Proof. For (1), $\mathbb{E}(X|\mathcal{D})$ is clearly $\sigma(\mathcal{D})$-measurable. Now from the uniqueness of conditional expectation (Definition 1.5.2), it suffices to check that for all $A \in \sigma(\mathcal{D})$,

$$\mathbb{E}[\mathbb{E}(X|\mathcal{D})\mathbf{1}_A] = \mathbb{E}[\mathbf{1}_A X].$$

We can see that this indeed holds as

$$\begin{aligned} \mathbb{E}[\mathbf{1}_A X] &= \mathbb{E}[\mathbb{E}[\mathbf{1}_A X|\mathcal{D}]] && \text{(tower property)} \\ &= \mathbb{E}[\mathbf{1}_A \mathbb{E}[X|\mathcal{D}]] && \text{(taking out what is known).} \end{aligned}$$

For (2), X is $\sigma(\mathcal{D}(X))$-measurable, so $\sigma(X) \subseteq \sigma(\mathcal{D}(X))$. On the other hand, $\mathcal{D}(X) \subseteq \sigma(X) \Rightarrow \sigma(\mathcal{D}(X)) \subseteq \sigma(X)$, hence the equality. ∎