Proof. We verify the axioms for a σ-algebra:

(i) $X^{-1}(\Psi) = \Omega \Rightarrow \Omega \in X^{-1}(\mathcal{G})$ as $\Psi \in \mathcal{G}$.

(ii) Let $S \in X^{-1}(\mathcal{G})$. Then $S = X^{-1}(G)$ for some $G \in \mathcal{G}$. We know that the pre-image of set difference equals the set difference of the respective pre-images, so

$$\Omega \setminus S = X^{-1}(\Psi) \setminus X^{-1}(G) = X^{-1}(\underbrace{\Psi \setminus G}_{\in \mathcal{G}}) \Rightarrow \Omega \setminus S \in X^{-1}(\mathcal{G}).$$

(iii) Let $(S_i)_{i \geq 1} \in X^{-1}(\mathcal{G})$. Then $S_i = X^{-1}(G_i)$ for some $G_i \in \mathcal{G}$ for all i. It follows that

$$\bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} X^{-1}(G_i) = X^{-1}\left(\bigcup_{i=1}^{\infty} G_i\right) \in X^{-1}(\mathcal{G}),$$

where we used the fact that the union of pre-images is the same as the pre-image of the union.

■