Proof. For (1), the trivial σ-algebra is obviously the smallest. Further, X is measurable w.r.t. $\mathcal{A}_0$ as

$$X^{-1}(B) = \begin{cases} \emptyset & \text{if } c \notin B \\ \Omega & \text{if } c \in B \end{cases}$$

for all $B \in \mathcal{B}(\mathbb{R})$. As $X^{-1}(\mathcal{B}(\mathbb{R})) \subseteq \mathcal{A}_0$, by definition $\sigma(X) = \mathcal{A}_0$.

For (2), let X be any random variable on $(\Omega, \mathcal{A}, \mathbb{P})$. The earlier result implies that $\mathbb{E}[X]$ is $\mathcal{A}_0$-measurable. Next, we check that

$$\begin{cases} \mathbb{E}(\mathbf{1}_\emptyset X) = \mathbb{E}(\mathbf{1}_\emptyset \mathbb{E}(X)) = 0 \\ \mathbb{E}(\mathbf{1}_\Omega X) = \mathbb{E}(\mathbf{1}_\Omega \mathbb{E}(X)) = \mathbb{E}(X) \end{cases}$$

That is, $\mathbb{E}[\mathbf{1}_A X] = \mathbb{E}[\mathbf{1}_A \mathbb{E}[X]]$ for all $A \in \mathcal{A}_0$. The required equality follows immediately from the uniqueness of conditional expectation. ■