Examples 0.0.1 (Ito's lemma, multidimensional). Let

$$R_t(d) := \sqrt{(W_t^1)^2 + (W_t^2)^2 + \cdots + (W_t^d)^2}$$

where $W = (W^1, W^2, \ldots, W^d)$ is a standard d-dimensional Brownian motion for $d \geq 3$. The process R is called a Bessel process. We want to show that $1/R_t(3)$ for $1 \leq t \leq T$ is a local martingale but not a martingale.

1. To show it is a local martingale, we apply Ito's lemma to compute the dynamics of $1/R_t(3)$ (and show that there is no drift term). Let

   $$Y_t = R_t^2(3) = (W_t^1)^2 + (W_t^2)^2 + (W_t^3)^2 \implies dY_t = \sum_{i=1}^{3} 2W_t^i\, dW_t^i + 3\, dt.$$

   But $1/R_t(3) = 1/\sqrt{Y_t} = g(Y_t)$ where $g(x) = 1/\sqrt{x}$. Strictly speaking, one cannot apply directly Ito's formula as g is not differentiable at zero. But for $d \geq 3$, the d-dimensional Brownian motion never return to the origin after time 0. So (heuristically) one can apply Ito formula again to obtain that

   $$\begin{aligned} d(1/R_t(3)) &= -\frac{1}{2}\frac{1}{Y_t^{3/2}}\, dY_t + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{Y_t^{5/2}}\, d\langle Y\rangle_t \\ &= -\frac{1}{2}\frac{1}{Y_t^{3/2}}\left(\sum_{i=1}^{3} 2W_t^i\, dW_t^i + 3\, dt\right) + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{Y_t^{5/2}}(4Y_t\, dt) \\ &= \frac{1}{R_t^3}\sum_{i=1}^{3} W_t^i\, dW_t^i = \frac{1}{R_t^3} W_t \cdot dW_t \end{aligned}$$

   as $d\langle Y\rangle_t = \sum_{i=1}^{3}(2W_t^i)^2\, dt = 4R_t^2\, dt$. This is not a true martingale as $\|1/R_t^3 \cdot W_t\|_W^2 = \infty$.

2. We can also show that

   $$\mathbb{E}[1/R_t(d)] = \frac{1}{\sqrt{t}}\mathbb{E}\left[\frac{1}{\sqrt{X}}\right] = \sqrt{\frac{2}{\pi t}}$$

   where $X \sim \chi_3^2$. As the expectation is decreasing in t, $1/R_t(3)$ is indeed a supermartingale (see also Theorem 5.1.1).