ocal martingales]

(i) Any non-negative $\mathcal{F}$-local martingale M is an $\mathcal{F}$-supermartingale[1]. If, in addition, $\mathbb{E}(M_t)$ is constant then M is an $\mathcal{F}$-martingale.

(ii) (corollary of above) Let M be a non-negative $\mathcal{F}$-local martingale. If $M_0 = 0$ then $M_t = 0$ for every $t \in [0, T]$.

(iii) (quadratic variation) Let M be a continuous $\mathcal{F}$-local martingale. Then the quadratic variation $\langle M \rangle$ is the unique continuous, increasing and $\mathcal{F}$-adapted process with $\langle M \rangle_0 = 0$ such that the process $M^2 - \langle M \rangle$ is a continuous $\mathcal{F}$-local martingale.

Remark 0.0.1. The quadratic variation is invariant with respect to an $\mathcal{F}_0$-measurable shift of M; specifically, if $N = \psi + M$ for some $\mathcal{F}_0$-measurable random variable ψ is a continuous local martingale, then $\langle N \rangle = \langle M \rangle$. In particular, $\langle M \rangle = \langle M - M_0 \rangle$.

(iv) Let M be a continuous $\mathcal{F}$-local martingale s.t. $\langle M \rangle_t = 0$ for $t \in [0, T]$. Then $M_t = M_0$ for every $t \in [0, T]$.

(v) Let M be a continuous $\mathcal{F}$-local martingale of finite variation[2]. Then $M_t = M_0$ for every $t \in [0, T]$.

(vi) (corollary of above) If $M_0 = 0$ and M is a continuous $\mathcal{F}$-local martingale of finite variation, then M vanishes (more precisely, it is indistinguishable from the null process).

Proof. (

[1]The assumption that M is non-negative can be replaced by the assumption that $M \geq \eta$ for some random variable η with $\mathbb{E}(\eta) > -\infty$.

[2]An $\mathcal{F}$-adapted stochastic process $X = (X_t)_{t \in [0,T]}$ is said to be of finite variation if almost all sample paths of X are functions of finite variation on $[0, T]$.