Proof. **Existence.** Suppose $X \in L^1(\Omega, \mathbb{P})$ then $X^{\pm} \in L^1(\Omega, \mathbb{P})$. Without loss of generality, we may assume that $X \geq 0$. Define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ by setting for any $A \in \mathcal{A}$,

$$\mathbb{Q}(A) := \frac{\mathbb{E}[\mathbf{1}_A X]}{\mathbb{E}[X]}$$

The probability $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{A}$ (i.e. for $A \in \mathcal{A}$, $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$). This implies that $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{G} \subseteq \mathcal{A}$ (Remark following Definition 7.1.1). Therefore from the Radon–Nikodym Theorem, there exists a positive $\mathcal{G}$-measurable Radon–Nikodym derivative $\eta = d\mathbb{Q}/d\mathbb{P} \in L^1(\Omega, \mathbb{P})$ such that

$$\begin{aligned} \mathbb{E}[\mathbf{1}_A X] &=: \mathbb{E}[X]\mathbb{Q}(A) \\ &= \mathbb{E}[X]\mathbb{E}[\eta \mathbf{1}_A] \\ &= \mathbb{E}[\mathbb{E}[X]\mathbb{E}[\eta \mathbf{1}_A]|\mathcal{G}] \\ &= \mathbb{E}[\eta \mathbb{E}[X]\mathbf{1}_A] \end{aligned}$$

then $\eta\mathbb{E}[X]$ is a version of the conditional expectation $\mathbb{E}[X|\mathcal{G}]$.

Uniqueness. Suppose Y and Y' are both $\mathcal{G}$-conditional expectation of X. Let $G = \{\omega : Y(\omega) > Y'(\omega)\}$ and we assume that $\mathbb{P}(G) > 0$. To this end, we note that

$$G := \{Y - Y' > 0\} = \bigcup_{n=1}^{\infty} \{Y - Y' > \tfrac{1}{n}\}$$

$$G_n := \{Y - Y' > \tfrac{1}{n}\} = \bigcup_{j=1}^{n} \{Y - Y' > \tfrac{1}{j}\}$$

By Theorem 1.2.1, $G_n \uparrow G \Rightarrow \mathbb{P}(G_n) \uparrow \mathbb{P}(G)$, so there exists $m > 0$ such that $\mathbb{P}(G_m) > 0$.

Since Y and Y' are both $\mathcal{G}$-conditional expectations, we have by (ii) of Definition 1.5.2 that for every $A \in \mathcal{G}$, in particular G, we have $\mathbb{E}[\mathbf{1}_G Y] = \mathbb{E}[\mathbf{1}_G Y'] \Rightarrow \mathbb{E}[\mathbf{1}_G(Y - Y')] = 0$. But

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_G(Y - Y')\right] &\geq \mathbb{E}\left[\mathbf{1}_{G_m}(Y - Y')\right] && (\mathbf{1}_G \geq \mathbf{1}_{G_m}) \\ &\geq \frac{1}{m}\mathbb{P}[G_m] > 0 && (\mathbf{1}_{G_m} = \mathbf{1}_{\{Y - Y' > 1/m\}}) \end{aligned}$$

This is a contradiction and hence $\mathbb{P}(G) = 0$. ■