Proof. (i) First note that the process $A_t = \int_0^t \alpha_u \, du$ is always a process of finite variation (thus zero quadratic variation). This is clear when $\alpha \geq 0$ and A_t is increasing, since the total variation on $[0, T]$ is just A_T (Theorem 3.3.1). If α is not positive, we can write $\alpha = \alpha^+ - \alpha^-$ and $A_t = A_t^+ - A_t^-$ where $\alpha^+ = \max(\alpha, 0)$, $\alpha^- = \max(-\alpha, 0)$ and

$$A_t^{\pm} = \int_0^t \alpha_s^{\pm} \, ds$$

and it is not difficult to see that the total variation of A is bounded by $A_T^+ + A_T^-$. The quadratic variation of the process $B_t = \int_0^t \beta_u \cdot dW_u$ is given in Theorem 4.1.1, thus the result.

(ii) We can use the polarisation formula:

$$\begin{aligned}
\langle X, Y \rangle_t &= \frac{1}{2} \left(\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t \right) \\
&= \frac{1}{2} \left(\left\langle \int (\alpha_u + \beta_u) \, dW_u, \int (\alpha_u + \beta_u) \, dW_u \right\rangle_t - \left\langle \int \alpha_u \, dW_u \right\rangle_t - \left\langle \int \beta_u \, dW_u \right\rangle_t \right) \\
&= \frac{1}{2} \left(\int_0^t (\alpha_u + \beta_u)^2 \, ds - \int_0^t \alpha_u^2 \, ds - \int_0^t \beta_u^2 \, ds \right) \\
&= \int_0^t \alpha_u \beta_u \, ds,
\end{aligned}$$

which concludes the proof. ■