Theorem 0.0.1 (Girsanov theorem for continuous semimartingales)**.** *Let $\widetilde{\mathbb{P}} \sim \mathbb{P}$ be two equivalent probability measures on $(\Omega, \mathcal{F}_T)$ with Radon–Nikodym density η_T. We assume, in addition, that the Radon–Nikodym density process $\eta_t := \mathbb{E}^{\mathbb{P}}(\eta_T \mid \mathcal{F}_t)$, $t \in [0,T]$ is continuous*[1]*. Then any continuous real-valued $\mathbb{P}$-semimartingale X is a continuous $\widetilde{\mathbb{P}}$-semimartingale. If the canonical decomposition of X under $\mathbb{P}$ is $X = X_0 + M + A$ then its canonical decomposition under $\widetilde{\mathbb{P}}$ is $X = X_0 + \widetilde{M} + \widetilde{A}$ where*

$$\widetilde{M}_t = M_t - \int_0^t \frac{1}{\eta_u}\, d\langle \eta, M\rangle_u = M_t - \langle \mathcal{L}(\eta), M\rangle_t$$

and

$$\widetilde{A}_t = A_t + \int_0^t \frac{1}{\eta_u}\, d\langle \eta, M\rangle_u = A_t + \langle \mathcal{L}(\eta), M\rangle_t$$

In particular, X follows a local martingale under $\widetilde{\mathbb{P}}$ if and only if the process $\widetilde{A} = A + \langle \mathcal{L}(\eta), M\rangle$ vanishes identically (Definition 5.2.1), that is, $A_t + \langle \mathcal{L}(\eta), M\rangle_t = 0$ for every $t \in [0,T]$.

The transform $\varphi : M \mapsto M - \langle \mathcal{L}(\eta), M\rangle$ is called ***Girsanov transform****.*

[1] We know already that this holds if $\mathcal{F} = \mathcal{F}^W$ for some Brownian motion W, see Proof of Proposition 7.4.4.