Proof. (Sketch) We consider

$$\mathbb{E}\left[(X-Z)^2\right] = \mathbb{E}\left[(X-\mathbb{E}(X|\mathcal{G})+\mathbb{E}(X|\mathcal{G})-Z)^2\right]$$

$$= \mathbb{E}\left[(X-\mathbb{E}(X|\mathcal{G}))^2\right] + \mathbb{E}\left[(\mathbb{E}(X|\mathcal{G})-Z)^2\right] + 2\underbrace{\mathbb{E}\left[(X-\mathbb{E}(X|\mathcal{G}))(\mathbb{E}(X|\mathcal{G})-Z)\right]}_{(*)}$$

But

$$(*) = \mathbb{E}\left[\mathbb{E}\left[(X-\mathbb{E}(X|\mathcal{G}))(\mathbb{E}(X|\mathcal{G})-Z)|\mathcal{G}\right]\right] \qquad \text{(tower property)}$$

$$= \mathbb{E}\left[(\mathbb{E}(X|\mathcal{G})-Z)\underbrace{\mathbb{E}\left[(X-\mathbb{E}(X|\mathcal{G}))|\mathcal{G}\right]}_{=\mathbb{E}(X|\mathcal{G})-\mathbb{E}(X|\mathcal{G})=0}\right] = 0 \qquad \text{(taking out what is known)}$$

and therefore $\mathbb{E}\left[(X-Z)^2\right] \geqslant \mathbb{E}\left[(X-\mathbb{E}[X|\mathcal{G}])^2\right]$. ■