need to prove that

(i) $\widetilde{M}$ is a local martingale under $\widetilde{\mathbb{P}}$.

(ii) $\widetilde{A}$ is a process of finite variation with $\widetilde{A}_0 = 0$ (i.e. almost all sample paths are of finite variation on $[0, T]$).

To show (ii), recall the polarisation formula

$$\langle X, Y \rangle = \frac{1}{4} \left(\langle X + Y \rangle - \langle X - Y \rangle \right),$$

which is the difference of two increasing processes and therefore must have finite variation (Theorem 3.3.1). Hence, $\widetilde{A} = A + \langle \mathcal{L}(\eta), M \rangle$ being the sum of two processes of finite variation, must be a process of finite variation itself.

To show (i), we check its equivalence: $\eta \widetilde{M}$ is a local martingale under $\mathbb{P}$. Consider the dynamics of $\eta \widetilde{M}$:

$$\begin{aligned}
d\eta_t \widetilde{M}_t &= \eta_t \, d\widetilde{M}_t + \widetilde{M}_t \, d\eta_t + d\langle \eta, \widetilde{M} \rangle_t && \text{(integration by parts)} \\
&= \eta_t \, d[M_t - \langle \mathcal{L}(\eta), M \rangle_t] + \widetilde{M}_t \, d\eta_t + d\langle \eta, M - \langle \mathcal{L}(\eta), M \rangle \rangle_t \\
&= \eta_t \, dM_t - \eta_t \, d\langle \mathcal{L}(\eta), M \rangle_t + \widetilde{M}_t \, d\eta_t + d\langle \eta, M \rangle_t && (\langle \mathcal{L}(\eta), M \rangle \text{ has finite variation}) \\
&= \eta_t \, dM_t + \widetilde{M}_t \, d\eta_t && (\text{observe } d\langle \eta, M \rangle_t = \eta_t \, d\langle \mathcal{L}(\eta), M \rangle_t)
\end{aligned}$$

where M and η are both $\mathbb{P}$-local martingales. Hence, $\eta \widetilde{M}$ is also a $\mathbb{P}$-local martingale, completing the proof.

+end$_s rc$

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