Proof. It can be easily checked that $\mathbb{E}^{\mathbb{P}}(\eta \mid \mathcal{G})$ is strictly positive $\mathbb{P}$-a.s. so that the RHS is well-defined. By our assumption, the random variable $\eta\psi$ is $\mathbb{P}$-integrable, it is therefore enough to show that

$$\mathbb{E}^{\mathbb{P}}(\eta\psi \mid \mathcal{G}) = \mathbb{E}^{\mathbb{Q}}(\psi \mid \mathcal{G})\mathbb{E}^{\mathbb{P}}(\eta \mid \mathcal{G}).$$

We want to verify that, for all $A \in \mathcal{G}$,

$$\mathbb{E}^{\mathbb{P}}(\eta\psi\mathbf{1}_A) = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}(\psi \mid \mathcal{G})\mathbb{E}^{\mathbb{P}}(\eta \mid \mathcal{G})\mathbf{1}_A\right].$$

Write $Y = \mathbb{E}^{\mathbb{Q}}(\psi \mid \mathcal{G})$. Then

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}(\psi \mid \mathcal{G})\mathbb{E}^{\mathbb{P}}(\eta \mid \mathcal{G})\mathbf{1}_A\right] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}(\eta Y \mid \mathcal{G})\mathbf{1}_A\right] && \text{(since } Y \text{ is } \mathcal{G}\text{-measurable)} \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}(\eta Y\mathbf{1}_A \mid \mathcal{G})\right] && \text{(since } \mathbf{1}_A \text{ is } \mathcal{G}\text{-measurable)} \\
&= \mathbb{E}^{\mathbb{P}}\left[\eta Y\mathbf{1}_A\right] && \text{(tower property)} \\
&= \mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_A\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}(\psi \mid \mathcal{G})\mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}(\psi\mathbf{1}_A \mid \mathcal{G})\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\psi\mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{P}}(\eta\psi\mathbf{1}_A)
\end{aligned}$$

The definition of conditional expectation then gives the desired result. ∎