*Proof.* By Definition 1.3.1, we just need to show that $X^{-1}(\mathcal{C}) \subseteq \mathcal{A}$ iff $X^{-1}(\sigma(\mathcal{C})) \subseteq \mathcal{A}$.

The backward direction is straightforward since $\mathcal{C} \subseteq \sigma(\mathcal{C})$. To show the forward direction, suppose $X^{-1}(\mathcal{C}) \subseteq \mathcal{A}$. Define

$$\mathcal{K} = \{B \in \sigma(\mathcal{C}) \mid X^{-1}(B) \in \mathcal{A}\}.$$

It is easy to verify that $\mathcal{K}$ is a $\sigma$-algebra (by using results on pre-image of complement, union, and intersection). By assumption, $\mathcal{C} \subseteq \mathcal{K}$ and therefore $\sigma(\mathcal{C}) \subseteq \mathcal{K}$. The definition of $\mathcal{K}$ then implies $X^{-1}(\sigma(\mathcal{C})) \subseteq \mathcal{A}$. ■