# Appendix F
# Liftings

Let $(X,\mathscr{A},\mu)$ be a measure space. Throughout this appendix we will assume that the measure $\mu$ is finite but not the zero measure (see Exercise 2). Recall that $\mathscr{L}^\infty(X,\mathscr{A},\mu,\mathbb{R})$ is the vector space of all bounded real-valued $\mathscr{A}$-measurable functions on $X$ and that $L^\infty(X,\mathscr{A},\mu,\mathbb{R})$ is the vector space of equivalence classes of functions in $\mathscr{L}^\infty(X,\mathscr{A},\mu,\mathbb{R})$, where two functions are considered equivalent if they are equal $\mu$-almost everywhere.[1] For simplicity, we will generally write $\mathscr{L}^\infty(X,\mathscr{A},\mu)$, instead of $\mathscr{L}^\infty(X,\mathscr{A},\mu,\mathbb{R})$. We will occasionally use the norm $\|\cdot\|_\infty$ on $\mathscr{L}^\infty(X,\mathscr{A},\mu)$ defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

Note that for this version of the norm $\|\cdot\|_\infty$ a function $f$ satisfies $\|f\|_\infty = 0$ only if $f$ vanishes *everywhere* on $X$; it is not enough for it to vanish almost everywhere.

It is natural to ask whether a function in $\mathscr{L}^\infty(X,\mathscr{A},\mu)$ can be chosen from each equivalence class in $L^\infty(X,\mathscr{A},\mu)$ in such a way the choice is linear and multiplicative. Since notation involving functions is simpler than notation involving equivalence classes, one generally deals with functions and makes the following definitions. A *lifting* of $\mathscr{L}^\infty(X,\mathscr{A},\mu)$ is a function $\rho\colon \mathscr{L}^\infty(X,\mathscr{A},\mu) \to \mathscr{L}^\infty(X,\mathscr{A},\mu)$ such that for all $f$, $g$ in $\mathscr{L}^\infty(X,\mathscr{A},\mu)$ and all real numbers $a$ and $b$ we have

[1] In the present context (i.e., in cases where the measure $\mu$ is finite), it is the same to say that two functions agree almost everywhere as to say that they agree locally almost everywhere. Thus, for our current discussion the definition of $\mathscr{L}^\infty(X,\mathscr{A},\mu,\mathbb{R})$ given here is consistent with the one in Chap. 4. We will use the current definition since it makes the exposition that follows simpler. If we were looking at liftings on very large measure spaces, we would speak of locally null sets and of equality locally almost everywhere; see [65].

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(a) if $f = g$ almost everywhere, then[2] $\rho(f) = \rho(g)$,
(b) $\rho(f) = f$ almost everywhere,
(c) $\rho(af + bg) = a\rho(f) + b\rho(g)$,
(d) $\rho(fg) = \rho(f)\rho(g)$, and
(e) $\rho(1) = 1$.

Conditions (a) and (b) say that $\rho$ can be interpreted as providing a choice of a function in $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ from each equivalence class in $L^\infty(X, \mathscr{A}, \mu)$.

The main theorem of this appendix (Theorem F.5, the lifting theorem) says that liftings of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ exist, if the measure $\mu$ is complete.

If $f$ is a nonnegative function in $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$, then $\sqrt{f}$ also belongs to $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$, and so $\rho(f) = \rho(\sqrt{f})\rho(\sqrt{f})$. It follows that

(f) if $f \geq 0$, then $\rho(f) \geq 0$.

A function $\rho\colon \mathscr{L}^\infty(X, \mathscr{A}, \mu) \to \mathscr{L}^\infty(X, \mathscr{A}, \mu)$ is called a *linear lifting* if it satisfies conditions (a), (b), (c), (e), and (f). We will encounter linear liftings while constructing liftings.

Recall that $\ell^\infty$ is the vector space of all bounded sequences of real numbers, with norm given by $\|\{x_n\}\|_\infty = \sup_n |x_n|$. Let $c$ be the subspace of $\ell^\infty$ consisting of the sequences $\{x_n\}$ for which $\lim_n x_n$ exists; give $c$ the norm it inherits from $\ell^\infty$.

**F.1. (Lemma)** *There is a linear functional $\Lambda\colon \ell^\infty \to \mathbb{R}$ such that*

(a) $\Lambda(\{x_n\}) = \lim_n x_n$ *for all* $\{x_n\}$ *in* $c$,
(b) $|\Lambda(\{x_n\})| \leq \|\{x_n\}\|_\infty$ *for all* $\{x_n\}$ *in* $\ell^\infty$, *and*
(c) $\Lambda(\{x_n\})$ *is positive, in the sense that* $\Lambda(\{x_n\}) \geq 0$ *whenever* $\{x_n\}$ *is a sequence in* $\ell^\infty$ *whose terms are nonnegative.*

In other words, if $L$ is the linear functional defined on the subspace $c$ of $\ell^\infty$ by $L(\{x_n\}) = \lim_n x_n$, then $L$ can be extended to a linear functional on all of $\ell^\infty$ that has norm 1 and is positive.

*Proof.* As in the previous paragraph, define a linear functional $L$ on $c$ by $L(\{x_n\}) = \lim_n x_n$. Then $L$ satisfies $|L(\{x_n\})| \leq \|\{x_n\}\|_\infty$ for all $\{x_n\}$ in $c$, and so the Hahn–Banach theorem (Theorem E.7 in Appendix E) gives a linear functional $\Lambda$ on $\ell^\infty$ that satisfies conditions (a) and (b). If $\{x_n\}$ is a sequence in $\ell^\infty$ whose terms are nonnegative, and if $s = \sup_n x_n$, then

$$|\Lambda(\{x_n\}) - s/2| = |\Lambda(\{x_n - s/2\})| \leq \|\{x_n - s/2\}\|_\infty = s/2,$$

from which it follows that $\Lambda(\{x_n\}) \geq 0$. □

[2]Note that when we say that two functions are equal, but don't give a qualification with the words "almost everywhere," then we are saying that the functions are identical. For example, condition (a) says that if $f(x) = g(x)$ for almost every $x$, then $\rho(f)(x) = \rho(g)(x)$ for *every* $x$.

(d) Define an operator $\rho$ on $\mathscr{L}^\infty([0,1],\mathscr{A},\lambda,\mathbb{R})$ by $\rho(f)(t) = \phi_t(f)$. Show that for each $f$ the function $\rho(f)$ is bounded and measurable, and moreover that $\rho$ is a lifting of $\mathscr{L}^\infty([0,1],\mathscr{A},\lambda,\mathbb{R})$.

## Notes

The existence of liftings was first proved by von Neumann [119] and by Maharam [87]. In the 1960s A. and C. Ionescu Tulcea were very active in studying liftings; see [64, 65]. The paper by Strauss et al. [115] surveys much more recent work.

(a) $\sigma(\cup_\alpha \mathscr{B}_\alpha) = \cup_\alpha \mathscr{B}_\alpha$, and
(b) $\mathscr{L}^\infty(X, \sigma(\cup_\alpha \mathscr{B}_\alpha), \mu) = \cup_\alpha \mathscr{L}^\infty(X, \mathscr{B}_\alpha, \mu)$.

6. In this exercise we look at a proof of the existence of liftings in the particular case of $\mathscr{L}^\infty([0,1], \mathscr{A}, \lambda)$, where $\mathscr{A}$ is the $\sigma$-algebra of Lebesgue measurable subsets of $[0,1]$. The proof outlined here has the advantage that it is simpler than the one given above and relates liftings to differentiation theory. However, it depends on a basic but nontrivial result about Banach algebras that is quoted below but not proved, and it only gives liftings in the case of certain measure spaces.

Let $A$ be a commutative Banach algebra (see Sect. 9.4). We assume that $A$ has a multiplicative identity element 1 that satisfies $\|1\| = 1$. Recall that an *ideal* in $A$ is a subset $I$ of $A$ that is a vector subspace of $A$, is a proper subset of $A$, and is such that $xy \in I$ whenever $x \in A$ and $y \in I$. A *maximal ideal* is an ideal that is included in no larger ideal. We will be looking at Banach algebras over the field $\mathbb{C}$, because complex-variable techniques are used in the proof of the result we quote below. We will assume that the Banach algebras that we consider have an *involution* $x \mapsto x^*$ that satisfies

(i) $(x+y)^* = x^* + y^*$,
(ii) $(xy)^* = x^* y^*$,
(iii) $(\alpha x)^* = \overline{\alpha} x^*$ (where $\overline{\alpha}$ is the complex conjugate of $\alpha$), and
(iv) $x^{**} = x$

for all $x$ and $y$ in $A$ and all $\alpha$ in $\mathbb{C}$. In the case where $A = \mathscr{L}^\infty([0,1], \mathscr{A}, \lambda, \mathbb{C})$, the operator that takes a function $f$ to the complex conjugate of $f$ is an involution (in fact, it is the only involution we will need to consider).

The result we need to quote says that if $A$ is a Banach algebra over $\mathbb{C}$ that has an involution, and if $M$ is a maximal ideal in $A$, then there is a linear functional $\phi$ on $A$ such that

(i) $\|\phi\| \le 1$,
(ii) $\phi(xy) = \phi(x)\phi(y)$ holds for all $x$, $y$ in $A$,
(iii) $\phi(1) = 1$,
(iv) $\phi(x^*) = \overline{\phi(x)}$ holds for all $x$ in $A$, and
(v) $M = \{x \in A : \phi(x) = 0\}$.

(see Simmons [109, Chapters 12 and 13], Hewitt and Ross [58, Appendix C], or Lax [82, Chapters 18 and 19]).

(a) Let $A$ be the Banach algebra $\mathscr{L}^\infty([0,1], \mathscr{A}, \lambda, \mathbb{C})$. For each $t$ in $[0,1]$ let $I_t$ be the subset of $A$ consisting of those functions $f$ such that $F'(t)$ exists and is equal to 0, where $F$ is the function defined by $F(u) = \int_0^u |f(s)|\,ds$ (note the absolute value signs around $f(s)$). Show that $I_t$ is an ideal in $A$.
(b) Show that for each $t$ there is a maximal ideal $M_t$ in $A$ that includes $I_t$. (Hint: Use Zorn's lemma.)
(c) Suppose that for each $t$ we apply the result quoted above to the maximal ideal $M_t$, thereby producing a family of function $\{\phi_t\}_t$. Show that if $f$ is a real-valued function in $\mathscr{L}^\infty([0,1], \mathscr{A}, \lambda, \mathbb{C})$, then for each t the value $\phi_t(f)$ is a real number.

Let $C$ be a convex subset of a vector space $E$. An *extreme point* of $C$ is a point $x$ in $C$ that cannot be written as a convex combination of points of $C$ different from $x$. In other words, we are requiring that if $x = ty + (1-t)z$, where $y$ and $z$ belong to $C$ and $0 < t < 1$, then $y = z = x$. More generally, an *extremal subset* of $C$ is a nonempty subset $C_0$ of $C$ such that if $x \in C_0$ and $x = ty + (1-t)z$, where $y$ and $z$ belong to $C$ and $0 < t < 1$, then $y$ and $z$ belong to $C_0$. Thus a point $x$ in $C$ is an extreme point of $C$ if and only if $\{x\}$ is an extremal subset of $C$.

As examples let us consider some subsets of $\mathbb{R}^2$. If $C_1$ is the disk defined by

$$C_1 = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\},$$

then $C_1$ has infinitely many extreme points, namely the points on the circle that forms the boundary of $C_1$. On the other hand, if $C_2$ is the square defined by

$$C_2 = \{(x_1,x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1 \text{ and } -1 \leq x_2 \leq 1\},$$

then $C_2$ has only four extreme points, namely its corner points $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. The remaining boundary points of $C_2$ are not extreme points. The four line segments that make up the boundary of $C_2$ (that is, the line segments that join adjacent corners of $C_2$) are extremal subsets of $C_2$, as is the set that consists of all the boundary points of $C_2$. Finally, the open disk $C_3$ defined by

$$C_3 = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

is convex, but it has no extreme points.

We will need to know that certain sets have extreme points. If we assumed a substantial amount of functional analysis in the reader's background, we would simply appeal to the Krein–Milman theorem, which says that if $K$ is a nonempty compact convex subset of a locally convex Hausdorff topological vector space, then $K$ has extreme points and is in fact the smallest closed convex set that contains all the extreme points of $K$. However, all we need is given by the following lemma, which we can prove without too much work.

**F.2. (Lemma)** *Let $S$ be a nonempty set and let $E$ be the product space $\mathbb{R}^S$, considered as a vector space and as a topological space with the product topology. Then each nonempty compact convex subset of $E$ has at least one extreme point.*

*Proof.* Let $K$ be a nonempty compact convex subset of $\mathbb{R}^S$, and let $\mathscr{E}$ be the collection of all nonempty closed extremal subsets of $K$. Then $\mathscr{E}$ contains $K$, and so is nonempty. Let us view $\mathscr{E}$ as a partially ordered set, with $E_1 \leq E_2$ holding if $E_2 \subseteq E_1$. (Be careful: sets that are larger with respect to the partial order $\leq$ are smaller with respect to set inclusion.) We will use Zorn's lemma (see A.13) to get an element of $\mathscr{E}$ that is maximal with respect to $\leq$ and hence minimal with respect to $\subseteq$. So suppose that $\mathscr{C}$ is a chain of elements of $\mathscr{E}$. The intersection of any finite subcollection of $\mathscr{C}$ belongs to $\mathscr{C}$ (it is a member of the subcollection), and so is nonempty. This, together with the compactness of $K$, implies that the intersection of all the members of $\mathscr{C}$ is nonempty (and, of course, closed). Furthermore, since each element of $\mathscr{C}$ is extremal, so is the intersection of the elements of $\mathscr{C}$. Thus $\mathscr{C}$, which was an arbitrary chain in $\mathscr{E}$, has an upper bound in $\mathscr{E}$. So we can apply Zorn's lemma, which gives a maximal element of $\mathscr{E}$, say $E_0$.

The maximality of $E_0$ says that $E_0$ has no subsets that belong to $\mathscr{E}$. What if $E_0$ contains more than one point? Each point in $E_0$ is a member of the product space $\mathbb{R}^S$ and so is a function from $S$ to $\mathbb{R}$. If there are two members of $E_0$, say $e_1$ and $e_2$, then there must be a point $s$ in $S$ such that $e_1(s) \neq e_2(s)$. Let $m = \inf\{e(s) : e \in E_0\}$. Then, since $E_0$ is compact and the function $e \mapsto e(s)$ is continuous, the set

$$\{e \in E_0 \colon e(s) = m\}$$

is a proper subset of $E_0$ that is nonempty, closed, and extremal. This contradicts the maximality of $E_0$, and we conclude that $E_0$ can contain only one element, say $e_0$. It follows that $e_0$ is an extreme point of $K$. □

The following two lemmas contain most of the technical details needed to prove the existence of liftings.

**F.3. (Lemma)** *Suppose that $(X, \mathscr{A}, \mu)$ is a probability*[3] *space, $\mathscr{A}_0$ is a sub-$\sigma$-algebra of $\mathscr{A}$, and $\rho$ is a lifting of $\mathscr{L}^\infty(X, \mathscr{A}_0, \mu)$. If $E_0$ is a member of $\mathscr{A}$ that does not belong to $\mathscr{A}_0$, then $\rho$ can be extended to a lifting of $\mathscr{L}^\infty(X, \sigma(\mathscr{A}_0 \cup \{E_0\}), \mu)$.*

Note the abuse of notation in the statement of Lemma F.3: $\mu$ first represents a measure on $\mathscr{A}$, then the restriction of that measure to the sub-$\sigma$-algebra $\mathscr{A}_0$, and finally the restriction of it to $\sigma(\mathscr{A}_0 \cup \{E_0\})$.

*Proof.* Recall that $\sigma(\mathscr{A}_0 \cup \{E_0\})$ consists of the sets of the form $(A \cap E_0) \cup (B \cap E_0^c)$, where $A$ and $B$ belong to $\mathscr{A}_0$ (see part (a) of Exercise 1.5.12), and that a function $f \colon X \to \mathbb{R}$ is $\sigma(\mathscr{A}_0 \cup \{E_0\})$-measurable if and only if there are $\mathscr{A}_0$-measurable real-valued functions $f_0$ and $f_1$ such that $f = f_0\chi_{E_0} + f_1\chi_{E_0^c}$ (see Exercise 2.1.9). It follows that the functions in $\mathscr{L}^\infty(X, \sigma(\mathscr{A}_0 \cup \{E_0\}), \mu)$ are those that have the form $f_0\chi_{E_0} + f_1\chi_{E_0^c}$ for some $f_0$, $f_1$ in $\mathscr{L}^\infty(X, \mathscr{A}_0, \mu)$.

Suppose that $\rho_1$ is a lifting of $\mathscr{L}^\infty(X, \sigma(\mathscr{A}_0 \cup \{E_0\}), \mu)$ that is an extension of $\rho$. Then there is a set $E_1$ in $\sigma(\mathscr{A}_0 \cup \{E_0\})$ such that $\rho_1(\chi_{E_0}) = \chi_{E_1}$ (see Exercise 3), and for each function of the form $f_0\chi_{E_0} + f_1\chi_{E_0^c}$, where $f_0$ and $f_1$ belong to $\mathscr{L}^\infty(X, \mathscr{A}_0, \mu)$, we have

$$\begin{aligned}\rho_1(f_0\chi_{E_0} + f_1\chi_{E_0^c}) &= \rho_1(f_0)\rho_1(\chi_{E_0}) + \rho_1(f_1)\rho_1(\chi_{E_0^c}) \\ &= \rho(f_0)\chi_{E_1} + \rho(f_1)\chi_{E_1^c}.\end{aligned}$$

[3] What we really want is for $\mu$ to be a finite measure such that $\mu(X) \neq 0$. It's easier, however, to say that we assume $\mu$ to be a probability measure, and if we prove our results for probability measures, we will also have proved them for all nonzero finite measures.

measurable. It is easy to check that $L$ is a linear lifting that extends each lifting $\rho_n$. Now use Lemma F.4 to get a lifting $\rho_\infty$ that satisfies

$$\chi_{\{L(\chi_A)=1\}} \le \rho_\infty(\chi_A) \le \chi_{\{L(\chi_A)>0\}} \tag{9}$$

for every $A$ in $\mathscr{B}_\infty$. If $A \in \mathscr{B}_n$, then, since $L$ is an extension of $\rho_n$ and since 0 and 1 are the only possible values for the function $\rho_n(\chi_A)$, we have $\{L(\chi_A) = 1\} = \{L(\chi_A) > 0\}$. It now follows from (9) that $\rho_\infty(\chi_A) = L(\chi_A) = \rho_n(\chi_A)$, and so $\rho_\infty$ is an extension of $\rho_n$ (to check this, approximate functions in $\mathscr{L}^\infty(X, \mathscr{B}_\infty, \mu)$ with simple functions—see the proof of Lemma F.4). Thus we have an upper bound $(\mathscr{B}_\infty, \rho_\infty)$ for the chain $\mathscr{C}$.

Finally, we need to produce an upper bound for the chain $\mathscr{C}$ in the case where $\mathscr{C}$ has no cofinal sequences. Suppose that $\mathscr{C}$ is the family $\{(\mathscr{B}_\alpha, \rho_\alpha)\}_\alpha$, where $\alpha$ ranges over some index set. Then $\cup_\alpha \mathscr{B}_\alpha$ is a $\sigma$-algebra and $\mathscr{L}^\infty(X, \cup_\alpha \mathscr{B}_\alpha, \mu) = \cup_\alpha \mathscr{L}^\infty(X, \mathscr{B}_\alpha, \mu)$ (see Exercise 5). We can define a lifting $\rho$ on $\mathscr{L}^\infty(X, \cup_\alpha \mathscr{B}_\alpha, \mu)$ by letting $\rho(f)$ be $\rho_\alpha(f)$, where $\alpha$ is an index such that $f \in \mathscr{L}^\infty(X, \mathscr{B}_\alpha, \mu)$ (the index $\alpha$ depends, of course, on $f$). With this we have an upper bound for the chain $\mathscr{C}$, and the proof is complete. □

## Exercises

1. Let $X = \{1,2,3\}$, let $\mathscr{A}$ be the set of all subsets of $X$, and let $\mu$ be the measure on $(X, \mathscr{A})$ defined by $\mu = \frac{1}{3}\delta_1 + \frac{2}{3}\delta_2$.
   (a) Find a lifting of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$.
   (b) Find all liftings of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$.
2. Suppose that $(X, \mathscr{A}, \mu)$ is a measure space such that $X$ is nonempty but $\mu(X) = 0$. Show that there are no liftings of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$.
3. Suppose that $\rho$ is a lifting of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$. Show that if $E \in \mathscr{A}$, then there is a set $E'$ in $\mathscr{A}$ such that $\rho(\chi_E) = \chi_{E'}$ and $\mu(E \triangle E') = 0$.
4. Let $(X, \mathscr{A}, \mu)$ be a measure space. A function $\rho'\colon \mathscr{A} \to \mathscr{A}$ is a *lifting* of $\mathscr{A}$ if

   (i) $\rho'(A) = \rho'(\mathrm{B})$ whenever $\mu(A \triangle B) = 0$,
   (ii) $\mu(A \triangle \rho'(A)) = 0$ for all $A$ in $\mathscr{A}$,
   (iii) $\rho'(\varnothing) = \varnothing$ and $\rho'(X) = X$,
   (iv) $\rho'(A \cup B) = \rho'(A) \cup \rho'(B)$ for all $A$ and $B$ in $\mathscr{A}$, and
   (v) $\rho'(A \cap B) = \rho'(A) \cap \rho'(B)$ for all $A$ and $B$ in $\mathscr{A}$.

   Suppose that for each lifting $\rho$ of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ we define a function $\rho'\colon \mathscr{A} \to \mathscr{A}$ by $\chi_{\rho'(A)} = \rho(\chi_A)$. Show that $\rho \mapsto \rho'$ is a bijection of the set of all liftings of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ onto the set of all liftings of $\mathscr{A}$.
5. Let $(X, \mathscr{A}, \mu)$ be a measure space and let $\{\mathscr{B}_\alpha\}_\alpha$ be a linearly ordered family of sub-$\sigma$-algebras of $\mathscr{A}$. Suppose that for each countable subfamily $\{\mathscr{B}_{\alpha_n}\}_n$ of $\{\mathscr{B}_\alpha\}_\alpha$ there is an element $\mathscr{B}_{\alpha'}$ of $\{\mathscr{B}_\alpha\}_\alpha$ such that $\mathscr{B}_{\alpha_n} \subseteq \mathscr{B}_{\alpha'}$ holds for every $n$. Show that

$L_{1-}(\chi_A)$ belong to $[0,1]$). Likewise, if $L_0(\chi_A)(x) = 0$, then $L_1(\chi_A)(x) = 0$ and so $L_{1+}(\chi_A)(x) = L_{1-}(\chi_A)(x) = 0$. It follows that $L_{1+}$ and $L_{1-}$ satisfy (7) and so correspond to elements of $C$. However, $L_1$ corresponds to an extreme point of $C$ and satisfies $L_1 = \frac{1}{2}L_{1+} + \frac{1}{2}L_{1-}$, and so we have $L_1 = L_{1+} = L_{1-}$. But this implies that $L_1(fg) - L_1(f)L_1(g) = 0$, and the multiplicativity of $L_1$ follows. Thus $L_1$ is a lifting that satisfies (5), and the proof is complete. □

**F.5. (Theorem)** *If $(X, \mathscr{A}, \mu)$ is a complete probability space, then there is a lifting of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$.*

*Proof.* Let $\mathscr{T}$ be the collection of all pairs $(\mathscr{B}, \rho)$, where $\mathscr{B}$ is a sub-$\sigma$-algebra of $\mathscr{A}$ that contains all the $\mu$-null sets in $\mathscr{A}$ and where $\rho$ is a lifting of $\mathscr{L}^\infty(X, \mathscr{B}, \mu)$. (Of course, by $\mathscr{L}^\infty(X, \mathscr{B}, \mu)$ we really mean $\mathscr{L}^\infty(X, \mathscr{B}, \mu_{\mathscr{B}})$, where $\mu_{\mathscr{B}}$ is the restriction of $\mu$ to the sub-$\sigma$-algebra $\mathscr{B}$ of $\mathscr{A}$. Such abuse of notation will occur often in this proof.) Let us define a relation $\le$ on $\mathscr{T}$ by defining $(\mathscr{B}_1, \rho_1) \le (\mathscr{B}_2, \rho_2)$ to mean that $\mathscr{B}_1 \subseteq \mathscr{B}_2$ and $\rho_1$ is the restriction of $\rho_2$ to $\mathscr{L}^\infty(X, \mathscr{B}_1, \mu)$. Then $\le$ is a partial order on $\mathscr{T}$.

We'll check that $\mathscr{T}$ is nonempty and that each chain in $\mathscr{T}$ has an upper bound in $\mathscr{T}$, and so Zorn's lemma will provide a maximal element $(\mathscr{B}', \rho')$ of $\mathscr{T}$. Then $\mathscr{B}'$ must be equal to $\mathscr{A}$ (and the proof will be complete), since otherwise Lemma F.3 would provide an extension of $\rho'$ to $\mathscr{L}^\infty(X, \mathscr{B}'', \mu)$ for some still larger sub-$\sigma$-algebra $\mathscr{B}''$ of $\mathscr{A}$, and $(\mathscr{B}', \rho')$ would not be maximal.

We turn to the details. First let us check that $\mathscr{T}$ is nonempty. Let $\mathscr{B}_0$ be the collection of $\mu$-null sets in $\mathscr{A}$, together with their complements. Then $\mathscr{B}_0$ is a $\sigma$-algebra, $\mathscr{L}^\infty(X, \mathscr{B}_0, \mu)$ consists of the bounded measurable functions that are almost everywhere constant, and the operator that assigns to each such function $f$ the constant function that is almost everywhere equal to $f$ is a lifting.

Next suppose that $\mathscr{C}$ is a chain in $\mathscr{T}$; we will produce an upper bound for $\mathscr{C}$. Let us consider two cases.

In the first case there is an increasing sequence $\{(B_n, \rho_n)\}_{n=1}^\infty$ in $\mathscr{C}$ that is *cofinal*, in the sense that for every $(\mathscr{B}, \rho)$ in $\mathscr{C}$ there is an $n$ such that $(\mathscr{B}, \rho) \le (\mathscr{B}_n, \rho_n)$. Let us construct an upper bound $(\mathscr{B}_\infty, \rho_\infty)$ of $\mathscr{C}$. We'll use conditional expectations and the martingale convergence theorem (see Sect. 10.4) to do so. Define $\mathscr{B}_\infty$ by $\mathscr{B}_\infty = \sigma(\cup_n \mathscr{B}_n)$. Choose a linear functional $\Lambda$ on $\ell^\infty$ as given by Lemma F.1, and note that for each $x$ in $X$ the sequence $\{\rho_n(E(f|\mathscr{B}_n))(x)\}$ belongs to $\ell^\infty$ (of course, $E(f|\mathscr{B}_n)$ is only determined up to a null set, but then $\rho_n$, as a lifting, gives the same result whatever version of $E(f|\mathscr{B}_n)$ is used). Thus we can define an operator $L$ on $\mathscr{L}^\infty(X, \mathscr{B}_\infty, \mu)$ by $L(f)(x) = \Lambda(\{\rho_n(E(f|\mathscr{B}_n))(x)\})$. It follows from Proposition 10.4.12 that if $f \in \mathscr{L}^\infty(X, \mathscr{B}_\infty, \mu)$, then the sequence $\{E(f|\mathscr{B}_n)\}$ converges almost everywhere to $f$. Hence $\{\rho_n(E(f|\mathscr{B}_n))\}$ also converges almost everywhere to $f$ and so $L(f) = f$ a.e. In particular, since $\mu$ is complete, $L(f)$ is

We need to construct such a lifting $\rho_1$; we'll do that by choosing a set $E_1$ in such a way that

$$\rho_1(f_0\chi_{E_0} + f_1\chi_{E_0^c}) = \rho(f_0)\chi_{E_1} + \rho(f_1)\chi_{E_1^c} \tag{1}$$

defines a lifting $\rho_1$ that is an extension of $\rho$.

Suppose that we produce a set $E_1$ in $\sigma(\mathscr{A}_0 \cup \{E_0\})$ such that

(a) $\chi_{E_0} = \chi_{E_1}$ a.e.,
(b) if functions $f$ and $f'$ in $\mathscr{L}^\infty(X, \mathscr{A}_0, \mu)$ agree almost everywhere on $E_0$, then $\rho(f)$ and $\rho(f')$ agree everywhere on $E_1$ (that is, if $(f - f')\chi_{E_0} = 0$ a.e., then $(\rho(f) - \rho(f'))\chi_{E_1} = 0$), and
(c) if functions $f$ and $f'$ in $\mathscr{L}^\infty(X, \mathscr{A}_0, \mu)$ agree almost everywhere on $E_0^c$, then $\rho(f)$ and $\rho(f')$ agree everywhere on $E_1^c$.

Then it follows that

$$\begin{aligned}&\text{if } f_0\chi_{E_0} + f_1\chi_{E_0^c} = f_0'\chi_{E_0} + f_1'\chi_{E_0^c} \text{ a.e., then}\\ &\qquad \rho(f_0)\chi_{E_1} + \rho(f_1)\chi_{E_1^c} = \rho(f_0')\chi_{E_1} + \rho(f_1')\chi_{E_1^c}\end{aligned}$$

and

$$f_0\chi_{E_0} + f_1\chi_{E_0^c} = \rho(f_0)\chi_{E_1} + \rho(f_1)\chi_{E_1^c} \text{ a.e.}$$

This implies that Eq. (1) gives a well-defined function $\rho_1$ that is an extension of $\rho$ and satisfies the first two conditions in the definition of a lifting. The remaining conditions (that $\rho_1$ is linear and multiplicative and that it satisfies $\rho_1(1) = 1$) are easy to check.

We turn to the construction of the set $E_1$. Choose a sequence $\{C_n\}$ of sets that belong to $\mathscr{A}_0$, satisfy $\chi_{C_n} \le \chi_{E_0}$ a.e. for each $n$, and are such that

$$\sup_n \mu(C_n) = \sup\{\mu(C) : C \in \mathscr{A}_0 \text{ and } \chi_C \le \chi_{E_0} \text{ a.e.}\};$$

then define a set $F_1$ by $F_1 = \cup_n C_n$. Then $F_1$ has maximal measure among the sets in $\mathscr{A}$ that are included (except perhaps for a null set) in $E_0$, and each $\mathscr{A}$-measurable set that is included (up to a null set) in $E_0$ is also included (up to a null set) in $F_1$. A similar construction produces an analogous set $F_2$ that is included (up to a null set) in $E_0^c$. Now let $G_1 = \rho(F_1)$ and $G_2 = \rho(F_2)$.

*Claim.* The sets $G_1$ and $G_2$ satisfy

$$G_1 \cap G_2 = \varnothing, \tag{2}$$

$$\mu(G_1 - E_0) = 0 \quad \text{(that is, } G_1 \subseteq E_0 \text{ to within a null set), and} \tag{3}$$

$$\mu(G_2 - E_0^c) = 0 \quad \text{(that is, } G_2 \subseteq E_0^c \text{ to within a null set).} \tag{4}$$

For (2), note that $\chi_{F_1}\chi_{F_2} = 0$ a.e., which implies that $\chi_{G_1}\chi_{G_2} = \rho(\chi_{F_1}\chi_{F_2}) = 0$. Relation (3) follows from the fact that $\chi_{G_1} = \chi_{F_1}$ a.e. and $\chi_{F_1} \le \chi_{E_0}$ a.e., and (4) has a similar proof.

Now define $E_1$ by $E_1 = (E_0 \cup G_1) \cap G_2^c$. Then condition (a) above follows from (2)–(4). We turn to condition (b). Suppose that $f$ and $f'$ belong to $\mathscr{L}^\infty(X, \mathscr{A}_0, \mu)$ and agree almost everywhere on $E_0$. We need to show that $\rho(f) = \rho(f')$ on $E_1$. Let $D = \{x \in X : f(x) \neq f'(x)\}$. Then $\chi_D \le \chi_{F_2}$ a.e., and so $\rho(\chi_D) \le \rho(\chi_{F_2}) = \chi_{G_2}$. Since $D$ was defined so that $(f - f')\chi_{D^c} = 0$, we have $(\rho(f) - \rho(f'))\rho(\chi_{D^c}) = 0$. It follows that $(\rho(f) - \rho(f'))\chi_{G_2^c} = 0$, and so $\rho(f)$ and $\rho(f')$ agree everywhere outside $G_2$ and hence on $E_1$. This completes the proof of (b). The proof of (c) is similar, and with that the lemma is proved. □

**F.4. (Lemma)** *Suppose that $(X, \mathscr{A}, \mu)$ is a complete probability space and that $L_0$ is a linear lifting of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$. Then there is a lifting $\rho$ of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ such that*

$$\chi_{\{L_0(\chi_A)=1\}} \le \rho(\chi_A) \le \chi_{\{L_0(\chi_A)>0\}} \tag{5}$$

*holds for each $A$ in $\mathscr{A}$.*

The significance of (5) will become clear when we use Lemma F.4 to prove Theorem F.5.

*Proof.* Let $S$ be the Cartesian product $\mathscr{L}^\infty(X, \mathscr{A}, \mu) \times X$. We will identify linear liftings of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ with functions from $S$ to $\mathbb{R}$, that is, we will identify a linear lifting $L$ of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ with the function $L' \colon S \to \mathbb{R}$ defined by $L'(f,x) = L(f)(x)$. Thus we will view linear liftings as members of the product space $\mathbb{R}^S$. The plan for the current proof is to define a certain subset $C$ of $\mathbb{R}^S$, to show that $C$ is nonempty, compact, and convex, and then to show that the extreme points of $C$ (which exist, according to Lemma F.2) are liftings that satisfy (5). That will complete the proof of the lemma.

Let us look at how the conditions defining liftings and linear liftings translate into conditions on elements of $\mathbb{R}^S$. For example, the condition that $L$ satisfies $L(af + bg) = aL(f) + bL(g)$ for all $a$, $b$, $f$, and $g$ becomes the condition that the corresponding function $L'$ satisfies

$$L'(af + bg, x) = aL'(f,x) + bL'(g,x) \text{ for all } a, b, f, g, \text{ and } x. \tag{6}$$

Note also that, since all the coordinate projections $L' \mapsto L'(f,x)$ of $\mathbb{R}^S$ are continuous, those elements of $\mathbb{R}^S$ that satisfy (6) form a closed subset of $\mathbb{R}^S$.

We now define the set $C$ to be the collection of all $L'$ in $\mathbb{R}^S$ that satisfy the translations into conditions on $L'$ of conditions (a), (c), (e), and (f) in the definition of a linear lifting, plus the translation of the relation

$$\chi_{\{L_0(\chi_A)=1\}} \le L(\chi_A) \le \chi_{\{L_0(\chi_A)>0\}}, \tag{7}$$

which is to hold for all $A$ in $\mathscr{A}$. It is easy to check that $C$ is closed and convex. Furthermore, conditions (c), (e), and (f) imply that

$$|L'(f,x)| \le \|f\|_\infty \tag{8}$$

holds for all $L'$ in $C$ and all $f$ and $x$; hence we can use Tychonoff's theorem to conclude that $C$ is compact. Finally, the function in $\mathbb{R}^S$ that corresponds to $L_0$ belongs to $C$, and so $C$ is nonempty.

It now follows from Lemma F.2 that $C$ has at least one extreme point, say $L_1'$. Let us reverse our translation from linear liftings to elements of $\mathbb{R}^S$, and let $L_1$ be the function from $\mathscr{L}^\infty(X,\mathscr{A},\mu)$ to functions[4] on $X$ that corresponds to the extreme point $L_1'$. We need to show that $L_1(f)$ is measurable and bounded, that $f = L_1(f)$ a.e., and that $L_1(fg) = L_1(f)L_1(g)$; the other conditions that $L_1$ must satisfy to be a lifting come from the conditions we placed on $C$.

It follows from (8) that $L_1(f)$ is bounded, and in fact that $\|L_1(f)\|_\infty \le \|f\|_\infty$. We turn to the measurability of $L_1(f)$ and the requirement that $f = L_1(f)$ a.e. If we use (7), plus the fact that $f = L_0(f)$ a.e. (recall that $L_0$ is a linear lifting), we find that each $f$ of the form $\chi_A$ satisfies $f = L_1(f)$ a.e. Since $\mu$ is complete, the measurability of $L_1(f)$ follows for such $f$. The measurability of $L_1(f)$ and the almost everywhere validity of $f = L_1(f)$ now follow first for simple $\mathscr{A}$-measurable functions and then for arbitrary $\mathscr{A}$-measurable functions (approximate an arbitrary function with simple functions, and use (8)).

We still need to show that $L_1$ is multiplicative,[5] in the sense that $L_1(fg) = L_1(f)L_1(g)$ holds for all $f$ and $g$. It is easy to see that we only need to check the identity $L_1(fg) = L_1(f)L_1(g)$ in the case where $0 \le g \le 1$ (use the linearity of $L_1$ and the fact that $L_1(1) = 1$). So assume that $g$ belongs to $\mathscr{L}^\infty(X,\mathscr{A},\mu)$ and satisfies $0 \le g \le 1$, and define functions $L_{1+}, L_{1-} \colon \mathscr{L}^\infty(X,\mathscr{A},\mu) \to \mathscr{L}^\infty(X,\mathscr{A},\mu)$ by

$$L_{1+}(f) = L_1(f) + (L_1(fg) - L_1(f)L_1(g)) \text{ and}$$
$$L_{1-}(f) = L_1(f) - (L_1(fg) - L_1(f)L_1(g)).$$

It is easy to check that $L_{1+}$ and $L_{1-}$ are linear liftings. We want to verify that they correspond to members of $C$, and for this we need to check that they satisfy (7). The keys to this will be the fact that $L_1 = \frac{1}{2}L_{1+} + \frac{1}{2}L_{1-}$, together with the fact that if $A \in \mathscr{A}$, then (since $L_{1+}$ and $L_{1-}$ are linear liftings) the values of the functions $L_{1+}(\chi_A)$ and $L_{1-}(\chi_A)$ belong to the interval $[0,1]$. Since $L_1$ corresponds to an element of $C$, it satisfies (7); thus if $L_0(\chi_A)(x) = 1$, then we can conclude that $L_1(\chi_A)(x) = 1$ (use (7)) and then that $L_{1+}(\chi_A)(x) = L_{1-}(\chi_A)(x) = 1$ (use the fact that $L_1 = \frac{1}{2}L_{1+} + \frac{1}{2}L_{1-}$, plus the fact that the values of $L_{1+}(\chi_A)$ and

[4] We cannot yet say "from $\mathscr{L}^\infty(X,\mathscr{A},\mu)$ to $\mathscr{L}^\infty(X,\mathscr{A},\mu)$," because we still need to verify that the functions $x \mapsto L_1(f,x)$ are measurable and bounded.

[5] Here is where we use the fact that $L_1$ corresponds to an extreme point in $C$.