Definition 0.0.1 (σ-algebra generated by families of sets).

- Let $\mathcal{C}$ be an arbitrary family of subsets of Ω. We denote by $\sigma(\mathcal{C})$ the smallest σ-algebra which contains every set in $\mathcal{C}$ (i.e. $\mathcal{C} \subseteq \sigma(\mathcal{C})$) and call this the σ-algebra generated by $\mathcal{C}$.

- (Borel σ-algebra) An important example is the Borel σ-algebra over any topological space Ω, denoted by $\mathcal{B}(\Omega)$, which is the σ-algebra generated by the open sets of Ω (or, equivalently, by the closed sets[1]). In other words, $\mathcal{B}(\Omega) := \sigma(\mathcal{O}(\Omega))$, where $\mathcal{O}(\cdot)$ denotes the collection of all open sets.

  - Recall that an open set of $\mathbb{R}$ is a subset $E \subseteq \mathbb{R}$ such that for every $x \in E$ there exists $\epsilon > 0$ such that $B_\epsilon(x) = \{y \in \mathbb{R} : |x - y| < \epsilon\}$ is contained in E.
  - A set $F \subseteq \mathbb{R}$ is said to be closed if F^c is open.
  - $\mathbb{R}$ and $\emptyset$ are simultaneously both open and closed sets.

- (σ-algebra generated by a random variable) Given $X : (\Omega, \mathcal{A}) \to (\Psi, \mathcal{G})$, the σ-algebra generated by X, denoted $\sigma(X)$ is the smallest σ-algebra on Ω such that X is a random variable, that is, X is measurable with respect to $\sigma(X)$ and $\mathcal{G}$. Equivalently, $\sigma(X) = X^{-1}(\mathcal{G}) = \{X^{-1}(S) \mid S \in \mathcal{G}\}$ (by Definition 1.3.1 and Theorem 1.3.2).

[1] To show this, we just need to show that $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\Omega)$ and $\mathcal{B}(\Omega) \subseteq \sigma(\mathcal{C})$. All closed sets are complements of open sets. Since $\mathcal{B}(\Omega)$ being a σ-algebra is closed under complement, it contains all the closed sets, i.e. $\mathcal{C} \subseteq \mathcal{B}(\Omega) \Rightarrow \sigma(\mathcal{C}) \subseteq \mathcal{B}(\Omega)$. A similar argument can be used to show that $\mathcal{O} \subseteq \sigma(\mathcal{C}) \Rightarrow \mathcal{B}(\Omega) = \sigma(\mathcal{O}) \subseteq \sigma(\mathcal{C})$.