, say with norm $\|\cdot\|$, and
(b) the relation $\|u\cdot v\| \le \|u\|\|v\|$ holds for all u and v in A.

Proposition 9.4.4. *Let G be a locally compact group, and let μ be a left Haar measure on G. Then $L^1(G,\mathscr{B}(G),\mu)$, with convolution as multiplication, is a Banach algebra.*

Proof. With the exception of the associative law for convolutions, the conditions that define a Banach algebra are either immediate or given by Theorem 3.4.1 and Proposition 9.4.1.

We turn to the associative law. Suppose that f, g, and h belong to $\mathscr{K}(G)$ (or to $\mathscr{K}^{\mathbb{C}}(G)$) and that x belongs to G. Then the functions involved in computing $f*(g*h)$ and $(f*g)*h$ are all integrable, and these convolutions are given by

$$\begin{aligned} (f*(g*h))(x) &= \int f(s)(g*h)(s^{-1}x)\,\mu(ds) \\ &= \int\int f(s)g(t)h(t^{-1}s^{-1}x)\,\mu(dt)\,\mu(ds) \end{aligned}$$

and

$$\begin{aligned} ((f*g)*h)(x) &= \int (f*g)(t)h(t^{-1}x)\,\mu(dt) \\ &= \int\int f(s)g(s^{-1}t)h(t^{-1}x)\,\mu(ds)\,\mu(dt). \end{aligned}$$

Consider the last of these integrals; in it reverse the order of integration and use the translation invariance of μ to replace t with st. It follows that $(f*(g*h))(x) = ((f*g)*h)(x)$. Thus the associative law holds for those elements of $L^1(G,\mathscr{B}(G),\mu)$ that are determined by functions in $\mathscr{K}(G)$ (or in $\mathscr{K}^{\mathbb{C}}(G)$); since these elements

9.2 The Existence and Uniqueness of Haar Measure

Let G be a locally compact group, and let μ be a nonzero regular Borel measure on G. Then μ is a *left Haar measure* (or simply a *Haar measure*) if it is *invariant under left translations* (or simply *translation invariant*), in the sense that $\mu(xA) = \mu(A)$ holds for each x in G and each A in $\mathscr{B}(G)$. Likewise, μ is a *right Haar measure* if $\mu(Ax) = \mu(A)$ holds for each x in G and each A in $\mathscr{B}(G)$. (Lemma 7.2.1 and Proposition 9.1.2 imply that if $x \in G$ and if A is a Borel subset of G, then xA and Ax are Borel subsets of G; hence the expressions $\mu(xA)$ and $\mu(Ax)$ appearing in the preceding definition are meaningful.)

In this section we prove that there is a left Haar measure on each locally compact group and that it is unique up to multiplication by a constant. A few properties of Haar measures, plus the relationship between left and right Haar measures, will be dealt with in Sect. 9.3. In Sect. 9.4 we will use these results to discuss some measure-theoretic tools for harmonic analysis.

Examples 9.2.1.

(a) Lebesgue measure on $\mathbb{R}$ (or on $\mathbb{R}^d$) is a left and a right Haar measure; see Proposition 1.4.4.
(b) If G is a group with the discrete topology (that is, with the topology that makes every subset of G open), then counting measure on G is a left and a right Haar measure; in particular, counting measure on the group $\mathbb{Z}$ of integers is a Haar measure.
(c) Let $\mathbb{T}$ be the set of complex numbers z such that $|z| = 1$, made into a topological group as in Example 9.1.1(d) in the previous section. Then linear Lebesgue measure on $\mathbb{T}$ is a Haar measure. More precisely, if λ_0 is Lebesgue measure on $\mathbb{R}$, restricted to the Borel subsets of the interval $[0, 2\pi)$, and if $F\colon [0, 2\pi) \to \mathbb{T}$ is defined by $F(\theta) = e^{i\theta}$, then $\lambda_0 F^{-1}$ is a left and a right Haar measure on $\mathbb{T}$. □

See Exercises 3 and 5 below and also Exercises 4 and 6 in Sect. 9.3, for additional examples of Haar measures.

We need a bit of notation. Let G be a group, let x be an element of G, and let f be a function on G. The *left translate of f by x*, written ${}_xf$, is defined by ${}_xf(t) = f(x^{-1}t)$, and the *right translate of f by x*, written f_x, is defined by $f_x(t) = f(tx^{-1})$. The function $\check{f}$ (or $f\check{}$) is defined by $\check{f}(t) = f(t^{-1})$. Note that if x, y, and t belong to G, then

$$ {}_{xy}f(t) = f((xy)^{-1}t) = f(y^{-1}x^{-1}t) = {}_yf(x^{-1}t) = {}_x({}_yf)(t); $$

hence

$$ {}_{xy}f = {}_x({}_yf). $$

A similar argument shows that

$$ f_{xy} = (f_x)_y. $$

If A is a subset of G, then the characteristic functions of the sets A, xA, and Ax are related by the identities

$$(\chi_A)_x = \chi_{Ax}$$

and

$$_x(\chi_A) = \chi_{xA}.$$

This gives one reason for defining $_xf(t)$ and $f_x(t)$ to be $f(x^{-1}t)$ and $f(tx^{-1})$, rather than $f(xt)$ and $f(tx)$. (The definitions of $_xf$ and f_x are not entirely standard; some authors use $f(xt)$ and $f(tx)$ where we used $f(x^{-1}t)$ and $f(tx^{-1})$.)

If G is a locally compact group and if μ is a left Haar measure on G, then

$$\int {_xf}\, d\mu = \int f\, d\mu \tag{1}$$

holds for each Borel function f that is either nonnegative or μ-integrable (note that $\int {_xf}\, d\mu = \mu(xA) = \mu(A) = \int f\, d\mu$ holds if f is the characteristic function of the Borel set A, and then use the linearity of the integral and the monotone convergence theorem).

Theorem 9.2.2. *Let G be a locally compact group. Then there is a left Haar measure on G.*

Proof. Let K be a compact subset of G, and let V be a subset of G whose interior V^o is nonempty. Then $\{xV^o\}_{x\in G}$ is an open cover of the compact set K, and so there are finite sequences $\{x_i\}_{i=1}^n$ of elements of G such that $K \subseteq \cup_{i=1}^n x_iV$. Let $\#(K:V)$ be the smallest nonnegative integer n for which such a sequence $\{x_i\}_{i=1}^n$ exists. Of course, $\#(K:V) = 0$ if and only if $K = \varnothing$.

Let us choose a compact set K_0 whose interior is nonempty; it will serve as a standard for measuring the sizes of various subsets of G and will remain fixed throughout this proof. Roughly speaking, we will measure the size of an arbitrary compact subset K of G by computing the ratio $\#(K:U)/\#(K_0:U)$ for each open neighborhood U of e and then finding a sort of limit of this ratio as the neighborhood U becomes smaller. We will use this “limit” to construct an outer measure μ^* on G, and then we will show that the restriction of μ^* to $\mathscr{B}(G)$ is the required measure.

We turn to the details. Let $\mathscr{C}$ be the family of all compact subsets of G, and let $\mathscr{U}$ be the family of all open neighborhoods of e. For each U in $\mathscr{U}$ define $h_U : \mathscr{C} \to \mathbb{R}$ by $h_U(K) = \#(K:U)/\#(K_0:U)$.

Lemma 9.2.3. *The relations*

(a) $0 \le h_U(K) \le \#(K:K_0)$,
(b) $h_U(K_0) = 1$,
(c) $h_U(xK) = h_U(K)$,
(d) $h_U(K_1) \le h_U(K_2)$ *if* $K_1 \subseteq K_2$,
(e) $h_U(K_1 \cup K_2) \le h_U(K_1) + h_U(K_2)$, *and*
(f) $h_U(K_1 \cup K_2) = h_U(K_1) + h_U(K_2)$ *if* $K_1U^{-1} \cap K_2U^{-1} = \varnothing$

hold for all U, K, K_1, K_2, and x.

We need the following two lemmas for the proof of Proposition 9.4.1.

Lemma 9.4.2. *Let G be a locally compact group, let μ be a left Haar measure on G, and let f belong to $\mathscr{L}^1(G,\mathscr{B}(G),\mu)$. Then there is a sequence $\{K_n\}$ of compact subsets of G such that f vanishes outside $\bigcup_n K_n$.*

Proof. We can use Corollary 2.3.11 and the regularity of μ to produce a sequence $\{U_n\}$ of open subsets of G that have finite measure under μ and are such that f vanishes outside $\bigcup_n U_n$. Let H be a subgroup of G that is open and σ-compact (see Proposition 9.1.8). Since each nonempty open subset of G has nonzero measure under μ (Lemma 9.2.5), it follows that each U_n meets at most countably many left cosets of H and hence that $\bigcup_n U_n$ is included in the union of a countable collection of left cosets of H. Since H, along with each of its cosets, is σ-compact, the lemma follows. □

Lemma 9.4.3. *Let G be a locally compact group, let μ be a left Haar measure on G, and let $F\colon G\times G\to G\times G$ be defined by $F(s,t)=(s,s^{-1}t)$. Then F is a measure-preserving homeomorphism of $G\times G$ onto itself. That is, F is a homeomorphism such that each Borel subset A of $G\times G$ satisfies $(\mu\times\mu)(A)=(\mu\times\mu)(F^{-1}(A))$.*

Proof. The inverse of F is given by $F^{-1}(s,t)=(s,st)$; thus F and F^{-1} are both continuous, and F is a homeomorphism. The regularity of the measure $(\mu\times\mu)F^{-1}$ follows. Now suppose that U is an open subset of $G\times G$. For each s in G the sections U_s and $(F^{-1}(U))_s$ are related by $(F^{-1}(U))_s = sU_s$, and so Proposition 7.6.5 and the translation invariance of μ imply that $(\mu\times\mu)(U)=(\mu\times\mu)(F^{-1}(U))$. It follows from this and the regularity of the measures $\mu\times\mu$ and $(\mu\times\mu)F^{-1}$ that $(\mu\times\mu)(A)=(\mu\times\mu)(F^{-1}(A))$ holds for each A in $\mathscr{B}(G\times G)$. □

Proof of Proposition 9.4.1. It follows from Exercise 7.6.4 that the function $(s,t)\mapsto f(s)g(t)$ belongs to $\mathscr{L}^1(G\times G,\mathscr{B}(G\times G),\mu\times\mu)$ and then from Lemma 9.4.3 that the function $(s,t)\mapsto f(s)g(s^{-1}t)$ belongs to $\mathscr{L}^1(G\times G,\mathscr{B}(G\times G),\mu\times\mu)$ (see Sect. 2.6). Since in addition $(s,t)\mapsto f(s)g(s^{-1}t)$ vanishes outside a σ-compact set (apply Lemma 9.4.2 to f and g, and then use Lemma 9.4.3), Theorem 7.6.7 implies part (a) and the first half of part (b). The second half of part (b) follows from the calculation

$$\begin{aligned}\int |(f*g)(t)|\,\mu(dt) &\le \int\int |f(s)g(s^{-1}t)|\,\mu(ds)\,\mu(dt)\\ &= \int\int |f(s)g(t)|\,\mu(dt)\,\mu(ds) = \|f\|_1\|g\|_1.\end{aligned}$$

□

Note that if f_1, f_2, g_1, and g_2 belong to $\mathscr{L}^1(G,\mathscr{B}(G),\mu)$, if $f_1=f_2$ μ-a.e., and if $g_1=g_2$ μ-a.e., then $f_1*g_1=f_2*g_2$ μ-a.e.; this follows, for example, from the calculation

$$\begin{aligned}\|f_1*g_1-f_2*g_2\|_1 &\le \|f_1*(g_1-g_2)\|_1+\|(f_1-f_2)*g_2\|_1\\ &\le \|f_1\|_1\|g_1-g_2\|_1+\|f_1-f_2\|_1\|g_2\|_1 = 0\end{aligned}$$

9.4 The Algebras $L^1(G)$ and $M(G)$

Since most of the topics dealt with in this section involve measures and integrals on products of locally compact groups, we begin by recalling some of the necessary facts.

Suppose that X and Y are locally compact Hausdorff spaces and that μ and ν are regular Borel measures on X and Y, respectively. If X and Y have countable bases for their topologies, then $\mathscr{B}(X \times Y)$ is equal to $\mathscr{B}(X) \times \mathscr{B}(Y)$, μ and ν are σ-finite, and the product measure $\mu \times \nu$ (as defined in Sect. 5.1) is a regular Borel measure (see Proposition 7.6.2). Thus the theory of product measures contained in Chap. 5 is adequate for the study of products of regular Borel measures on second countable locally compact Hausdorff spaces.[3]

We dealt with products of arbitrary locally compact Hausdorff spaces in Sect. 7.6; there we showed that if μ and ν are regular Borel measures on X and Y, then

$$\int\int f(x,y)\,\mu(dx)\,\nu(dy) = \int\int f(x,y)\,\nu(dy)\,\mu(dx)$$

holds for each f in $\mathscr{K}(X \times Y)$, and we used the Riesz representation theorem (applied to the functional $f \mapsto \int\int f(x,y)\,\mu(dx)\,\nu(dy)$) to construct a regular Borel measure $\mu \times \nu$ on $X \times Y$ such that

$$\int f\,d(\mu \times \nu) = \int\int f(x,y)\,\mu(dx)\,\nu(dy) = \int\int f(x,y)\,\nu(dy)\,\mu(dx) \qquad (1)$$

holds for each f in $\mathscr{K}(X \times Y)$. We proved that (1) also holds for many other functions on $X \times Y$ (see Theorem 7.6.7 and Exercises 7.6.3 and 7.6.4).

Now let G be an arbitrary locally compact group, let μ be a left Haar measure on G, and let f and g belong to $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$. The *convolution* of f and g is the function $f * g$ from G to $\mathbb{R}$ (or to $\mathbb{C}$) defined by

$$(f * g)(t) = \begin{cases} \int f(s)g(s^{-1}t)\,\mu(ds) & \text{if } s \mapsto f(s)g(s^{-1}t) \text{ is integrable,} \\ 0 & \text{otherwise.} \end{cases}$$

Some basic properties of convolutions are given by the following propositions.

Proposition 9.4.1. *Let G be a locally compact group, let μ be a left Haar measure on G, and let f and g belong to $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$.*

(a) *The function $s \mapsto f(s)g(s^{-1}t)$ belongs to $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$ for μ-almost every t in G.*
(b) *The convolution $f * g$ of f and g belongs to $\mathscr{L}^1(G, \mathscr{B}(G), \mu)$ and satisfies $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.*

[3] In particular, the reader who is interested only in second countable locally compact groups can ignore the references to Sect. 7.6 in what follows.

Proof. The relation

$$\#(K:U) \le \#(K:K_0)\#(K_0:U) \tag{2}$$

holds for all K and U, as we can see by noting that if $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ are sequences in G such that $K \subseteq \cup_{i=1}^m x_i K_0$ and $K_0 \subseteq \cup_{j=1}^n y_j U$, then $K \subseteq \cup_{i=1}^m \cup_{j=1}^n x_i y_j U$. Dividing both sides of (2) by $\#(K_0:U)$ gives assertion (a). Assertions (b), (c), (d), and (e) are clear. In view of (e), we can prove (f) by checking that

$$\#(K_1 \cup K_2 : U) \ge \#(K_1:U) + \#(K_2:U) \tag{3}$$

holds whenever

$$K_1 U^{-1} \cap K_2 U^{-1} = \varnothing. \tag{4}$$

So suppose that (4) holds and that $\{x_i\}_{i=1}^n$ is a sequence of points such that $n = \#(K_1 \cup K_2 : U)$ and $K_1 \cup K_2 \subseteq \cup_{i=1}^n x_i U$. Then each set $x_i U$ meets at most one of K_1 and K_2 (for if $x_i U$ met both K_1 and K_2, then x_i would belong to $K_1 U^{-1} \cap K_2 U^{-1}$), and so we can partition the sequence $\{x_i\}_{i=1}^n$ into sequences $\{y_i\}_{i=1}^j$ and $\{z_i\}_{i=1}^k$ such that $K_1 \subseteq \cup_{i=1}^j y_i U$ and $K_2 \subseteq \cup_{i=1}^k z_i U$. Relation (3) and part (f) of the lemma follow. □

We now turn to the "limit" of the ratios $\#(K:U)/\#(K_0:U)$—that is, of the functions $\{h_U\}_{U \in \mathscr{U}}$. We will find this "limit" by constructing a certain product space that contains all the functions h_U and then using a compactness argument to produce the "limit" function.

For each K in $\mathscr{C}$ let I_K be the subinterval $[0, \#(K:K_0)]$ of $\mathbb{R}$. Let X be the product space $\prod_{K \in \mathscr{C}} I_K$, endowed with the product topology. Since each interval I_K is compact, Tychonoff's theorem (Theorem D.20) implies that X is compact. According to part (a) of Lemma 9.2.3, each function h_U belongs to X. For each open neighborhood V of e let $S(V)$ be the closure in X of the set $\{h_U : U \in \mathscr{U} \text{ and } U \subseteq V\}$. If $V_1, \ldots, V_n$ belong to $\mathscr{U}$ (that is, if they are open neighborhoods of e) and if V is defined by $V = \cap_{i=1}^n V_i$, then $h_V \in \cap_{i=1}^n S(V_i)$; since $V_1, \ldots, V_n$ were arbitrary, this implies that the closed sets $\{S(V)\}_{V \in \mathscr{U}}$ satisfy the finite intersection property. The compactness of X now implies that $\cap_{V \in \mathscr{U}} S(V)$ is nonempty. Let us choose, once and for all, an element $h_\bullet$ of $\cap_{V \in \mathscr{U}} S(V)$. This function $h_\bullet$ is our "limit" of the functions h_U.

Lemma 9.2.4. *The function $h_\bullet$ satisfies*

(a) $0 \le h_\bullet(K)$,
(b) $h_\bullet(\varnothing) = 0$,
(c) $h_\bullet(K_0) = 1$,
(d) $h_\bullet(xK) = h_\bullet(K)$,
(e) $h_\bullet(K_1) \le h_\bullet(K_2)$ *if* $K_1 \subseteq K_2$,
(f) $h_\bullet(K_1 \cup K_2) \le h_\bullet(K_1) + h_\bullet(K_2)$, *and*
(g) $h_\bullet(K_1 \cup K_2) = h_\bullet(K_1) + h_\bullet(K_2)$ *if* $K_1 \cap K_2 = \varnothing$

for all x in G and all K, K_1, and K_2 in $\mathscr{C}$.

Proof. Let us begin with part (f). Recall that X, as the product space $\prod_{K \in \mathscr{C}} I_K$, is a certain set of functions on $\mathscr{C}$, with its topology defined so that for each compact subset K of G (i.e., for each element K of the index set $\mathscr{C}$) the projection from X to $\mathbb{R}$ defined by $h \mapsto h(K)$ is continuous. Hence for each choice of compact subsets K_1 and K_2 of G the map from X to $\mathbb{R}$ defined by

$$h \mapsto h(K_1) + h(K_2) - h(K_1 \cup K_2) \tag{5}$$

is continuous. Since this map is, in addition, nonnegative at each h_U (see part (e) of Lemma 9.2.3), it is nonnegative at each point in each set $S(V)$. In particular, it is nonnegative at $h_\bullet$, and so part (f) is proved.

Property (a) is clear, and properties (b) through (e) can be proved with arguments similar to the one given above for part (f). We turn to part (g). Suppose that K_1 and K_2 are disjoint compact subsets of G. According to Proposition 7.1.2 there are disjoint open sets U_1 and U_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$, and according to Proposition 9.1.4 there are open neighborhoods V_1 and V_2 of e such that $K_1V_1 \subseteq U_1$ and $K_2V_2 \subseteq U_2$. Let $V = V_1 \cap V_2$. Then K_1V and K_2V are disjoint, and so for each U that belongs to $\mathscr{U}$ and satisfies $U \subseteq V^{-1}$ we have

$$h_U(K_1 \cup K_2) = h_U(K_1) + h_U(K_2)$$

(see part (f) of Lemma 9.2.3). Consequently the map defined by (5) vanishes at each element of $S(V^{-1})$. Since $h_\bullet \in S(V^{-1})$, part (g) follows. □

Let us return to the proof of Theorem 9.2.2. We are now in a position to construct the promised outer measure on G. Define μ^* on the collection of open subsets of G by

$$\mu^*(U) = \sup\{h_\bullet(K) : K \subseteq U \text{ and } K \in \mathscr{C}\}, \tag{6}$$

and extend it to the collection of all subsets of G by

$$\mu^*(A) = \inf\{\mu^*(U) : A \subseteq U \text{ and } U \text{ is open}\}. \tag{7}$$

It is clear that μ^* is nonnegative, that it is monotone, and that $\mu^*(\varnothing) = 0$.

In view of (7), we can verify the countable subadditivity of μ^* by checking that each sequence $\{U_i\}$ of open subsets of G satisfies

$$\mu^*\Big(\bigcup_i U_i\Big) \leq \sum_i \mu^*(U_i). \tag{8}$$

So suppose that $\{U_i\}$ is a sequence of open subsets of G. Let K be a compact subset of $\cup_i U_i$. Then there is a positive integer n such that $K \subseteq \cup_{i=1}^n U_i$, and there are compact subsets $K_1, \ldots, K_n$ of $U_1, \ldots, U_n$ such that $K = \cup_{i=1}^n K_i$ (use Lemma 7.1.10 and mathematical induction). It follows that

2. Let G be the group considered in Exercises 9.1.9 and 9.2.5, and let Δ be the modular function of G. Show that $\Delta\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = 1/a$ holds for each $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ in G.
3. Let G be as in the preceding exercise. Find a Borel subset of G that has finite measure under the left Haar measures on G but infinite measure under the right Haar measures on G.
4. Show that the formula

$$\mu(A) = \int_A \frac{1}{|\det(u)|^d}\,\lambda(du),$$

where λ is Lebesgue measure on $\mathbb{R}^{d^2}$, defines a left and a right Haar measure on $GL(d,\mathbb{R})$. Hence $GL(d,\mathbb{R})$ is unimodular (note, however, that it is neither compact nor abelian). (Hint: See Exercise 9.2.4.)
5. Let G be a locally compact group and let μ be a left Haar measure on G. Show that G is unimodular if and only if $\mu = \check{\mu}$.
6. Let H be $\{0,1\}$, with the discrete topology and with addition modulo 2 as the group operation. Let G be $H^{\mathbb{N}}$, with the product topology and with the group operation defined component-by-component in terms of the operation on H.
   (a) Show that G is a compact group.
   (b) Let μ be the Haar measure on G for which $\mu(G) = 1$ (see Proposition 9.3.3 and the remark following it). Show that

$$\mu(\{\{a_j\} \in G : a_{n_i} = b_i \text{ for } i = 1, \dots, k\}) = \frac{1}{2^k}$$

   holds for each sequence $n_1, \dots, n_k$ of distinct positive integers and each sequence $b_1, \dots, b_k$ of elements of $\{0,1\}$.
   (c) Show that there are compact subsets K and L of G such that $\mu(K) = \mu(L) = 0$, but $KL = G$.
   (d) Let $f\colon G \to [0,1]$ be the map that takes the sequence $\{a_i\}$ to the number $\sum_{i=1}^{\infty} a_i 2^{-i}$. Show that $\lambda(B) = \mu(f^{-1}(B))$ holds for each Borel subset B of $[0,1]$.
7. Let G be a locally compact group, and let μ be a left Haar measure on G. Show that μ is σ-finite if and only if G is σ-compact.
8. Let G be a locally compact group that is not unimodular, let μ be a left Haar measure on G, and let ν be a right Haar measure on G. Show that there is a Borel subset A of G such that $\mu(A) < +\infty$ and $\nu(A) = +\infty$. (Hint: See Proposition 9.3.6 or Exercise 9.3.1.)
9. Let G be a locally compact group, let μ be a left Haar measure on G, and let ν be a right Haar measure on G. Suppose that outer measures μ^* and ν^* and measures μ_1 and ν_1 are associated to μ and ν as in Sect. 7.5.
   (a) Show that $\mathscr{M}_{\mu^*} = \mathscr{M}_{\nu^*}$.
   (b) Show that a subset of G is locally μ_1-null if and only if it is locally ν_1-null.

proved. It is easy to see that each compact subset K of G satisfies $\nu(K) < +\infty$ (note that $\mu(K)$ is finite and that the function $x \mapsto \Delta(x^{-1})$ is bounded on K). With this the proof of the regularity of ν is complete.

Since ν is regular and nonzero, the calculation

$$\begin{aligned}
\nu(Ay) &= \int \chi_{A_y}(x)\Delta(x^{-1})\,\mu(dx) \\
&= \int \chi_{A_y}(x)\Delta(y^{-1})\Delta((xy^{-1})^{-1})\,\mu(dx) \\
&= \Delta(y^{-1}) \int (\chi_A)_y(x)\Delta((xy^{-1})^{-1})\,\mu(dx) \\
&= \Delta(y^{-1})\Delta(y) \int \chi_A(x)\Delta(x^{-1})\,\mu(dx) \\
&= \nu(A)
\end{aligned}$$

(here we used (2) and part (b) of Proposition 9.3.4) implies that ν is a right Haar measure.

Thus there is a positive number c such that $\nu = c\check{\mu}$ (see Proposition 9.3.1 and Corollary 9.3.2), and so

$$c = \frac{\nu(A)}{\check{\mu}(A)} = \frac{\nu(A)}{\mu(A^{-1})} = \frac{1}{\mu(A^{-1})} \int_A \Delta(x^{-1})\,\mu(dx)$$

holds whenever A is a Borel set that satisfies $0 < \check{\mu}(A) < +\infty$. Since Δ is continuous and has value 1 at e, we can make the right side of the equation arbitrarily close to 1 by letting A be a sufficiently small symmetric neighborhood of e. Thus $c = 1$, and so $\nu = \check{\mu}$. □

Corollary 9.3.7. *Let G be a locally compact group, let μ be a left Haar measure on G, and let ν be a right Haar measure on G. Then a Borel subset A of G satisfies $\mu(A) = 0$ if and only if it satisfies $\nu(A) = 0$.*

Proof. The formula $A \mapsto \int_A \Delta(t^{-1})\,\mu(dt)$ defines a right Haar measure on G (Proposition 9.3.6), and so there is a positive constant c such that for each A in $\mathscr{B}(G)$ we have $\nu(A) = c\int_A \Delta(t^{-1})\,\mu(dt)$. Since Δ is positive everywhere on G, it follows that A satisfies $\nu(A) = 0$ if and only if it satisfies $\mu(A) = 0$ (see Corollary 2.3.12). □

Exercises

1. Let G be a locally compact group, and let μ be a right Haar measure on G. Show that $\mu(xA) = \Delta(x^{-1})\mu(A)$ holds for each x in G and each A in $\mathscr{B}(G)$.

$$h_\bullet(K) \le \sum_{i=1}^{n} h_\bullet(K_i) \le \sum_{i=1}^{n} \mu^*(U_i) \le \sum_{i=1}^{\infty} \mu^*(U_i)$$

(see Lemma 9.2.4 and Eq. (6)); since K was an arbitrary compact subset of $\cup_i U_i$, another application of (6) gives (8).

We can prove that each Borel subset of G is μ^*-measurable by verifying that if U and V are open subsets of G and if $\mu^*(V) < +\infty$, then

$$\mu^*(V) \ge \mu^*(V \cap U) + \mu^*(V \cap U^c) \tag{9}$$

(see the proof of Proposition 7.2.9). We proceed to check this inequality. Let ε be a positive number. Choose a compact subset K of $V \cap U$ such that

$$h_\bullet(K) > \mu^*(V \cap U) - \varepsilon, \tag{10}$$

and then choose a compact subset L of $V \cap K^c$ such that $h_\bullet(L) > \mu^*(V \cap K^c) - \varepsilon$. Then K and L are disjoint, and, since $V \cap U^c \subseteq V \cap K^c$, L satisfies

$$h_\bullet(L) > \mu^*(V \cap U^c) - \varepsilon. \tag{11}$$

It follows from these inequalities and Lemma 9.2.4 that

$$h_\bullet(K \cup L) = h_\bullet(K) + h_\bullet(L) \ge \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon.$$

Since ε is arbitrary and $h_\bullet(K \cup L) \le \mu^*(V)$, inequality (9) follows. Consequently $\mathscr{B}(G)$ is included in the σ-algebra of μ^*-measurable sets, and the restriction of μ^* to $\mathscr{B}(G)$ is a measure (Theorem 1.3.6). Call this measure μ.

We turn to the regularity of μ. Note that if K is compact, if U is open, and if $K \subseteq U$, then $h_\bullet(K) \le \mu(U)$. It follows from this and (7) that

$$h_\bullet(K) \le \mu(K). \tag{12}$$

Furthermore, if K is a compact set and U is an open set that includes K and has a compact closure (see Proposition 7.1.4), then

$$h_\bullet(L) \le h_\bullet(U^-)$$

holds for each compact subset L of U, and so

$$\mu(K) \le \mu(U) \le h_\bullet(U^-).$$

It follows that μ is finite on the compact subsets of G. The outer regularity of μ follows from (7), and the inner regularity follows from (6) and (12).

It is easy to check that μ is nonzero and translation-invariant (use Lemma 9.2.4 and Eqs. (6) and (7)). Thus μ is the required measure. □

The following lemma gives a fundamental elementary property of Haar measures; we will need it for our proof of Theorem 9.2.6.

Lemma 9.2.5. *Let G be a locally compact group, and let μ be a left Haar measure on G. Then each nonempty open subset U of G satisfies $\mu(U) > 0$, and each nonnegative function f that belongs to $\mathscr{K}(G)$ and is not identically zero satisfies $\int f\,d\mu > 0$.*

Proof. Since μ is regular and not the zero measure, we can choose a compact set K such that $\mu(K) > 0$. Let U be a nonempty open subset of G. Then $\{xU\}_{x\in G}$ is an open cover of the compact set K, and so there is a finite collection, say $x_1, \ldots, x_n$, of elements of G such that the sets x_iU, $i = 1, \ldots, n$, cover K. The relation $\mu(K) \le \sum_i \mu(x_iU)$ and the translation invariance of μ imply that $\mu(K) \le n\mu(U)$ and hence that $\mu(U) > 0$. Thus the first half of the lemma is proved.

Now suppose that f is a nonnegative function that belongs to $\mathscr{K}(G)$ and does not vanish identically. Then there is a positive number ε and a nonempty open set U such that $f \ge \varepsilon\chi_U$. It follows that $\int f\,d\mu \ge \varepsilon\mu(U) > 0$. □

Theorem 9.2.6. *Let G be a locally compact group, and let μ and ν be left Haar measures on G. Then there is a positive real number c such that $\nu = c\mu$.*

Proof. Let g be a nonnegative function that belongs to $\mathscr{K}(G)$ and does not vanish identically (g will be held fixed throughout this proof), and let f be an arbitrary function in $\mathscr{K}(G)$. Since $\int g\,d\mu \ne 0$ (Lemma 9.2.5), we can form the ratio $\int f\,d\mu \,/ \int g\,d\mu$. We will show that this ratio depends only on the functions f and g and not on the particular Haar measure μ used in its computation. It follows that the Haar measure ν satisfies

$$\frac{\int f\,d\nu}{\int g\,d\nu} = \frac{\int f\,d\mu}{\int g\,d\mu}$$

and hence satisfies $\int f\,d\nu = c\int f\,d\mu$, where c is defined by $c = \int g\,d\nu \,/ \int g\,d\mu$. Since this holds for each f in $\mathscr{K}(G)$, Theorem 7.2.8 implies that $\nu = c\mu$.

We turn to the ratio $\int f\,d\mu \,/ \int g\,d\mu$. If $h \in \mathscr{K}(G\times G)$, then Proposition 7.6.4 implies that the iterated integrals $\int\int h(x,y)\,\mu(dx)\,\nu(dy)$ and $\int\int h(x,y)\,\nu(dy)\,\mu(dx)$ exist and are equal. If in the second of these integrals we reverse the order of integration, use the translation invariance of the Haar measure μ to replace x with $y^{-1}x$ (see (1)), again reverse the order of integration, and finally replace y with xy, we find that

$$\begin{aligned}\int\int h(x,y)\,\nu(dy)\,\mu(dx) &= \int\int h(y^{-1}x,y)\,\mu(dx)\,\nu(dy)\\ &= \int\int h(y^{-1},xy)\,\nu(dy)\,\mu(dx).\end{aligned} \tag{13}$$

Let us apply this identity to the function h defined by

$$h(x,y) = \frac{f(x)g(yx)}{\int g(tx)\,\nu(dt)}.$$

$\Delta(x) > 1$ or that satisfies $0 < \Delta(x) < 1$ (for then x^{-1} satisfies $\Delta(x^{-1}) > 1$). However the continuity of Δ and the compactness of G imply that Δ is bounded; thus $\Delta(x) = 1$ must hold at each x in G. □

The remaining results in this section will be needed only for a few exercises and for the definition and study of $M_a(G)$ in Sect. 9.4; they can be omitted on a first reading.

Proposition 9.3.6. *Let G be a locally compact group, and let μ be a left Haar measure on G. Then each Borel subset A of G satisfies*

$$\check{\mu}(A) = \int_A \Delta(x^{-1})\,\mu(dx).$$

Proof. Define a measure ν on $\mathscr{B}(G)$ by

$$\nu(A) = \int_A \Delta(x^{-1})\,\mu(dx).$$

We will show that ν is regular, that ν is a right Haar measure, and finally that $\nu = \check{\mu}$.

We begin with the regularity of ν. For each positive integer n let G_n be the open subset of G defined by

$$G_n = \left\{x \in G : \frac{1}{n} < \Delta(x^{-1}) < n\right\}.$$

Let U be an open subset of G. Since $\nu(U) = \lim_n \nu(U \cap G_n)$ (Proposition 1.2.5), we can show that

$$\nu(U) = \sup\{\nu(K) : K \subseteq U \text{ and } K \text{ is compact}\}$$

by checking that

$$\nu(U \cap G_n) = \sup\{\nu(K) : K \subseteq U \cap G_n \text{ and } K \text{ is compact}\}$$

holds for each n. However this last equation is an easy consequence of the regularity of μ and the fact that $1/n < \Delta(x^{-1}) < n$ holds at each x in G_n (consider the cases where $\mu(U \cap G_n) = +\infty$ and where $\mu(U \cap G_n) < +\infty$ separately). Now suppose that A is an arbitrary Borel subset of G. We need to show that

$$\nu(A) = \inf\{\nu(U) : A \subseteq U \text{ and } U \text{ is open}\}. \tag{3}$$

We can certainly restrict our attention to the case where $\nu(A)$ is finite. Let ε be a positive number. Then for each n we can choose an open subset U_n of G_n that includes $A \cap G_n$ and satisfies $\nu(U_n) < \nu(A \cap G_n) + \varepsilon/2^n$ (use the regularity of μ and the fact that $1/n < \Delta(x^{-1}) < n$ holds at each x in G_n). The set U defined by $U = \cup_n U_n$ then includes A and satisfies $\nu(U) < \nu(A) + \varepsilon$; since ε is arbitrary, (3) is

Let G be a locally compact group, and let μ be a left Haar measure on G. The maps $u \mapsto ux$ are homeomorphisms of G onto itself (Proposition 9.1.2), and so for each x in G the formula $\mu_x(A) = \mu(Ax)$ defines a regular Borel measure μ_x on G. The translation invariance of μ implies that μ_x satisfies $\mu_x(yA) = \mu(yAx) = \mu(Ax) = \mu_x(A)$ for each y in G and each A in $\mathscr{B}(G)$. Thus μ_x is a left Haar measure, and so Theorem 9.2.6 implies that for each x there is a positive number, say $\Delta(x)$, such that $\mu_x = \Delta(x)\mu$. The function $\Delta\colon G \to \mathbb{R}$ defined in this way is called the *modular function* of G. See Exercises 2 and 4 for some examples.

If ν is another left Haar measure on G, then there is a positive constant c such that $\nu = c\mu$, and so $\nu_x = c\mu_x = c\Delta(x)\mu = \Delta(x)\nu$ holds for each x in G. Thus the modular function Δ is determined by the group G and does not depend on the particular left Haar measure used in its definition.

Recall that $(\chi_A)_x = \chi_{Ax}$ holds for each member x and subset A of G. It follows that

$$\int f_x \, d\mu = \Delta(x) \int f \, d\mu \tag{2}$$

holds if f is the characteristic function of a Borel subset of G and hence if f is a Borel function that is nonnegative or μ-integrable.

Proposition 9.3.4. *Let G be a locally compact group, and let Δ be the modular function of G. Then*

(a) Δ *is continuous, and*
(b) $\Delta(xy) = \Delta(x)\Delta(y)$ *holds for each x and y in G.*

Proof. Let μ be a left Haar measure on G, and let f be a nonnegative function that belongs to $\mathscr{K}(G)$ and does not vanish identically. Then $\int f \, d\mu \neq 0$ (Lemma 9.2.5), and so Corollary 9.1.6 and Eq. (2) imply the continuity of Δ. The relation $\Delta(xy) = \Delta(x)\Delta(y)$ follows from the calculation

$$\Delta(xy)\mu(A) = \mu(Axy) = \Delta(y)\mu(Ax) = \Delta(y)\Delta(x)\mu(A).$$

□

A locally compact group G is *unimodular* if its modular function satisfies $\Delta(x) = 1$ at each x in G. Thus a locally compact group G is unimodular if and only if each left Haar measure on G is a right Haar measure and so if and only if the collection of all left Haar measures on G coincides with the collection of all right Haar measures on G. Of course every commutative locally compact group is unimodular.

Proposition 9.3.5. *Every compact group is unimodular.*

Proof. Let G be a compact group, and let Δ be its modular function. The relation $\Delta(x^n) = (\Delta(x))^n$ holds for each positive integer n and each element x of G (Proposition 9.3.4); hence Δ is unbounded if there is an element x of G that satisfies

(Note that h does belong to $\mathscr{K}(G\times G)$: Corollary 9.1.6 and Lemma 9.2.5 imply that $x \mapsto \int g(tx)\,\nu(dt)$ is continuous and never vanishes; furthermore, if $K = \operatorname{supp}(f)$ and $L = \operatorname{supp}(g)$, then $\operatorname{supp}(h) \subseteq K \times LK^{-1}$.) For this function h we have $h(y^{-1}, xy) = f(y^{-1})g(x)/\int g(ty^{-1})\,\nu(dt)$, and so Eq. (13) implies that

$$\int f(x)\,\mu(dx) = \int g(x)\,\mu(dx) \int \frac{f(y^{-1})}{\int g(ty^{-1})\,\nu(dt)}\,\nu(dy).$$

Thus the ratio of $\int f\,d\mu$ to $\int g\,d\mu$ depends on f and g, but not on μ, and the proof is complete. □

The reader should note that if the locally compact group G is commutative (and if, for convenience, the group operation is written additively), then a simpler proof of Theorem 9.2.6 can be given. In fact, it is easy to check that if f and g belong to $\mathscr{K}(G)$, then

$$\begin{aligned}
\int f\,d\mu \int g\,d\nu &= \int\int f(x)g(y)\,\mu(dx)\,\nu(dy) \\
&= \int\int f(x+y)g(y)\,\mu(dx)\,\nu(dy) \\
&= \int\int f(y)g(y-x)\,\nu(dy)\,\mu(dx) \\
&= \int\int f(y)g(-x)\,\mu(dx)\,\nu(dy) \\
&= \int f\,d\nu \int \check{g}\,d\mu.
\end{aligned}$$

Thus if we let g be a nonnegative function that belongs to $\mathscr{K}(G)$ and does not vanish identically and if we define c by $c = \int g\,d\nu / \int \check{g}\,d\mu$, then $\int f\,d\nu = c\int f\,d\mu$ holds for each f in $\mathscr{K}(G)$. It follows that $\nu = c\mu$.

Exercises

1. Let G be the set of rational numbers, with addition as the group operation and with the topology inherited from $\mathbb{R}$. Show that there is no nonzero translation-invariant regular Borel measure on G.
2. Let G be a locally compact group, let μ be a left Haar measure on G, and let f and g be continuous real-valued functions on G. Show that if f and g are equal μ-almost everywhere, then they are equal everywhere.

3. Let G be the multiplicative group of positive real numbers, with the topology it inherits as a subspace of $\mathbb{R}$. Show that the formula

$$\mu(A) = \int_A \frac{1}{x}\,\lambda(dx)$$

defines a Haar measure on G.

4. Let G be a locally compact group that is homeomorphic to an open subset (say U) of $\mathbb{R}^d$, and let φ be a homeomorphism of G onto U.

   (a) Show that if for each a in G the function $u \mapsto \varphi(a\varphi^{-1}(u))$ is the restriction to U of an affine[2] map $L_a \colon \mathbb{R}^d \to \mathbb{R}^d$, then the formula

$$\mu(A) = \int_{\varphi(A)} \frac{1}{|\det(L_{\varphi^{-1}(u)})|}\,\lambda(du)$$

   defines a left Haar measure on G.

   (b) Likewise, show that if for each a in G the function $u \mapsto \varphi(\varphi^{-1}(u)a)$ is the restriction to U of an affine map $R_a \colon \mathbb{R}^d \to \mathbb{R}^d$, then the formula

$$\mu(A) = \int_{\varphi(A)} \frac{1}{|\det(R_{\varphi^{-1}(u)})|}\,\lambda(du)$$

   defines a right Haar measure on G.

5. Let G be the group defined in Exercise 9.1.9. Suppose that we identify G with the right half-plane in $\mathbb{R}^2$ by associating the point (a,b) with the matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Show that the formula

$$\mu(A) = \iint_A \frac{1}{a^2}\,da\,db$$

defines a left Haar measure on G and that the formula

$$\mu(A) = \iint_A \frac{1}{a}\,da\,db$$

defines a right Haar measure on G. (Hint: Use the preceding exercise.)

6. Let G be a locally compact group, and let μ be a left Haar measure on G. Show that the topology of G is discrete if and only if $\mu(\{x\}) \neq 0$ holds for some (and hence for each) x in G.

---

[2] A map $F \colon \mathbb{R}^d \to \mathbb{R}^d$ is *affine* if there exist a linear map $G \colon \mathbb{R}^d \to \mathbb{R}^d$ and an element b of $\mathbb{R}^d$ such that $F(x) = G(x) + b$ holds for each x in $\mathbb{R}^d$. If F is affine, then G and b are uniquely determined by F, and we will (for simplicity) denote by $\det(F)$ the determinant of the linear part G of F (see Sect. 6.1).

9.3 Properties of Haar Measure

Let G be a locally compact group, and let μ be a regular Borel measure on G. The map $x \mapsto x^{-1}$ is a homeomorphism of G onto itself (Proposition 9.1.2), and so the subsets A of G that belong to $\mathscr{B}(G)$ are exactly those for which A^{-1} belongs to $\mathscr{B}(G)$ (Lemma 7.2.1). Define a function $\check{\mu}$ on $\mathscr{B}(G)$ by $\check{\mu}(A) = \mu(A^{-1})$. It is easy to check that $\check{\mu}$ is a regular Borel measure on G. The relation

$$\int f\, d\check{\mu} = \int \check{f}\, d\mu \tag{1}$$

holds if f is a Borel function that is nonnegative or $\check{\mu}$-integrable; this is clear if f is a characteristic function and follows in general from the linearity of the integral and the monotone convergence theorem.

Proposition 9.3.1. *Let G be a locally compact group, and let μ be a regular Borel measure on G. Then μ is a left Haar measure if and only if $\check{\mu}$ is a right Haar measure, and is a right Haar measure if and only if $\check{\mu}$ is a left Haar measure.*

Proof. The identity $(Ax)^{-1} = x^{-1}A^{-1}$ implies that $\check{\mu}(Ax) = \check{\mu}(A)$ holds for each x in G and each A in $\mathscr{B}(G)$ if and only if $\mu(x^{-1}A^{-1}) = \mu(A^{-1})$ holds for each x in G and each A in $\mathscr{B}(G)$. The first half of the proposition follows. We can derive the second half from the first by replacing μ with $\check{\mu}$ and noting that $\check{\check{\mu}} = \mu$. □

Corollary 9.3.2. *Let G be a locally compact group. Then there is one and, up to a constant multiple, only one right Haar measure on G.*

Proof. In view of Proposition 9.3.1, this is an immediate consequence of Theorems 9.2.2 and 9.2.6. □

Proposition 9.3.3. *Let G be a locally compact group, and let μ be a left Haar measure on G. Then μ is finite if and only if G is compact.*

Proof. The regularity of μ implies that μ is finite if G is compact.

We turn to the converse. Suppose that μ is finite. Let K be a compact subset of G such that $\mu(K) > 0$ (for instance, K can be a compact set whose interior is nonempty; see Lemma 9.2.5). The finiteness of $\mu(G)$ implies that there is an upper bound, for instance $\mu(G)/\mu(K)$, for the lengths of those finite sequences $\{x_i\}_1^n$ for which the sets x_iK, $i = 1, \ldots, n$, are disjoint. Thus we can choose a positive integer n and points $x_1, \ldots, x_n$ such that the sets x_iK, $i = 1, \ldots, n$, are disjoint, but such that for no choice of x_{n+1} are the sets x_iK, $i = 1, \ldots, n+1$, disjoint. It follows that if $x \in G$, then xK meets $\cup_{i=1}^n x_iK$, and so x belongs to $(\cup_{i=1}^n x_iK)K^{-1}$; hence G is equal to the compact set $(\cup_{i=1}^n x_iK)K^{-1}$. □

It follows that each compact group G has a Haar measure μ such that $\mu(G) = 1$. In dealing with compact groups one often assumes that the corresponding Haar measures have been "normalized" in this way.