Examples 0.0.1 (Ito's lemma, 1D). We can use Ito formula to determine whether a process is a martingale (no drift), submartingale (drift > 0) or supermartingale (drift < 0).

1. $f(W_t) = (W_t)^2$. Then we have for all $t \in [0, T]$,

$$df(W_t) = 2W_t\,dW_t + \frac{1}{2} \cdot 2\,dt \implies W_t^2 = W_0^2 + \int_0^t 2W_s\,dW_s + \int_0^t ds = \int_0^t 2W_s\,dW_s + t.$$

   This also gives the Doob–Meyer decomposition (Theorem 2.4.2) of the positive submartingale W^2. Also recall that $W_t^2 - t = \int_0^t 2W_s\,dW_s$ is a martingale (e.g. Theorem 3.3.3, or Example 4.1.2).

2. $Y_t = g(t, W_t) = e^{\mu t + W_t}$. Let $g(t, x) = e^{\mu t + x}$ and then $g_t = \mu Y_t$, $g_x = Y_t$, $g_{xx} = Y_t$ and so

$$\begin{aligned} dY_t &= \mu Y_t\,dt + Y_t\,dW_t + \frac{1}{2} \cdot Y_t\,d\langle W\rangle_t = \left(\mu + \frac{1}{2}\right) Y_t\,dt + Y_t\,dW_t \\ \implies Y_t &= Y_0 + \int_0^t \underbrace{\left(\mu + \frac{1}{2}\right) Y_u\,du}_{\text{increasing}} + \int_0^t \underbrace{Y_u\,dW_u}_{\text{a martingale, as } Y \in L^2_{\mathbb{P}}(W)}. \end{aligned}$$

   Therefore, we conclude that Y is a submartingale.

3. (integration by parts, Corollary 4.4.1) $Z_t = \sin(W_t)\cos(W_t)$. Then

$$dZ_t = \sin W_t\,d(\cos W_t) + \cos W_t\,d(\sin W_t) + d\langle \sin W, \cos W\rangle_t.$$

   By Ito's lemma,

$$\begin{aligned} d(\cos W_t) &= -\sin W_t\,dW_t - \frac{1}{2}\cos W_t\,dt, \\ d(\sin W_t) &= \cos W_t\,dW_t - \frac{1}{2}\sin W_t\,dt. \end{aligned}$$

   Also $d\langle \sin W, \cos W\rangle_t = -\sin W_t \cos W_t\,dt$, so

$$dZ_t = -\sin^2(W_t)\,dW_t + \cos^2(W_t)\,dW_t - 2\sin W_t \cos W_t\,dt$$

   The sign of the drift term is not always positive or negative, and therefore Z is none of a martingale, submartingale or supermartingale.