Proof. This is a direct application of the abstract Bayes formula. Assume first that ηX is an $\mathcal{F}$-martingale under $\mathbb{P}$ so that $\mathbb{E}^{\mathbb{P}}(\eta_t X_t \mid \mathcal{F}_u) = \eta_u X_u$ for any $0 \le u \le t \le T$. Then the Bayes formula yields,

$$\mathbb{E}^{\mathbb{Q}}(X_t \mid \mathcal{F}_u) = \frac{\mathbb{E}^{\mathbb{P}}(\eta_T X_t \mid \mathcal{F}_u)}{\mathbb{E}^{\mathbb{P}}(\eta_T \mid \mathcal{F}_u)} = \frac{\mathbb{E}^{\mathbb{P}}(X_t \mathbb{E}^{\mathbb{P}}(\eta_T \mid \mathcal{F}_t) \mid \mathcal{F}_u)}{\mathbb{E}^{\mathbb{P}}(\eta_T \mid \mathcal{F}_u)} = \frac{\mathbb{E}^{\mathbb{P}}(X_t \eta_t \mid \mathcal{F}_u)}{\eta_u} = \frac{X_u \eta_u}{\eta_u} = X_u$$

for any $0 \le u \le t \le T$. We conclude that X is an $\mathcal{F}$-martingale under $\mathbb{Q}$. The proof of the converse implication goes along the same lines. ∎