Appendix A
Notation and Set Theory

See van Dalen et al. [118], Halmos [55], Hrbacek and Jech [63], or Moschovakis [90] for further information on the topics discussed in this appendix.

A.1. Let A and B be sets. We write $x \in A$, $x \notin A$, and $A \subseteq B$ to indicate that x is a member of A, that x is not a member of A, and that A is a subset of B, respectively. We will denote the union, intersection, and difference of A and B by $A \cup B$, $A \cap B$, and $A - B$, respectively (of course $A - B = \{x : x \in A \text{ and } x \notin B\}$). In case we are dealing with subsets of a fixed set X, the complement of A will be denoted by A^c; thus $A^c = X - A$.

The empty set will be denoted by $\varnothing$.

The symmetric difference of the sets A and B is defined by

$$A \triangle B = (A - B) \cup (B - A).$$

It is clear that $A \triangle A = \varnothing$ and that $A \triangle B = A^c \triangle B^c$. Furthermore, x belongs to $A \triangle (B \triangle C)$ if and only if it belongs either to exactly one, or else to all three, of A, B, and C; since a similar remark applies to $(A \triangle B) \triangle C$, we have

$$A \triangle (B \triangle C) = (A \triangle B) \triangle C.$$

Suppose that $A_1, \ldots, A_n$ is a finite sequence of sets. The union and intersection of these sets are defined by

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i \text{ in the range } 1, \ldots, n\}$$

and

$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for each } i \text{ in the range } 1, \ldots, n\}$$

D.L. Cohn, *Measure Theory: Second Edition*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4614-6956-8,

The union and intersection of an infinite sequence $\{A_i\}_{i=1}^{\infty}$ of sets, written $\cup_{i=1}^{\infty} A_i$ and $\cap_{i=1}^{\infty} A_i$ respectively, are defined in a similar way. (To simplify notation we will sometimes write $\cup_i A_i$ in place of $\cup_{i=1}^{n} A_i$ or $\cup_{i=1}^{\infty} A_i$, and $\cap_i A_i$ in place of $\cap_{i=1}^{n} A_i$ or $\cap_{i=1}^{\infty} A_i$.)

The union and intersection of an arbitrary family $\mathscr{S}$ of subsets of a set X are defined by

$$\cup\mathscr{S} = \{x \in X : x \in S \text{ for some } S \text{ in } \mathscr{S}\}$$

and

$$\cap\mathscr{S} = \{x \in X : x \in S \text{ for each } S \text{ in } \mathscr{S}\}.$$

De Morgan's laws hold: $(\cup\mathscr{S})^c = \cap\{S^c : S \in \mathscr{S}\}$ and $(\cap\mathscr{S})^c = \cup\{S^c : S \in \mathscr{S}\}$.

The set of all subsets of the set X is called the *power set of* X; we will denote it by $\mathscr{P}(X)$.

A.2. We will use $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ to denote the sets of positive integers, of nonnegative integers, of integers (positive, negative, or zero), of rational numbers, of real numbers, and of complex numbers, respectively. The subintervals $[a,b]$ and (a,b) of $\mathbb{R}$ are defined by

$$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$$

and

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}.$$

Other types of intervals, such as $(a,b]$, $(-\infty,b)$, and $(-\infty,+\infty)$, are defined analogously.

A.3. We write $f: X \to Y$ to indicate that f is a function whose *domain* is X and whose values lie in Y (Y is then sometimes called the *codomain* of f); thus f associates a unique element $f(x)$ of Y to each element x of X. We will sometimes define a function $f: X \to Y$ by using the arrow $\mapsto$ to show the action of f on an element of X. For example, if we are dealing with real-valued functions on $\mathbb{R}$, it is often easier to say "the function $x \mapsto x+2$" than to say "the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x+2$." Be careful to distinguish between $\to$ and $\mapsto$: the arrow $\to$ is used to specify the domain and codomain of a function, while the arrow $\mapsto$ is used to describe the action of a function on an element of its domain.

Let X, Y, and Z be sets, and consider functions $g: X \to Y$ and $f: Y \to Z$. Then $f \circ g: X \to Z$ is the function defined by $(f \circ g)(x) = f(g(x))$; it is called the *composition* of f and g.

Suppose that f is a function from X to Y, that A is a subset of X, and that B is a subset of Y. The *image* of A under f, written $f(A)$, is defined by

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \text{ in } A\},$$

and the *inverse image* of B under f, written $f^{-1}(B)$, is defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

A proof of Theorem A.13 can be found in [55, 63, 90, 118]. (Gödel and Cohen showed that the axiom of choice is consistent with and independent of the remaining standard axioms for set theory; again see [30, 50].)

A.14. The reader will need some experience with ordinal numbers in order to do a few of the exercises in Chaps. 7 through 9. It is enough to know a bit about the countable ordinals and the first uncountable ordinal and to have some facility with transfinite recursion and induction. Once again, see [55, 63, 90, 118].

If A is a subset of the domain of f, then the *restriction* of f to A is the function that agrees with f on A and is undefined elsewhere.

A function $f: X \to Y$ is *injective* (or *one-to-one*) if $f(x_1) \neq f(x_2)$ holds whenever x_1 and x_2 are distinct elements of X, and is *surjective* (or *onto*) if each element of Y is of the form $f(x)$ for some x in X. A function is *bijective* if it is both injective and surjective. A function that is injective (or surjective, or bijective) is sometimes called an *injection* (or a *surjection*, or a *bijection*).

If $f: X \to Y$ is bijective, then the *inverse* of f, written f^{-1}, is the function from Y to X that is defined by letting $f^{-1}(y)$ be the unique element of X whose image under f is y; thus $x = f^{-1}(y)$ holds if and only if $y = f(x)$.

Let A be a subset of the set X. The *characteristic function* (or *indicator function*) of A is the function $\chi_A: X \to \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

A.4. The *product* (or *Cartesian product*) of sets X and Y, written $X \times Y$, is the set of all ordered pairs (x, y) for which $x \in X$ and $y \in Y$.

A.5. Notation such as $\{A_i\}_{i \in I}$ or $\{A_i\}$ will be used for an indexed family of sets; here I is the *index set* and A_i is the set associated to the element i of I. An infinite sequence of sets is, of course, an indexed family of sets for which the index set is $\mathbb{N}$ (or perhaps $\mathbb{N}_0$). The *product* $\prod_i A_i$ of the indexed family of sets $\{A_i\}_{i \in I}$ is the set of all functions $a: I \to \cup\{A_i : i \in I\}$ such that $a(i) \in A_i$ holds for each i in I (here one usually writes a_i in place of $a(i)$). If each A_i is equal to the set A, we often write A^I instead of $\prod_i A_i$.

A.6. Sets X and Y have the *same cardinality* if there is a bijection of X onto Y. A set is *finite* if it is empty or has the same cardinality as $\{1, 2, \ldots, n\}$ for some positive integer n; it is *countably infinite* if it has the same cardinality as $\mathbb{N}$. An *enumeration* of a countably infinite set X is a bijection of $\mathbb{N}$ onto X. Thus an enumeration of X can be viewed as an infinite sequence $\{x_n\}$ such that

(a) each x_n belongs to X, and
(b) each element of X is of the form x_n for exactly one value of n.

A set is *countable* if it is finite or countably infinite.

It is easy to check that every subset of a countable set is countable. We should also note that if X and Y are countable, then

(a) $X \cup Y$ is countable, and
(b) $X \times Y$ is countable.

Let us check (b) in the case where X and Y are both countably infinite. Let f be an enumeration of X, and let g be an enumeration of Y. Then $(m, n) \mapsto (f(m), g(n))$ is a bijection of $\mathbb{N} \times \mathbb{N}$ onto $X \times Y$, and so we need only construct an enumeration of $\mathbb{N} \times \mathbb{N}$. This, however, can be done if we define $h: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by following

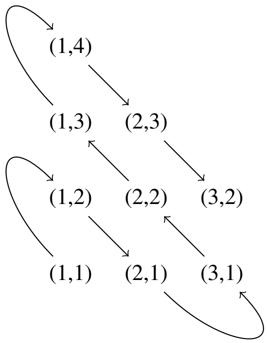

Fig. A.1 Enumeration of $\mathbb{N} \times \mathbb{N}$

the path indicated in Fig. A.1, letting $h(1) = (1,1)$, $h(2) = (1,2)$, $h(3) = (2,1)$, and so forth. (Alternatively, one can define an enumeration h of $\mathbb{N} \times \mathbb{N}$ by letting $h(n) = (r+1, s+1)$, where r and s are the nonnegative integers appearing in the factorization $n = 2^r(2s+1)$ of n into a product of a power of 2 and an odd integer.)

A similar argument can be used to show that the set $\mathbb{Q}$ of rational numbers is countable.

A.7. Suppose that A and B are sets. The Schröder–Bernstein theorem says that if A has the same cardinality as some subset of B and if B has the same cardinality as some subset of A, then A has the same cardinality as B; this can be proved with a version of the arguments used in Proposition G.2 and suggested in part (c) of Exercise 8.3.5 (alternatively, see Halmos [55, Section 22]).

A.8. The set $\mathbb{R}$ is not countable. To say that a set *has the cardinality of the continuum*, or *has cardinality* c, is to say that it has the same cardinality as $\mathbb{R}$. The product sets $\{0,1\}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{N}}$ both have the cardinality of the continuum.

The *continuum hypothesis* says that if A is an infinite subset of $\mathbb{R}$, then either A is countably infinite or else A has the cardinality of the continuum. K. Gödel proved that the continuum hypothesis is consistent with the usual axioms for set theory, and P. J. Cohen proved that it is independent of these axioms (see [30, 50]).

A.9. A *relation* on a set X is a property that holds for some (perhaps none, perhaps all) of the ordered pairs in $X \times X$. For instance, $=$ and $\leq$ are relations on $\mathbb{R}$. If $\sim$ is a relation on X, we write $x \sim y$ to indicate that $\sim$ holds for the pair (x,y). A relation $\sim$ is usually represented by (or is considered to be) the set of ordered pairs (x,y) for which $x \sim y$ holds. Thus a relation on X is a subset of $X \times X$.

A.10. An *equivalence relation* on X is a relation $\sim$ that is reflexive ($x \sim x$ holds for each x in X), symmetric (if $x \sim y$, then $y \sim x$), and transitive (if $x \sim y$ and $y \sim z$, then

$x \sim z$). If $\sim$ is an equivalence relation on X, and if $x \in X$, then the *equivalence class* of x under $\sim$ is the set E_x defined by

$$E_x = \{y \in X : y \sim x\}.$$

Of course, x belongs to E_x. It is easy to check that if $x, y \in X$, then E_x and E_y are either equal or disjoint. Thus the equivalence classes under $\sim$ form a *partition* of X (i.e., a collection of nonempty disjoint sets whose union is X).

A.11. A *partial order* on a set X is a relation $\leq$ that is reflexive ($x \leq x$ holds for each x in X), antisymmetric (if $x \leq y$ and $y \leq x$, then $x = y$), and transitive (if $x \leq y$ and $y \leq z$, then $x \leq z$). A *partially ordered set* is a set, together with a partial order on it. A *linear order* on a set X is a partial order $\leq$ on X such that if $x, y \in X$, then either $x \leq y$ or $y \leq x$. The relation $\leq$ (with its usual meaning) is a linear order on $\mathbb{R}$. If X is a set with at least two elements, and if $\mathscr{P}(X)$ is the set of all subsets of X, then $\subseteq$ is a partial order, but not a linear order, on $\mathscr{P}(X)$.

If $\leq$ is a partial order on a set X, then $x < y$ means that x and y satisfy $x \leq y$ but are not equal.

Let X be a partially ordered set, with partial order $\leq$. A *chain* in X is a subset C of X such that if $x, y \in C$, then either $x \leq y$ or $y \leq x$. An element x of X is an *upper bound* for a subset A of X if $y \leq x$ holds for each y in A; a *lower bound* for A is defined analogously. An element x of X is *maximal* if $x \leq y$ can hold only if $y = x$ (in other words, x is maximal if there are no elements of X larger than x; there may be elements y of X for which neither $x \leq y$ nor $y \leq x$ holds).

A linear order on a set X is a *well ordering* if each nonempty subset of X has a smallest element (that is, if each nonempty subset A of X has a lower bound *that belongs to* A). A set X *can be well ordered* if there is a well ordering on X.

The set $\mathbb{N}$ of positive integers (with the usual order relation on it) is well ordered, but the set $\mathbb{Q}$ of rationals is not. However, one can easily define a well ordering on $\mathbb{Q}$, as follows: Let $f\colon \mathbb{N} \to \mathbb{Q}$ be a bijective function (that is, an enumeration of $\mathbb{Q}$), and let f^{-1} be its inverse. Define a binary relation $\prec$ on $\mathbb{Q}$ by declaring that $x \prec y$ holds if and only if $f^{-1}(x) < f^{-1}(y)$ (here $<$ is the usual less-than relation on $\mathbb{N}$). Since $<$ is a well ordering of $\mathbb{N}$, $\prec$ is a well ordering of $\mathbb{Q}$.

A.12. The *axiom of choice* says that if $\mathscr{S}$ is a set of disjoint nonempty sets, then there is a set that has exactly one element in common with each set in $\mathscr{S}$. Another (equivalent) formulation of the axiom of choice says that if $\{A_i\}_{i \in I}$ is an indexed family of nonempty sets, then $\prod_i A_i$ is nonempty.

A.13. **(Theorem)** *The following are equivalent:*

(a) *The axiom of choice holds.*
(b) *(Zorn's lemma) If X is a partially ordered set such that each chain in X has an upper bound in X, then X has a maximal element.*
(c) *(The well-ordering theorem) Every set can be well ordered.*