Proof. **Existence.** We prove the existence of Doob decomposition by construction. Define the processes A and M by setting, for every $n \in I = \{0, 1, \dots, N\}$,

$$A_n = \sum_{k=1}^{n} \left(\mathbb{E}[X_k|\mathcal{F}_{k-1}] - X_{k-1}\right)$$

and

$$M_n = X_0 + \sum_{k=1}^{n} \left(X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]\right).$$

It is easy to check that $X_n = M_n + A_n$ for all $n \in I$, by writing $X_n - X_0$ as a telescoping sum.

It is clear that A is increasing since $\mathbb{E}[X_k|\mathcal{F}_{k-1}] - X_{k-1} \geq 0$. To check that M is a martingale, we note that

$$\begin{aligned} \mathbb{E}[M_n|\mathcal{F}_{n-1}] &= \mathbb{E}[M_{n-1} + X_n - \mathbb{E}[X_n|\mathcal{F}_{n-1}] \mid \mathcal{F}_{n-1}] \\ &= M_{n-1} + \mathbb{E}[X_n|\mathcal{F}_{n-1}] - \mathbb{E}[X_n|\mathcal{F}_{n-1}] \\ &= M_{n-1} \end{aligned}$$

for all $n = 1, \dots, N$.

Uniqueness. Let $X = M' + A'$ be an additional decomposition. Then the process $Y := M - M' = A' - A$ is a martingale, implying that

$$\mathbb{E}[Y_n|\mathcal{F}_{n-1}] = Y_{n-1},$$

and also predictable (as A is predictable), implying that

$$\mathbb{E}[Y_n|\mathcal{F}_{n-1}] = Y_n,$$

for any $n = 1, \dots, N$. Since $Y_0 = A'_0 - A_0 = 0$ by the convention about the starting point of the predictable processes, this implies iteratively that $Y_n = 0$ almost surely for all $n = 0, 1, \dots, N$. ∎