Appendix C
Calculus and Topology in $\mathbb{R}^d$

C.1. Recall that $\mathbb{R}^d$ is the set of all d-tuples of real numbers; it is a vector space over $\mathbb{R}$. (The d in $\mathbb{R}^d$ is for dimension; we write $\mathbb{R}^d$, rather than $\mathbb{R}^n$, in order to have n available for use as a subscript.) Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ be elements of $\mathbb{R}^d$. The *norm* of x is defined by

$$\|x\| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2}$$

and the *distance* between x and y is defined to be $\|x - y\|$.

C.2. If $x \in \mathbb{R}^d$ and if r is a positive number, then the *open ball* $B(x, r)$ with center x and radius r is defined by

$$B(x, r) = \{y \in \mathbb{R}^d : \|y - x\| < r\}.$$

A subset U of $\mathbb{R}^d$ is *open* if for each x in U there is a positive number r such that $B(x, r) \subseteq U$. A subset of $\mathbb{R}^d$ is *closed* if its complement is open. A point x in $\mathbb{R}^d$ is a *limit point* of the subset A of $\mathbb{R}^d$ if for each positive r the open ball $B(x, r)$ contains infinitely many points of A (this is equivalent to requiring that for each positive r the ball $B(x, r)$ contain at least one point of A distinct from x). It is easy to check that a subset of $\mathbb{R}^d$ is closed if and only if it contains all of its limit points.

If A is a subset of $\mathbb{R}^d$, then the *closure* of A is the set $\overline{A}$ (or A^-) that consists of the points in A, together with the limit points of A; $\overline{A}$ is closed and is, in fact, the smallest closed subset of $\mathbb{R}^d$ that includes A.

C.3. A subset A of $\mathbb{R}^d$ is *bounded* if there is a real number M such that $\|x\| \le M$ holds for each x in A.

C.4. (Proposition) *Let U be an open subset of $\mathbb{R}$. Then there is a countable collection $\mathscr{U}$ of disjoint open intervals such that $U = \cup\mathscr{U}$.*

D.L. Cohn, *Measure Theory: Second Edition*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4614-6956-8,

Proof. Let $\mathscr{U}$ consist of those open subintervals I of U that are maximal, in the sense that the only open interval J that satisfies $I \subseteq J \subseteq U$ is I itself. Of course $\cup\mathscr{U} \subseteq U$. One can verify the reverse inclusion by noting that if $x \in U$, then the union of those open intervals that contain x and are included in U is an open interval that contains x and belongs to $\mathscr{U}$. It is easy to check (do so) that the intervals in $\mathscr{U}$ are disjoint from one another. If for each I in $\mathscr{U}$ we choose a rational number x_I that belongs to I, then (since the sets in $\mathscr{U}$ are disjoint from one another) the map $I \mapsto x_I$ is an injection; thus $\mathscr{U}$ has the same cardinality as some subset of $\mathbb{Q}$, and so is countable (see item A.6 in Appendix A). □

C.5. A sequence $\{x_n\}$ of elements of $\mathbb{R}^d$ *converges* to the element x of $\mathbb{R}^d$ if $\lim_n \|x_n - x\| = 0$; x is then called the *limit* of the sequence $\{x_n\}$ (note that here x and $x_1, x_2, \ldots$ are elements of $\mathbb{R}^d$; in particular, $x_1, x_2, \ldots$ are *not* the components of x). A sequence in $\mathbb{R}^d$ is *convergent* if it converges to some element of $\mathbb{R}^d$.

C.6. Let A be a subset of $\mathbb{R}^d$, and let x_0 belong to A. A function $f\colon A \to \mathbb{R}$ is *continuous at* x_0 if for each positive number ε there is a positive number δ such that $|f(x) - f(x_0)| < \varepsilon$ holds whenever x belongs to A and satisfies $\|x - x_0\| < \delta$; f is *continuous* if it is continuous at each point in A. The function $f\colon A \to \mathbb{R}$ is *uniformly continuous* if for each positive number ε there is a positive number δ such that $|f(x) - f(x')| < \varepsilon$ holds whenever x and x' belong to A and satisfy $\|x - x'\| < \delta$. A function $f\colon A \to \mathbb{R}$ is *continuous on* (or *uniformly continuous on*) the subset A_0 of A if the restriction of f to A_0 is continuous (or uniformly continuous).

C.7. Let A be a subset of $\mathbb{R}^d$, let f be a real- or complex-valued function on A, and let a be a limit point of A. Then $f(x)$ has *limit* L as x approaches a, written $\lim_{x\to a} f(x) = L$, if for every positive ε there is a positive δ such that $|f(x) - f(a)| < \varepsilon$ holds whenever x is a member of A that satisfies $0 < \|x - a\| < \delta$.

One can check that $\lim_{x\to a} f(x) = L$ if and only if $\lim_n f(x_n) = L$ for every sequence $\{x_n\}$ of elements of A, all different from a, such that $\lim_n x_n = a$. (Let us consider the more difficult half of that assertion, namely that if $\lim_n f(x_n) = L$ for every sequence $\{x_n\}$ of elements of A, all different from a, such that $\lim_n x_n = a$, then $\lim_{x\to a} f(x) = L$. We prove this by proving its contrapositive. So assume that $\lim_{x\to a} f(x) = L$ is not true. Then there exists a positive ε such that for every positive δ there is a value x in A such that $0 < \|x - a\| < \delta$ and $|f(x) - L| \ge \varepsilon$. If for each n we let $\delta = 1/n$ and choose an element x_n of A such that $0 < \|x_n - a\| < 1/n$ and $|f(x_n) - L| \ge \varepsilon$, we will have a sequence $\{x_n\}$ of elements of A, all different from a, that satisfy $\lim_n x_n = a$ but not $\lim_n f(x_n) = L$.)

C.8. Let A be a subset of $\mathbb{R}^d$. An *open cover* of A is a collection $\mathscr{S}$ of open subsets of $\mathbb{R}^d$ such that $A \subseteq \cup\mathscr{S}$. A *subcover* of the open cover $\mathscr{S}$ is a subfamily of $\mathscr{S}$ that is itself an open cover of A.

Proofs of the following results can be found in almost any text on advanced calculus or basic analysis (see, for example, Bartle [4], Hoffman [60], Rudin [104], or Thomson et al. [117]).

C.9. (Theorem) *Let A be a closed bounded subset of $\mathbb{R}^d$. Then every open cover of A has a finite subcover.*

Theorem C.9 is often called the *Heine–Borel* theorem.

C.10. (Theorem) *Let A be a closed bounded subset of $\mathbb{R}^d$. Then every sequence of elements of A has a subsequence that converges to an element of A.*

C.11. It is easy to check that the converses of Theorems C.9 and C.10 hold: if A satisfies the conclusion of Theorem C.9 or of Theorem C.10, then A is closed and bounded. The subsets of $\mathbb{R}^d$ that satisfy the conclusion of Theorem C.9 (hence the closed bounded subsets of $\mathbb{R}^d$) are often called *compact*. See also Appendix D.

C.12. (Theorem) *Let C be a nonempty closed bounded subset of $\mathbb{R}^d$, and let $f\colon C \to \mathbb{R}$ be continuous. Then*

(a) *f is uniformly continuous on C, and*
(b) *f is bounded on C. Moreover, there are elements x_0 and x_1 of C such that $f(x_0) \le f(x) \le f(x_1)$ holds at each x in C.*

C.13. (The Intermediate Value Theorem) *Let A be a subset of $\mathbb{R}$, and let $f\colon A \to \mathbb{R}$ be continuous. If the interval $[x_0, x_1]$ is included in A, then for each real number y between $f(x_0)$ and $f(x_1)$ there is an element x of $[x_0, x_1]$ such that $y = f(x)$.*

C.14. (The Mean Value Theorem) *Let a and b be real numbers such that $a < b$. If $f\colon [a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b]$ and differentiable at each point in the open interval (a,b), then there is a number c in (a,b) such that $f(b) - f(a) = f'(c)(b-a)$.*