Proof. To establish the proposition, we make use of the abstract Bayes formula (Lemma 7.3.1) and Lévy's characterisation theorem (Lemma 5.3.1). In view of the latter, it suffices to show that for any $\lambda \in \mathbb{R}$ the process

$$M_t^\lambda = \exp\left(\lambda \widetilde{W}_t - \frac{1}{2}\lambda^2 t\right), \quad \forall t \in [0,T],$$

is an $\mathcal{F}$-martingale under $\widetilde{\mathbb{P}}$. By Lemma 7.3.2, we can equivalently check that ηM^λ is an $\mathcal{F}$-martingale under $\mathbb{P}$. But

$$\eta_t M_t^\lambda = \exp\left(\gamma W_t - \frac{1}{2}\gamma^2 t\right) \exp\left(\lambda(W_t - \gamma t) - \frac{1}{2}\lambda^2 t\right) = \exp\left(\alpha W_t - \frac{1}{2}\alpha^2 t\right)$$

where $\alpha = \lambda + \gamma$. Clearly $\mathbb{E}^{\mathbb{P}}(\eta_t M_t^\lambda) = 1$ for all $t \in [0,T]$ and thus this process follows an $\mathcal{F}$-martingale under $\mathbb{P}$. From the Lévy characterisation theorem we conclude that $\widetilde{W}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. ■